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SIGNATURE QUANTIZATION

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Abstract

We associate to the action of a compact Lie group G on a line bundle over a compact oriented even-dimensional manifold a virtual representation of G using a twisted version of the signature operator. We obtain analogues of various theorems in the more standard theory of geometric quantization. Some of these results were announced in Guillemin, Sternberg and Weitsman, 2003.

1. Introduction

Let M be a compact oriented manifold of dimension $2d$. It has its de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots .$$

If we equip M with a Riemann metric we get a Hodge $\star$ operator coming from the metric and orientation:

$$\star : \Omega^q \to \Omega^{2d-q}$$

and a δ operator

$$\delta : \Omega^q \to \Omega^{q-1}, \qquad \delta := -\star d \star .$$

If we modify $\star$ by defining

$$\alpha := (-1)^{q(q-1)/2} \star$$

then

$$\alpha^2 = (-1)^d$$

and

$$\alpha(d+\delta) = -(d+\delta)\alpha.$$

(We shall review these basic facts together with the application of the Atiyah-Bott fixed formula in Appendix I to this paper.) Hence if we

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define $\Omega^{\pm}$ to be the ± 1 eigenspaces of $\alpha \otimes i^d$ acting on $\Omega_{\mathbb{C}} = \Omega \otimes \mathbb{C}$ and let D denote the restriction of $d + \delta$ to Ω^+ then

$$D : \Omega^+ \to \Omega^-. \tag{1}$$

This is the definition of the **signature operator** [**5**] and its index is the **signature** [**17**] of the manifold M.

Let $\mathbb{L} \to M$ be a Hermitian line bundle with compatible connection ∇. This induces a covariant differential

$$d_{\mathbb{L}} = d_{\mathbb{L},\nabla} : \mathbb{L} \otimes \Omega \to \mathbb{L} \otimes \Omega$$

with the defining property

$$d_{\mathbb{L}}(s \otimes \omega) = \nabla s \otimes \omega + s \otimes d\omega.$$

We then also get

$$\delta_{\mathbb{L}} =: -(\mathrm{id} \otimes \star) d_{\mathbb{L}} (\mathrm{id} \otimes \star) : \quad \mathbb{L} \otimes \Omega \to \mathbb{L} \otimes \Omega$$

and therefore a **twisted signature operator** [**5**]

$$D_{\mathbb{L}} := d_{\mathbb{L}} + \delta_{\mathbb{L}} : \quad \mathbb{L} \otimes \Omega^+ \to \mathbb{L} \otimes \Omega^-.$$

If G is a compact Lie group acting as bundle automorphisms of $\mathbb{L}$ we can choose the metric on M and the connection to be G-invariant. This gives representations of G on $\mathbb{L} \otimes \Omega^{\pm}$ which are intertwined by $D_{\mathbb{L}}$. We thus get a representation of G on the (finite dimensional) vector spaces kernel $D_{\mathbb{L}}$ and cokernel $D_{\mathbb{L}}$. We will let $Q(M)$ denote the element of the Grothendieck group $K_G(\mathrm{pt})$ given by

$$Q(M) := -[\text{cokernel } D_{\mathbb{L}}] \oplus [\text{kernel } D_{\mathbb{L}}]. \tag{2}$$

We will call $Q(M)$ the **signature quantization** of the action of G on $(M, \mathbb{L})$. As is usual, we will call an element of $K_G(\mathrm{pt})$ a "virtual representation of G" and usually omit the [ ] while still continuing to work with equivalence classes of G-representations. The definition (2) depends on the choices made — the metric and connection. However it is clear that up to isomorphism $Q(M)$ is independent of these choices.

The aim of this article is to describe the analogues for signature quantization of a number of theorems involving the Dolbeault operator quantization. In comparing these two methods of quantization some formulas look simpler in the signature quantization setting — for example the Kostant formula appears without ρ shifts, and some look more complicated — for example the signature analogue of the Bott-Borel-Weil theorem does have ρ making its appearance in contrast to the Dolbeault quantization. But, it seems to us, the principal virtue of signature quantization is its additivity under cutting (see Section 7 below). We suspect that this is a determining property of signature quantization.

Signature quantization has been studied in the context of a general study of the relation between quantization and reduction by Tian and Zhang in [**27**]; in particular formula (31) occurs as formula (3.4) of their paper.

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2. *q*-weighted partition functions

Let q be any complex number. Then

$$\frac{1+(q-1)z}{1-z} = 1 + qz + qz^2 + qz^3 + \cdots \tag{3}$$

as a formal power series in z.

Let G be an n-dimensional torus and $i\mathbb{Z}_G^*$ its weight lattice. A set of weights $\alpha_1, \ldots, \alpha_d$ is said to be **polarized** if there exists a $\xi \in \mathfrak{g}$, the Lie algebra of G such that $\alpha_j(\xi) \in i\mathbb{R}^+$, $j = 1, \ldots, d$. Then for any $\mu \in i\mathbb{Z}_G^*$ there are only finitely many solutions of

$$\mu = \sum_{i=1}^{d} k_i \alpha_i, \qquad k_i \in \mathbb{Z}, \quad k_i \geq 0. \tag{4}$$

Taking the product of d-versions of (3) shows that

$$\prod_j \frac{1+(q-1)e^{\alpha_j}}{1-e^{\alpha_j}} = \sum_\mu N^{(q)}(\mu) e^\mu \tag{5}$$

where $N^{(q)}(\mu)$is a weighted count of the number of solutions of (4): A solution is counted with weight q^d if all the k_i are positive; it is counted with weight q^{d-1} if exactly one of the $k_i = 0$ and the rest are positive etc.

For signature quantization we will be interested in the case $q = 2$:

$$\prod_j \frac{1+e^{\alpha_j}}{1-e^{\alpha_j}} = \sum_\mu N^{(2)}(\mu) e^\mu. \tag{6}$$

For Dolbeault quantization one is interested in the case $q = 1$ and the corresponding partition function is unweighted. Recently, Jose Agapito

[**1**] has shown that for toric varieties the generalized Hirzebruch q-Todd genus can be computed by a q-weighted count of lattice points in the moment polytope — interior points being counted with weight q^d, points on codimension one faces being counted with weight q^{d-1} etc. This suggests that the other theorems that we prove in this paper might have q analogues for manifolds with line bundles, analogues which for $q = 1$ reduce to Dolbeault quantization and for $q = 2$ to signature quantization.

3. The generalized Kostant formula

Let $G = \mathbb{T}^n$ be the standard n-torus and let $i\mathbb{Z}_G^*$ be the weight lattice of G. Let $(M, \mathbb{L})$ be as in Section 1. Suppose that the action of G on M has isolated fixed points. For any weight μ of G and any virtual representation $\mathbf{r}$ of G let

$$\#(\mu, \mathbf{r})$$

denote the multiplicity with which μ occurs in $\mathbf{r}$. (It is understood that $\#(\mu, -\mathbf{r}) = -\#(\mu, \mathbf{r})$.) The Kostant formula for signature quantization asserts that

$$\#(\mu, Q(M, \mathbb{L})) = \sum (-1)^p N_p^{(2)}(\mu - \nu_p). \tag{7}$$

In this formula the $N_p^{(2)}$ are weighted versions of the Kostant partition function associated with the "polarized weights" of the isotropy representation of G on TM_p. These are defined as follows: Since TM_p does not have a complex structure, the weights

$$\alpha_{i,p}, \quad i = 1, \ldots, d$$

of the isotropy representation are only determined up to sign. Since p is isolated, none of these weights are zero. So by fixing a $\xi \in \mathfrak{g}$ on which none of them vanish for any p, we can arrange that for all p and j we have $\alpha_{j,p}(\xi) \in i\mathbb{R}^+$. Then $N_p^{(2)}(\beta)$ is defined to be the weighted number of solutions of the equation

$$\beta = \sum_{i=1}^{d} k_i \alpha_{i,p} \qquad k_i \in \mathbb{Z}_+. \tag{8}$$

where one assigns the weight 2^d to a solution if all the k_i are positive, the weight 2^{d-1} if exactly one of the k_i is zero etc. The sign $(-1)^p$ occurring in (7) is defined as follows: The choice of a polarization of the weights of the isotropy representation of G on TM_p gives a complex structure on TM_p and hence an orientation. Then $(-1)^p$ is defined to be $+1$ if this orientation coincides with the orientation coming from that of M and -1 otherwise. Finally ν_p is the weight of the action of G on $\mathbb{L}_p$.

Notice that there are no "ρ shifts" in (7) in contrast to the usual Kostant formula for spin-$\mathbb{C}$ quantization[**14**] or in the original theorem of Kostant [**22**].

Proof of the Kostant theorem. The Atiyah-Bott fixed point theorem applied to the twisted signature operator implies that the character of G acting on $Q(M)$ is given by

$$\sum_p (-1)^p e^{\nu_p} \prod_{k=1}^{d} \frac{1+e^{\alpha_{k,p}}}{1-e^{\alpha_{k,p}}}. \tag{9}$$

In the appendix we review the ingredients of this key formula, see especially (47). By (6), the expression (9) is equal to

$$\sum (-1)^p N_p^{(2)}(\mu - \nu_p) e^{\mu}.$$

By definition, the character of G on $Q(M)$ is equal to

$$\sum_\mu \#(\mu, Q(M)) e^{\mu}$$

where $\#(\mu, Q(M))$ is the multiplicity with which the character e^μ occurs in $Q(M)$. Comparing the last two expressions gives the signature version (7) of Kostant's formula.

4. The signature version of the Bott-Borel-Weil theorem

We recall the set up of the classical version of this theorem. Let G be a simply connected compact Lie group, T its maximal torus and μ a weight of T which then determines a line bundle $\mathbb{L}_\mu \to M$ where $M := G/T$. The standard Bott-Borel-Weil theorem says that if μ is in the positive Weyl chamber the spin-$\mathbb{C}$ quantization of $(M, \mathbb{L}_\mu)$ is the irreducible representation $\mathrm{Irr}(\mu)$ with maximal weight μ. For signature quantization we get the following version of this theorem:

Proposition 4.1. *If μ is in the interior of the positive Weyl chamber then*

$$Q(M, \mathbb{L}_\mu) = (-1)^d \mathrm{Irr}(\mu - \rho) \otimes \mathrm{Irr}(\rho) \tag{10}$$

where ρ is one-half the sum of the positive roots of G. If μ is on the boundary of the positive Weyl chamber, $Q(M, \mathbb{L}_\mu) = 0$.

Proof. The idea is to apply the Atiyah-Bott twisted signature formula (see (46) and (47)) to T acting on $\mathbb{L}_\mu \to M$ where $M := G/T$. A point $a^{-1}T$ is fixed by T if and only if $aTa^{-1} = T$ which says that $a \in N(T)$ the normalizer of T and so the fixed points are in one to one correspondence with $N(T)/T = W(T)$ the Weyl group of G. So the fixed points are $\{w \cdot T\}$ given by the action of the Weyl group on the identity coset T. Let us choose the orientation on $M = G/T$ so that the positive roots

$$\alpha_1, \dots, \alpha_d$$

are the weights of the isotropy representation at the identity coset T, and that these give an orientation that is coherent with the chosen one. The Atiyah-Bott fixed point formula (9) (with no polarization) applied to the group T acting on G/T gives

$$\sum_{w \in W} e^{\mu^w} \prod_j \frac{1 + e^{\alpha_j^w}}{1 - e^{\alpha_j^w}} \tag{11}$$

where $\mu^w := w(\mu)$ denotes the image under the Weyl group element w of the weight μ with similar notation for the α.

We are going to show that (11) is the character of the virtual representation given by the right-hand side of Equation (10). To do so, it will be convenient to temporarily consider an expression e^λ as belonging to the group ring of $\Lambda/2$ where Λ is the lattice of weights as in [**18**], pp. 124 and 135–136. (We write e^λ instead of $e(\lambda)$ or ϵ_λ which are Humphreys' notation.)

Let us pull out $\prod_j e^{\alpha_j^w/2}$ from the numerator and denominator of each factor of each summand in (11). The result is

$$\chi(\mathrm{Irr}(\rho)) \cdot \sum_{w \in W} \frac{e^{\mu^w}}{\prod_j (e^{-\alpha_j^w/2} - e^{\alpha_j^w/2})} \tag{12}$$

where

$$\chi(\mathrm{Irr}(\rho)) = \prod_j (e^{\alpha_j^w/2} + e^{-\alpha_j^w/2})$$

is independent of w and the character of the irreducible representation with highest weight

$$\rho := \frac{1}{2} \sum_j \alpha_j.$$

The denominator in the summand of (12) at $w = e$ is

$$\prod_j (e^{-\alpha_j/2} - e^{\alpha_j/2}) = (-1)^d A_\rho$$

where

$$A_\rho = \prod_j (e^{\alpha_j/2} - e^{-\alpha_j/2})$$

and where for any weight μ

$$A_\mu = \sum_{w \in W} (-1)^w e^{\mu^w}.$$

This is anti-symmetric under the action of the Weyl group; in particular, if μ is on the boundary of the positive Weyl chamber, $A_\mu = 0$, so that $Q(M, \mathbb{L}_\mu) = 0$. Taking $\mu = \rho$, we see that A_ρ is anti-symmetric under the action of the Weyl group, and so we can write (12) as

$$(-1)^d \chi(\mathrm{Irr}(\rho)) \cdot \frac{A_\mu}{A_\rho}.$$

If μ lies in the interior of the Weyl chamber so that $\mu - \rho$ is a dominant weight, then the Weyl character formula says that

$$\chi(\mathrm{Irr}(\mu - \rho)) = \frac{A_\mu}{A_\rho}$$

so that (12) is indeed the character of $(-1)^d \mathrm{Irr}(\mu - \rho) \otimes \mathrm{Irr}(\rho)$ proving the proposition.

Note that our convention for the signature operator differs from that of [**24**]; the choice of sign conventions as in [**24**] would eliminate the sign in Proposition 4.1.

5. The Khovanskii theorem

Let Δ be a compact convex polytope. Δ can be written as an intersection of half-spaces

$$\Delta = H_1 \cap \cdots \cap H_n, \quad \text{where} \quad H_i = \{x \mid \langle u_i, x \rangle + \mu_i \geq 0\} \quad \text{for } i = 1, \ldots, n \tag{13}$$

and n is the number of facets of Δ. The vector $u_i \in \mathbb{R}^{d^*}$ can be thought of as the inward normal to the ith facet of Δ; a-priori it is determined up to multiplication by a positive number. If all the vertices of Δ are integral, then the u_i's can be chosen to belong to the dual lattice $\mathbb{Z}^{d^*}$, and we can fix our choice of the u_i's by imposing the normalization condition that the u_i's be primitive lattice elements, that is, that no u_i can be expressed as a multiple of a lattice element by an integer greater than one. (The fact that a normal vector u to a facet σ can be chosen to be integral is a consequence of Cramer's rule. Indeed, we can choose integral edge vectors $\beta_1, \dots, \beta_d$ that emanate from a vertex on σ such that $\beta_1, \dots \beta_{d-1}$ span the tangent plane to σ and β_d is transverse to σ. Solving the linear equations $\langle u, \beta_1 \rangle = \cdots = \langle u, \beta_{d-1} \rangle = 0$ and $\langle u, \beta_d \rangle = 1$, we get an inward normal vector u with rational entries; clearing denominators, we may assume that u is actually integral.)

We can then consider the "dilated polytope" $\Delta(h_1, \dots, h_n)$, which is obtained by shifting the ith facet outward by a "distance" h_i. More precisely,

$$\Delta(h) = \bigcap_{i=1}^{n} \{x \mid \langle u_i, x \rangle + \mu_i + h_i \geq 0\} \quad \text{where} \quad h = (h_1, \dots, h_n).$$

A polytope in $\mathbb{R}^d$ is called *integral* if its vertices are in the lattice $\mathbb{Z}^d$; it is called *simple* if exactly n edges emanate from each vertex; it is called *regular* if, additionally, the edges emanating from each vertex lie along lines which are generated by a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^d$.

The Khovanskii formula (applied to the constant function one) counts the number of lattice points in a regular integral polytope by applying a certain differential operator in the h_i to the volume of the expanded polytope. Here is the proof of the Khovanskii formula using the Dolbeault Dirac quantization given in [**11**] (but specialized to the regular case): A regular integral polytope $\Delta \subset \mathbb{R}^d$ determines a smooth Kähler toric variety (M, ω), and geometric quantization gives rise to a virtual representation $Q(M)$ of the torus T^d. The dimension $\dim Q(M)$ of this quantization is equal to the number of lattice points in Δ. (This fact, which is a well-known folk theorem in the toric variety literature, is an expression of the "quantization commutes with reduction" principle in symplectic geometry [**15**]. According to this principle, $\dim Q(M)^c = \dim Q(M_c)$ for each lattice point $c \in \mathbb{Z}^d \subset \mathrm{Lie}(T^d)^*$, where $Q(M)^c$ is the subspace of $Q(M)$ on which T^d acts through the character given by c, and where M_c is the reduced space of M at c. Because M is a toric variety, M_c is a point if $c \in \Delta$ and is empty otherwise.) On the other hand, by the Hirzebruch-Atiyah-Singer generalization of the classical Riemann-Roch formula, we have $\dim Q(M) = \int_M \exp(c_1(L))\mathrm{Td}(TM)$, where $c_1(L) = [\omega]$ is the Chern class of the pre-quantization line bundle and $\mathrm{Td}(TM)$ is the Todd class of the tangent bundle. Expressing M as a reduction of a linear torus action on $\mathbb{C}^n$ (where n is the number of facets of Δ), the tangent bundle stably splits into line bundles $L_1, \ldots, L_n$, and the above integral is obtained by applying the Khovanskii-Pukhlikov differential operator $\prod \mathrm{Td}(\frac{\partial}{\partial h_i})$ to the integral $\int_M \exp(\omega + \sum h_i c_1(L_i))$. The Duistermaat-Heckman theorem on the variation of reduced symplectic structures implies that this integral is equal to the volume of the polytope $\Delta(h)$ that is obtained from Δ by shifting the ith facet by a distance h_i, for $i = 1, \ldots, n$. Hence, the number of lattice points in Δ is obtained by applying the Khovanskii-Pukhlikov operator to the volume of $\Delta(h)$.

This proof can be taken over directly to the signature quantization case by replacing the Riemann Roch theorem by the Atiyah-Singer generalization [**5**] (to the twisted case) of Hirzebruch's formula [**17**] for the signature of a manifold of dimension $4n$. The result can be stated as follows: Define the series $L(x)$ by the power series expansion

$$L(x) = \frac{x}{\tanh(x)} = \sum_{i=0}^{\infty} \frac{b_{2i} 2^{2i}}{(2i)!} x^{2i} \tag{14}$$

where b_i denotes the $i-$th Bernoulli number. Let $\#'(\Delta)$ denote the number of points in $\mathbb{Z}^n \cap \Delta$ counted with weights as above. Then

$$\dim Q(M) = (-1)^d \#'(\Delta) = (-1)^d \prod_{i=1}^{n} L\left(\frac{\partial}{\partial h_i}\right)|_{h_i=0} \mathrm{vol}(\Delta_h),$$

where $L(\frac{\partial}{\partial h_i})$ is the infinite-order, constant coefficient differential operator defined by the series (14). In fact, up to an overall factor of 2^d, this is the content of the Euler-Maclaurin formula in [**20**] when applied to the constant function one. A purely combinatorial proof of this result for integral regular polytopes is given in [**20**] and for simple integral polytopes in [**21**]. In this context the use of L-classes (rather than the Todd classes that appear in the Khovanskii formula) was a key idea of Cappell and Shaneson [**9**]. For an alternative proof of this formula using the Kostant formula above, see [**1**].

6. The Kostant formula for non-isolated fixed points

We return to the notation of Section 3 but drop the condition that the fixed points of G be isolated. We then can generalize the argument of Section 3 if we replace the Atiyah-Bott fixed point formula by the

Atiyah-Segal-Singer formula [**4**]. Before recalling the statement of this formula we make some definitions:

Let F denote a component of the fixed point set of G. Let TF and NF denote the tangent bundle and normal bundle of F. Let $\mathrm{Ch}(\mathbb{L}, F)$ denote the Chern character of the restriction of $\mathbb{L}$ to F. Let $L(TF)$ denote the L-class of the tangent bundle TF. We can form the "virtual splitting"

$$N(F) = \bigoplus_{j=1}^{r} \mathbb{L}_j(F)$$

into line bundles. Let

$$i\alpha_{j,F} + \omega_{j,F}$$

be the equivariant curvature forms of $\mathbb{L}_j(F)$; the $\alpha_{j,F}$'s are the weights of the isotropy representation of G on NF. For $\xi \in \mathfrak{g}$ define the **equivariant** L-class of the normal bundle to be

$$\widetilde{L}_{\exp \xi}(NF) := \prod_j \frac{1 + \exp(i\alpha_{j,F}(\xi) + \omega_{j,F})}{1 - \exp(i\alpha_{j,F}(\xi) + \omega_{j,F})}. \tag{15}$$

The line bundle $\mathbb{L} \to M$ with its connection determine an abstract moment map $\phi : M \to \mathfrak{g}^*$. See[**12**]. See also Appendix II below for a summary of the relevant facts. By the definition of an abstract moment map, ϕ is constant on the components of the fixed point set of G. We let ϕ_F denote its value on F. Finally let

$$\chi_F(\exp \xi) := (-1)^F (-1)^{(\mathrm{codim}\ F)/2} e^{i\langle \phi_F, \xi\rangle} \cdot \int_F Ch(\mathbb{L}, F) L(TF) \widetilde{L}_{\exp(\xi)}(NF). \tag{16}$$

The $(-1)^F$ in this formula is defined as follows: The virtual splitting of NF depends on a choice of orientation of NF and the integration in (16) depends on a choice of orientation of F (hence of TF).Then $(-1)^F$ is $+1$ if these orientations fit together so as to agree with the orientation of $TM|_F$ and is -1 if they do not.

Finally let χ denote the character of the virtual representation of G on $Q(M)$. Then the Atiyah-Segal-Singer formula asserts that

$$\chi(\exp \xi) = \sum_F \chi_F(\exp \xi). \tag{17}$$

We now proceed as in Section 3: Choose a polarizing vector $\xi_0 \in \mathfrak{g}$ and choose the orientations so that the $\alpha_{j,F}(\xi_0)$ are positive for all j and

F, Let ω be the curvature form of $(\mathbb{L}, \nabla)$ and for each set of nonnegative integers $k_1, \ldots, k_r$ where r is the codimension of F set

$$p_{k,F} := (-1)^{(\text{codim } F)/2} \int_F \exp\left(\omega + \sum_{j=1}^{r} k_j \omega_{j,F}\right) L(TF). \tag{18}$$

For each $\mu \in i\mathbb{Z}_G^*$ let

$$\#(\mu, Q(M, \mathbb{L}))$$

denote the multiplicity with which μ occurs in $Q(M, \mathbb{L})$. The generalization of the Kostant formula says that

$$\#(\mu, Q(M, \mathbb{L})) = \sum_F (-1)^F N_F^{(2)}(\mu) \tag{19}$$

where $N_F^{(2)}(\mu)$ is the weighted sum

$$N_F^{(2)}(\mu) := \sum_k{}' p_{k,F} \tag{20}$$

over all r-tuplets k of nonnegative integers satisfying

$$\mu = \sum_{j=1}^{r} k_j \alpha_{j,F} + \phi_F \tag{21}$$

and the sum in (20) is weighted in the usual signature fashion: if all the k_j are > 0 we assign to $p_{k,F}$ the weight 2^r, if exactly one of the $k_j = 0$ we to $p_{k,F}$ the weight 2^{r-1} etc.

The formula above is the analog for signature quantization of the generalized Kostant formula for manifolds with non-isolated fixed points obtained by Canas da Silva and Guillemin [**8**].

Remark. If $\mu = \phi_F$ then

$$N_F^{(2)}(\mu) = -\dim Q(F). \tag{22}$$

7. Additivity of signature quantization under cutting

Let $G = S^1$ and let $\phi : M \to \mathbb{R}$ be the abstract moment map associated to the line bundle $\mathbb{L} \to M$ and its connection. Suppose that S^1 acts freely on $\phi^{-1}(0)$ and that $\phi^{-1}(0)$ is connected. Then the spaces $\phi^{-1}([0, \infty))$ and $\phi^{-1}((-\infty, 0])$ are compact manifolds with boundary with S^1 acting freely on the boundary. By collapsing the orbits on the boundary to points we get compact manifolds without boundary (see

Section 11.7) which we denote by M^+ and M^-. The manifold M is cobordant to the disjoint union of M^+ and M^- and the operation

$$M \mapsto M^+ \sqcup M^-$$

is called (following Lerman[**25**]) **cutting**. We will review some of the facts concerning this operation in Section 11.7. In particular we will show that from the action of S^1 on $\mathbb{L}$ we get line bundles $\mathbb{L}^\pm \to M^\pm$ and actions of $G = S^1$ on them. Also, the orientation of M induces orientations on $M^\pm$. We claim that the signature quantization of these three spaces are related by

$$Q(M, \mathbb{L}) = Q(M^+, \mathbb{L}^+) \oplus Q(M^-, \mathbb{L}^-) \tag{23}$$

For the analog of Equation (23) in the case of Dolbeault quantization, see [**10**].

Proof. We break the proof into several steps:

Step 1. Suppose that $m \in \mathbb{Z}$ is negative. Then by (19) and (21) we have

$$\#(m, Q(M, \mathbb{L})) = \sum_{\phi_F < 0} (-1)^F N_F^{(2)}(m) = \#(m, Q(M^-, \mathbb{L}^-)) \tag{24}$$

and

$$\#(m, Q(M^+, \mathbb{L}^+)) = 0. \tag{25}$$

This shows that (23) holds at the characters corresponding to the negative integers. Reversing the polarization shows that the same holds for $m > 0$. So the issue boils down to examining the case $m = 0$.

Step 2. ($m = 0$). By (19) and (21) and the assumption that S^1 acts freely on $\phi^{-1}(0)$ we have

$$\#(0, Q(M, \mathbb{L})) = \sum_{\phi_F < 0} (-1)^F N_F^{(2)}(0). \tag{26}$$

Let us now apply (19) to M^-. By our assumption that $\phi^{-1}(0)$ is connected, we know that

$$F^- := M_{\text{red}} := \phi^{-1}(0)/S^1$$

is the unique connected component of $(M^-)^{S^1}$ on which the moment map is zero. Hence it follows from (22) and (26) that

$$\#(0, Q(M^-, \mathbb{L}^-)) = \#(0, Q(M, \mathbb{L})) - (-1)^{F^-} \dim Q(M_{\text{red}}, \mathbb{L}_{\text{red}}). \tag{27}$$

Similarly,

$$F^+ := M_{\text{red}} := \phi^{-1}(0)/S^1$$

is the unique connected component of $(M^+)^{S^1}$ on which $\phi = 0$, so we get from (22) and (26) that

$$\#(0, Q(M^+, \mathbb{L}^+)) = -(-1)^{F^+} \dim Q(M_{\text{red}}, \mathbb{L}_{\text{red}}). \tag{28}$$

Step 3. We claim that $(-1)^{F^-}$ (occurring in (27)) is -1 and that $(-1)^{F^+}$ (occurring in (28)) is $+1$. This would complete the proof of (23). Actually, we will prove this here under the assumption that the curvature ω of ∇ on M is symplectic, where the proof is straightforward, deferring the general case to the appendix (Section 11.7).

If ω is symplectic, then without loss of generality we may assume that the orientation of M is induced from ω. Furthermore, each connected component F of the fixed point set is a symplectic submanifold, so TF and NF are symplectic sub-bundles of $(TM)|_F$. If we give them their symplectic orientations, these orientations will be compatible with the orientation of M. Since F^+ is the subset of M^+ on which ϕ takes its minimum value, the normal weight α_{F^+} will be polarized (relative to the positive direction on $\mathbb{R}$) and hence $(-1)^{F^+} = 1$. Since F^- is the subset of M^- where ϕ takes on its maximum value, we see that $(-1)^{F^-} = -1$. This completes the proof of (23). But we can get something more from this argument:

8. The relation between quantization and reduction for circle actions.

We have proved that

$$\#(0, Q(M^+, \mathbb{L}^+)) = -\dim Q(M_{\text{red}}, \mathbb{L}_{\text{red}}).$$

Reversing the polarization shows equally well that

$$\#(0, Q(M^-, \mathbb{L}^-)) = -\dim Q(M_{\text{red}}, \mathbb{L}_{\text{red}}).$$

But from (23) it follows that

$$\begin{aligned}\#(0, Q(M^+, \mathbb{L}^+)) + \#(0, Q(M^-, \mathbb{L}^-)) &= \#(0, Q(M, \mathbb{L})) \\ &= \dim Q(M, \mathbb{L})^{S^1}.\end{aligned}$$

Hence we have proved that

$$\dim Q(M, \mathbb{L})^{S^1} = -2 \dim Q(M_{\text{red}}, \mathbb{L}_{\text{red}}). \tag{29}$$

This gives the desired relation between signature quantization and reduction.

Remark. Let $H = T^{n-1}$ and set $G = S^1 \times H$. Suppose we have an action of G on $\mathbb{L} \to M$. Then we can replace the χ_F that occur in (16) by their H-equivariant counterparts,

$$\widetilde{\chi}_F = (-1)^F (-1)^{(\text{codim } F)/2} e^{i\langle \phi_F, \cdot \rangle} \int_F \widetilde{\mathrm{Ch}}(\mathbb{L}, F) \widetilde{L}(TF) \widetilde{L}(NF). \tag{30}$$

the tildes denoting the equivariant Chern and L classes etc. By essentially the same argument as given above, it will then follow that the isomorphisms (23) and (29) are H-isomorphisms. Since (23) is an S^1 isomorphism it is then also a G-isomorphism.

9. The relation between quantization and reduction for torus actions.

Let M now be a G-space where $G = T^n$ is a torus. The identity (29) becomes

$$\dim Q(M, \mathbb{L})^G = (-2)^n \dim Q(M_{\text{red}}, \mathbb{L}_{\text{red}}) \tag{31}$$

where $M_{\text{red}} = \phi^{-1}(0)/G$.

This result was proved by Tian and Zhang [**27**]. It should be possible to prove this by induction from (29), but this proof would be complicated by the fact that the reduced spaces appearing in the stages of the induction will in general be orbifolds rather than manifolds. We give instead an alternative proof for the case where M^G is finite. By the results of Appendix II, there exists in this case a cobordism of G-manifolds with abstract moment maps

$$\partial W = M - \sqcup_{p \in M^G} TM_p. \tag{32}$$

If $\mu \in \mathbb{Z}_G^*$ is a regular value of the moment map $\phi : M \to \mathfrak{g}^*$, *and* of the moment maps $\phi_p : TM_p \to \mathfrak{g}^*$, then, by slightly perturbing the cobording moment map on W, one can arrange for μ to be a regular value of this moment map as well; and hence one gets a cobordism of compact manifolds

$$\phi^{-1}(\mu) \sim \sqcup (-1)^p \phi_p^{-1}(\mu),$$

and, by quotienting by G, a cobordism of compact orbifolds,

$$M_{\text{red}}(\mu) \sim \sqcup_p (-1)^p M^p_{\text{red}}(\mu) \tag{33}$$

where $M_{\mathrm{red}}(\mu) = \phi^{-1}(\mu)/G$ and $M^p_{\mathrm{red}}(\mu) = \phi_p^{-1}(\mu)/G$. The spaces $M^p_{\mathrm{red}}(\mu)$ are toric varieties, so that by our version of Khovanskii's theorem (see also [**1**]) it is easy to see that

$$\dim\, Q(M^p_{\mathrm{red}}(\mu)) = \frac{1}{(-2)^n} N_p^{(2)}(\mu - \phi(p)). \tag{34}$$

However, the index of the signature operator is invariant under cobordism; so one gets from (33)

$$\dim\, Q(M_{\mathrm{red}}(\mu)) = \frac{1}{(-2)^n} \sum_p (-1)^p N_p^{(2)}(\mu - \phi(p)),$$

and hence, by (34)

$$\dim\, Q(M_{\mathrm{red}}(\mu)) = \frac{1}{(-2)^n} \#(\mu, Q(M)),$$

which, for $\mu = 0$, becomes the identity (31). If μ is not a regular value of all of the moment maps ϕ_p a somewhat more delicate argument is required. We hope to present this elsewhere.

10. Appendix I. Facts about the signature operator

In this section we follow the classic treatment of Atiyah-Bott [**3**] very closely with some twists.

10.1. The Hodge $\star$ operator. Let V be a vector space with a positive definite scalar product and an orientation. This picks out a basis element τ of $\wedge^p V$ where $p = \dim V$ and then a linear map

$$\star : \wedge^j(V) \to \wedge^{p-j}(V)$$

defined by

$$u \wedge \star v = \langle u, v\rangle \tau \qquad u, v \in \wedge^j(V)$$

and where the scalar product $\langle u, v\rangle$ on the right is the one induced from the scalar product on V. So if $e_1, \ldots, e_p$ is an oriented orthonormal basis of V then

$$\tau = e_1 \wedge \cdots \wedge e_p.$$

Let $J = (i_1, \ldots, i_j)$ be a subset of $\{1, \ldots, p\}$ with its elements arranged in increasing order so that the

$$e_J := e_{i_1} \wedge \cdots \wedge e_{i_j}$$

form an orthonormal basis of $\wedge^j(V)$ as J ranges over all ordered subsets of cardinality j. Then

$$e_J \wedge e_L = 0 \quad \text{if} \;\; L \neq J^c \text{ and } |L| = p - j$$

while

$$e_J \wedge e_{J^c} = (-1)^\pi \tau$$

where $(-1)^\pi$ is the sign of the permutation π required to bring the entries of $e_J \wedge e_{J^c}$ back to increasing order. So

$$\star e_J = (-1)^\pi e_{J^c}. \tag{35}$$

So

$$\star^2(e_1 \wedge \cdots e_j) = \star(e_{j+1} \wedge \cdots \wedge e_p) = (-1)^{j(p-j)} e_1 \wedge \cdots \wedge e_j$$

since we have to move $(p-j)$ elements e_{j+r} past $e_1 \wedge \cdots \wedge e_j$ to get to increasing order. Hence

$$\star^2 = (-1)^{j(p-j)} \mathrm{id} \quad \text{on} \quad \wedge^j(V).$$

If $p = 2d$ is even, this simplifies to

$$\star^2 = (-1)^j \mathrm{id} \quad \text{on} \quad \wedge^j(V). \tag{36}$$

10.2. The operator α. Let V and W be oriented even-dimensional vector spaces with positive definite inner products. Then the direct sum $V \oplus W$ becomes an oriented vector space with inner product if we choose the direct sum scalar product and the orientation such that $e_1, \ldots, e_p, f_1, \ldots, f_q$ is an oriented basis of $V \oplus W$ where $e_1, \ldots, e_p$ is an oriented basis of V and $f_1, \ldots, f_q$ is an oriented basis of W. We now have three $\star$ operators — the $\star$ operator $\star_V$ of V, the $\star$ operator $\star_W$ of W, and the $\star$ operator $\star_{V \oplus W}$ of $V \oplus W$. We have the decomposition

$$\wedge^r(V \oplus W) = \bigoplus_{j+k=r} \wedge^j(V) \otimes \wedge^k(W)$$

and it follows from (35) that

$$\star_{V \oplus W} = (-1)^{jk} \star_V \otimes \star_W \quad \text{on} \;\; \wedge^j(V) \otimes \wedge^k(W).$$

Due to the presence of the factor $(-1)^{jk}$ the $\star$ operator is not "multiplicative" under the identification $\wedge(V \oplus W) \sim \wedge(V) \otimes \wedge(W)$. However we can get a multiplicative operator by modifying the $\star$ operator: Define $\alpha : \wedge(V) \to \wedge(V)$ by

$$\alpha := (-1)^{\frac{j(j-1)}{2}} \star \quad \text{on} \;\; \wedge^j(V). \tag{37}$$

Since

$$\frac{(j+k)(j+k-1)}{2} = \frac{j(j-1)}{2} + \frac{k(k-1)}{2} + jk$$

we see that

$$\alpha_{V\oplus W} = \alpha_V \otimes \alpha_W \tag{38}$$

so we can drop the subscripts on α and just remember that it is multiplicative.

By Equation (36) and the definition of α (37) we see that α satisfies the relation

$$\alpha^2 = (-1)^d \mathrm{id} \qquad \text{if} \quad \dim V = 2d. \tag{39}$$

Thus the eigenvalues of α on $\wedge(V)$ are $(\pm i)^d$. So if we consider the operator $\alpha \otimes i^d$ on the complexification $\wedge(V)_{\mathbb{C}} := \wedge(V) \otimes \mathbb{C}$ we have $(\alpha \otimes i^d)^2 = 1$ and hence a decomposition

$$\wedge(V)_{\mathbb{C}} = \wedge(V)^+_{\mathbb{C}} \oplus \wedge(V)^-_{\mathbb{C}} \tag{40}$$

where $\wedge(V)^\pm_{\mathbb{C}}$ are the eigenspaces of $\alpha \otimes i^d$ corresponding to the eigenvalues ± 1. This makes $\wedge(V)_{\mathbb{C}}$ into a super vector space, i.e., a vectors space with a $\mathbb{Z}/2\mathbb{Z}$ gradation. The projections of $\wedge(V)_{\mathbb{C}}$ onto $\wedge(V)^\pm_{\mathbb{C}}$ are given by $\pi^\pm$ where

$$\pi^+ := \frac{1}{2}[\mathrm{id} + \alpha \otimes i^d], \qquad \pi^- := \frac{1}{2}[\mathrm{id} - \alpha \otimes i^d]. \tag{41}$$

For future use we record the "supertrace" of an "even" linear operator β, one which preserves the two subspaces $\wedge(V)^\pm_{\mathbb{C}}$: We claim that

$$\mathrm{tr}\,\beta|_{\wedge(V)^+_{\mathbb{C}}} - \mathrm{tr}\,\beta|_{\wedge(V)^-_{\mathbb{C}}} = \mathrm{tr}\,\beta \circ (\alpha \otimes i^d) \quad \text{on } \wedge(V)_{\mathbb{C}}. \tag{42}$$

Indeed, the difference on the left is the trace on $\wedge(V)_{\mathbb{C}}$ of $\beta \circ \pi^+ - \beta \circ \pi^-$. So the result follows from (41).

If X is an even dimensional oriented Riemannian manifold all of the above applies to $\wedge T^*X$ as it applies pointwise, So we can consider the operator α as mapping $\Omega^j(X) = \Gamma(\wedge^j(T^*X))$ to $\Omega^{2d-j}(X)$ where $2d = \dim X$.

10.3. The Hodge-Dirac operator D. Let X be a compact oriented Riemannian manifold. The (Dirichlet) global inner product on j-forms is defined as

$$(u, u') := \int_X \langle u, u' \rangle \tau = \int_X u \wedge \star u'. \tag{43}$$

This is a positive definite inner product and therefore the operator d has a well-defined adjoint δ:

$$(du, v) = (u, \delta v), \qquad u \in \Omega^j(X), \quad v \in \Omega^{j+1}(X).$$

We have

$$d(u \wedge \star v) = du \wedge \star v + (-1)^j u \wedge d \star v = du \wedge \star v + (-1)^j u \wedge \star^{-1}(\star d \star v).$$

Now $\star d \star v \in \Omega^j(X)$. If X is even dimensional then $\star^{-1} = (-1)^j \mathrm{id}$ on $\Omega^j(X)$ and so by Stokes' theorem

$$\delta = - \star d \star . \tag{44}$$

We have

$$d \circ \alpha = (-1)^{\frac{j(j-1)}{2}} d \star \qquad \text{on } \Omega^j(X)$$

while

$$\star\delta = - \star^2 d\star = -(-1)^{2m-j+1} d\star = (-1)^j d\star$$

so

$$\alpha \circ \delta = (-1)^{\frac{(j-1)(j-2)}{2}} \star \delta = (-1)^{\frac{(j-1)(j-2)}{2}+j} d\star$$

and therefore

$$\alpha \circ \delta = -d \circ \alpha$$

which implies that

$$d \circ \alpha = -\delta \circ \alpha.$$

So (dropping the $\circ$) we have

$$(d + \delta)\alpha = -\alpha(d + \delta). \tag{45}$$

So $d+\delta$ interchanges the spaces $\Omega(X)^{\pm}_{\mathbb{C}}$. It is an "odd" operator relative to the $\mathbb{Z}/2\mathbb{Z}$ grading. We define

$$D^+ := d + \delta : \quad \Omega(X)^+_{\mathbb{C}} \to \Omega(X)^-_{\mathbb{C}}$$

and

$$D^- := d + \delta : \quad \Omega(X)^-_{\mathbb{C}} \to \Omega(X)^+_{\mathbb{C}}.$$

As explained in the introduction, if $\mathbb{L} \to M$ is a Hermitian line bundle with compatible connection ∇ then we get a twisted signature operator

$$D_{\mathbb{L}} : \mathbb{L} \otimes \Omega^+ \to \mathbb{L} \otimes \Omega^-.$$

10.4. The Atiyah-Bott fixed point theorem. We want to apply this theorem to the operators $D^{\pm}_{\mathbb{L}}$. We first recall the general formulation of this famous theorem [**2**] and then follow the computation in [**3**] but extend it to the twisted case:

A **morphism** from a vector bundle $E \to X$ to a vector bundle $F \to M$ is a pair $f = (\phi, r)$ where

$$\phi : X \to M$$

is a smooth map and where r is a smooth section of $\mathrm{Hom}(\phi^\sharp F, E)$. Then f defines a pull-back operation f^* from sections u of F to sections of E by

$$f^* u(x) = r(x)u(\phi(x)).$$

We will let f depend on a parameter, i.e., $\phi : Y \times X \to M$ and r be a section of $\mathrm{Hom}(\phi^\sharp F, E)$ as before. For each $y \in Y$ we then get a pull back which we shall denote by f^*_y from smooth sections of F to sections of E. We will be especially interested in the case where $M = X$.

Suppose that we are given a sequence of differential operators on vector bundles over X

$$0 \to C^\infty(E_0) \overset{D_0}{\to} C^\infty(E_1) \overset{D_1}{\to} C^\infty(E_2) \overset{D_2}{\to} \cdots \overset{D_{N-1}}{\to} C^\infty(E_N) \to 0$$

which is a complex (i.e., $D_{i+1} \circ D_i = 0$) whose cohomology groups are finite dimensional.

A morphism of this complex is a sequence of morphisms $f^i = (\phi^i, r^i)$ of each E_i such that the induced maps on sections satisfy $(f^{i+1})^* D_i = D_i \circ (f^i)^*$. We will usually drop the subscript or superscript i and simply write $f^* \circ D = D \circ f^*$.

Then f induces a linear map on each of the cohomology groups $H^i(E)$ which we shall denote by $f^\flat$ and the Lefschetz number $L(f)$ is defined to be

$$L(f) := \sum_i (-1)^i \operatorname{tr} f^\flat_{H^i(E)}.$$

Let us assume that all of the morphisms f^i have the same underlying geometrical map $\phi : X \to X$.

A map $\phi : X \to X$ is called a **Lefschetz** map if graph (ϕ) is transversal to the diagonal. This amounts to the assertion that at every fixed point p of ϕ the map $I - d\phi_p$ of TX_p to itself is invertible. The Atiyah-Bott fixed point theorem asserts that under certain hypotheses

$$L(f) = \sum_{p|\phi(p)=p} \frac{\sum_{j=0}^{N}(-1)^j \operatorname{tr} r^j(p)}{|\det(I - d\phi_p)|}. \tag{46}$$

Two hypotheses will make this work: i) We can assume that the complex is elliptic, or ii) We can assume that the morphism f can be embedded in a transitive family of morphisms.

In the case of the twisted signature operators we have a two-step complex. Our underlying map ϕ is an isometry. The map is Lefschetz, meaning that the map $d\phi_p : TX_p \to TX_p$ does not have 1 as an eigenvalue. This means that we can decompose TX_p into a direct sum of orthogonal two-dimensional subspaces each invariant under ϕ_p and such that the restriction of $d\phi_p$ to each such subspace is a nontrivial rotation. The angle of rotation is only determined up to sign — the choice of sign depends on the orientation that we choose on the two-dimensional subspaces. The orientations are called "coherent" if the direct product of these orientations is the orientation on TX_p induced from the orientation of X. But we may want to modify the choice of orientation on each two-dimensional subspace.

Now the contribution of each fixed point p to Atiyah-Bott fixed point formula for $D^{\pm}$ can be calculated as follows: The denominator is

$$|\det(I - d\phi_p)| = \prod_{i=1}^{m}(1 - e^{i\theta_i})(1 - e^{-i\theta_i})$$

where the product is over all the two-dimensional subspaces and is clearly independent of the choice of the signs of the angles on each subspace. Let us turn to the numerator $\operatorname{tr} r(p)$.

An endomorphism of a complex line is just multiplication by a complex number. So the contribution of $\mathbb{L}_p$ to $\operatorname{tr} r(p)$ is an overall factor of a complex number $z_p = e^{i\gamma_p}$. This multiplies

$$\pm[\operatorname{tr}(d\phi_p)_{\wedge^+ T_p} - \operatorname{tr}(d\phi_p)_{\wedge^- T_p})].$$

By (42) this is the same as

$$\pm \operatorname{tr} d\phi_p \circ (\alpha \otimes i^d)$$

where the trace is taken over all of $\wedge(TX_p)_{\mathbb{C}}$. Now $\operatorname{tr}(A \otimes B) = \operatorname{tr} A \cdot \operatorname{tr} B$ and the action of $d\phi_p$ preserves the decomposition into the two-dimensional subspaces V_i and hence is multiplicative under the decomposition

$$\wedge(TX_p) = \prod \wedge(V_i).$$

We also know that α is multiplicative. So we are reduced to a two-dimensional computation where u, v is an orthonormal basis of V and the restriction of $d\phi_p$ to V is given by

$$\begin{array}{rcl} u & \mapsto & \cos\theta\ u - \sin\theta\ v \\ v & \mapsto & \sin\theta\ u + \cos\theta\ v. \end{array}$$

This is of course the same as the induced action on $\wedge^1(V) = V$ while $d\phi_p$ acts as the identity on $\wedge^0(V)$ and on $\wedge^2(V)$. As to α we have to consider two cases:

1) u, v is an oriented basis. Then

$$\begin{array}{rcl} \star 1 & = & u \wedge v \\ \star u & = & v \\ \star v & = & -u \\ \star(u \wedge v) & = & 1 \\ (-1)^{j(j-1)/2} & = & 1, \quad j = 0, 1 \\ & = & -1, \quad j = 2 \end{array}$$

so $1 + i(u \wedge v), u + iv$ is an eigenbasis of $\wedge(V)^+_{\mathbb{C}}$ for $d\phi_p$ with corresponding eigenvalues $1, e^{i\theta}$ while $1 - i(u \wedge v), u - iv$ is an eigenbasis of $\wedge(V)^-_{\mathbb{C}}$ with eigenvalues $1, e^{-i\theta}$. Thus the overall contribution of V to the numerator is

$$e^{i\theta} - e^{-i\theta}.$$

2) v, u is an oriented basis. Then there is a change of sign in the right-hand side of the first four displayed equations above and the overall contribution of V to the numerator is

$$e^{-i\theta} - e^{i\theta}.$$

If we combine the numerator and denominator contributions of V we get

$$\frac{e^{i\theta} - e^{-i\theta}}{(1 - e^{i\theta})(1 - e^{-i\theta})} = \frac{1 + e^{i\theta}}{1 - e^{i\theta}}$$

in the first case and the negative of this expression in the second case. So the overall contribution of the point p to $L(f)$ is

$$(-1)^p e^{i\gamma_p} \prod_{k=1}^{d} \frac{1+e^{i\theta_{k,p}}}{1-e^{i\theta_{k,p}}}. \tag{47}$$

where $(-1)^p = 1$ if the orientation of TX_p coincides with the orientation induced from all our choices of orientations on the two-dimensional subspaces and $= -1$ if it doesn't.

11. Appendix II. Facts about prequantization

All the material in this section is taken from [**12**]. We refer to this book for a discussion of the general subject of geometric quantization, in particular for spin-$\mathbb{C}$ quantization and for the vast literature on this subject.

11.1. Abstract moment maps. Let G be a torus and let M be an oriented (not necessarily compact) G-manifold. A map

$$\phi : M \to \mathfrak{g}^*$$

is called an **abstract moment map** if:

- ϕ is equivariant.
- For any subgroup H of G with corresponding injection $\mathfrak{h} \to \mathfrak{g}$ and hence projection
  $$\mathfrak{g}^* \to \mathfrak{h}^*$$
  the composite map
  $$M \xrightarrow{\Phi} \mathfrak{g}^* \to \mathfrak{h}^*$$
  is constant on connected components of M^H, the set of points of M fixed by H.

11.2. Moment maps associated to a closed two form. Let ω be a closed two form invariant under G. To each $\xi \in \mathfrak{g}$ we get a vector field ξ_M on M generating the action of the one parameter group $t \mapsto \exp t\xi$ and the map $\xi \to \xi_M$ is an anti-Lie algebra homomorphism. The invariance of ω implies that

$$di(\xi_M)\omega = i(\xi_M)d\omega + di(\xi_M)\omega = L_{\xi_M}\omega = 0$$

since ω is closed. A **moment map for** ω is an equivariant map Φ : $M \to \mathfrak{g}^*$ such that

$$d\langle\phi, \xi\rangle = i(\xi_M)\omega. \tag{48}$$

If N is a connected submanifold on which ξ_M vanishes, the above equation implies that $\langle \phi, \xi \rangle$ is constant on N. This proves that a moment map associated to ϕ is an abstract moment map.

11.3. Poisson algebras and moment maps. Let ω be a closed two form on a manifold M. The **Poisson algebra associated to** ω is defined to be the vector space

$$\mathcal{P} = \mathcal{P}(M, \omega) := \{(f, v) \in C^\infty(M) \times \text{Vect }(M) | df = i(v)\omega\}$$

equipped with the associative commutative multiplication

$$(f, v) \cdot (g, u) := (fg, fu + gv)$$

and the Lie bracket

$$[(f, v), (g, u)] := (D_u f, -[u, v]).$$

Notice that

$$D_u f = i(u)df = i(u)i(v)\omega = \omega(v, u) = -\omega(u, v) = -D_v g$$

so the bracket is indeed anti-symmetric. Straightforward computation shows that Jacobi's identity holds and that the Lie bracket acts as derivation of the commutative multiplication. If ω is nondegenerate the projection $(f, v) \to f$ is an isomorphism so $\mathcal{P}(M, \omega)$ is isomorphic to $C^\infty(M)$. At the other extreme, if $\omega = 0$ and M is connected, then $\mathcal{P}(M, \omega) = \mathbb{R} \times \text{Vect}(M)$ as a vector space.

In all cases projection onto the second component, $\text{Vect}(M)$ is an anti-homomorphism of Lie algebras by definition. Suppose that we have an action of G on M which preserves ω and suppose further that we have an equivariant homomorphism from $\mathfrak{g}$ to $\mathcal{P}(M, \omega)$. Then projection onto the first component is a moment map in that we can write the homomorphism in the form

$$\xi \mapsto (\langle \phi, \xi \rangle, \xi_M).$$

11.4. Prequantization. Let M be a manifold with a closed two form ω. **Prequantization data** for (M, ω) are defined to be a Hermitian line bundle $\mathbb{L} \to M$ with a Hermitian connection ∇ whose curvature is ω. Equivalently, the prequantization data consists of a circle bundle $\pi : \mathbb{P} \to M$ with connection form Θ, so that

$$\Theta \left(\frac{\partial}{\partial \theta} \right) = 1$$

where $\frac{\partial}{\partial \theta}$ is the vector field generating the circle action and

$$\pi^* \omega = -d\Theta.$$

The relation between these two definitions is as follows. We can regard $\mathbb{P}$ as the unit circle sub-bundle of $\mathbb{L}$. The covariant derivative gives a map

$$\nabla : \quad \Gamma(\mathbb{L}) \to \Omega^1(M, \mathbb{L})$$

and for any section of $\mathbb{P} \subset \mathbb{L}$

$$\nabla(s) = is \otimes s^*\Theta.$$

Given an action of a Lie algebra $\mathfrak{g}$ on M (i.e., an anti-homomorphism $\xi \mapsto \xi_M$ from $\mathfrak{g}$ to $\mathrm{Vect}(M)$) a **lifting** of this action to $\mathbb{P}$ is an action $\xi \to \xi_{\mathbb{P}}$ of $\mathfrak{g}$ on $\mathbb{P}$ such that the vector fields $\xi_{\mathbb{P}}$ are invariant under the circle action and such that $\pi_*\xi_{\mathbb{P}} = \xi_M$.

The Lie algebra $\mathcal{P} = \mathcal{P}(M, \omega)$ acts on M via the assignment $(f, v) \mapsto v$. For any vector field v on M let v_{hor} denote the horizontal lift of v.

Proposition 11.1. *The map*

$$(f, v) \mapsto v_{\mathrm{hor}} + f \cdot \frac{\partial}{\partial \theta} \tag{49}$$

is lifting of the action of $\mathcal{P}$ on M to an action of $\mathcal{P}$ on $\mathbb{P}$ and gives an isomorphism of $\mathcal{P}$ with the algebra of infinitesimal isomorphisms of $(\mathbb{P}, \Theta)$.

Proof. We have

$$\Theta([v_{\mathrm{hor}}, u_{\mathrm{hor}}]) = d\Theta(v_{\mathrm{hor}}, u_{\mathrm{hor}}) = -\omega(v, u)$$

so

$$[v_{\mathrm{hor}}, u_{\mathrm{hor}}] = [v, u]_{\mathrm{hor}} + \omega(v, u)\frac{\partial}{\partial \theta}$$

and hence

$$\left[v_{\mathrm{hor}} + f\frac{\partial}{\partial \theta}, u_{\mathrm{hor}} + f\frac{\partial}{\partial \theta}\right] = [v, u]_{\mathrm{hor}} + D_u f \cdot \frac{\partial}{\partial \theta}.$$

This proves that (49) is an isomorphism. Vector fields v_{hor} and $f \cdot \frac{\partial}{\partial \theta}$ commute with the circle action and hence are infinitesimal automporphisms of $\mathbb{P}$. We must show that the right-hand side of (49) preserves Θ. Now

$$D_{v_{\mathrm{hor}}}\Theta = i(v_{\mathrm{hor}})d\Theta = -\pi^*(i(v)\omega)$$

while

$$D_{f \cdot \frac{\partial}{\partial \theta}}\Theta = d\left(i\left(f \cdot \frac{\partial}{\partial \theta}\right)\Theta\right) = df = \pi^*(i(v)\omega).$$

Finally, we must prove that the map (49) is surjective: Let ξ be an infinitesimal symmetry of $(\mathbb{P}, \Theta)$. Since it is invariant under the action of $U(1)$ we can write $\xi = v_{\text{hor}} + f \cdot \frac{\partial}{\partial \theta}$. The preceding argument shows that $D_\xi \Theta = 0$ is equivalent to $df = i(v)\omega$. q.e.d.

11.5. Proper cobordisms of abstract moment maps. Let M_1 and M_2 be oriented (not necessarily compact) G-manifolds equipped with proper abstract moment maps ϕ_1 and ϕ_2. The pairs (M_1, ϕ_1) and (M_2, ϕ_2) are said to be (properly) **cobordant** if there exists an oriented G-manifold W with boundary and with a proper abstract moment map $\phi : W \to \mathfrak{g}^*$ such that

$$\partial W = -M_1 \sqcup M_2 \tag{50}$$

and such that the restriction of ϕ to M_2 is ϕ_2, and the restriction of ϕ to M_1 is $-\phi_1$. The cobordism (32) is of this type where the M occurring on the right-hand side of (32) is compact. Each other component on the right-hand side is a vector space with prequantization data and hence a (proper) abstract moment map. In fact, since the prequantization data determine an abstract moment map, we will obtain a cobordism of the abstract moment map once we have a cobordism of the prequatization data as described in the next subsection.

11.6. Proper cobordisms of quantization data. We formulate the theorem that we need: Suppose that M^G is finite. Choose a polarization of the weights at all the fixed point on the right-hand side of (50). For each fixed point p let $\alpha_1, \ldots, \alpha_d$ be the polarized weights at p. (We drop the subscript p.) This gives an identification of TM_p with $\mathbb{C}^d$ intertwining the action of G on M_p with the action

$$\kappa_p(\exp \xi) z = (e^{i\alpha_1(\xi)} z_1, \ldots, e^{i\alpha_d(\xi)} z_d)$$

of G on $\mathbb{C}^d$. The line bundle is the trivial line bundle with with its Bargmann metric which assigns to the trivializing section s the square Hermitian length

$$\langle s.s \rangle (z) = e^{-|z|^2}.$$

The corresponding connection $\nabla = \nabla_p$ has as its moment map

$$\phi_p : \mathbb{C}^d \to \mathfrak{g}^*, \qquad \phi_p(z) = \sum_{j=1}^{d} |z_j|^2 \alpha_j.$$

The following theorem is proved in Chapter 7 of [**12**]:

There exists a G-manifold W with boundary, a Hermitian line bundle

$$\widetilde{\mathbb{L}} \to W,$$

an action $\widetilde{\tau}$ of G on $\widetilde{\mathbb{L}}$ lifting the action on W and a G- invariant connection $\widetilde{\nabla}$ on $\widetilde{\mathbb{L}}$ such that:

- $$\partial W = -M \sqcup \left(\bigsqcup_p (-1)^p TM_p \right).$$
- The restriction of $(\widetilde{\mathbb{L}}, \widetilde{\nabla})$ to M is $(\mathbb{L}, \nabla)$.
- The line bundle and connection induced on TM_p from $(\mathbb{L}, \nabla)$ is $(\widetilde{\mathbb{L}}, \widetilde{\nabla})$ for each fixed point p.
- The moment map associated to $(\widetilde{\mathbb{L}}, \widetilde{\nabla})$ is proper.

11.7. Cutting. We define an analog of the cutting operation of Lerman [**25**].

Let M be a smooth manifold and $\phi : M \to \mathbb{R}$ a proper map. If 0 is a regular value of ϕ then
$$\phi^{-1}([0,\infty))$$
is a manifold with boundary
$$\phi^{-1}(0).$$
Suppose that we have an S^1 action on M and that ϕ is S^1 invariant.

Proposition 11.2. *If S^1 acts freely on $\phi^{-1}(0)$ then the topological space*
$$M^+ := \phi^{-1}([0,\infty))/\sim$$
obtained by collapsing the S^1 orbits on the boundary $\phi^{-1}(0)$ is a smooth manifold without boundary.

Proof. Let S^1 act on $\mathbb{C}$ by $\tau(e^{i\theta})z = e^{i\theta}z$ and consider the product action of S^1 on $M \times \mathbb{C}$. The map
$$\widetilde{\phi}_+ \; M \times \mathbb{C} \to \mathbb{R}, \quad \widetilde{\phi}_+(m,z) := \phi(m) - |z|^2$$
is invariant under the action of S^1. The action of S^1 on the set $\widetilde{\phi}_+^{-1}(0)$ is free. Indeed, this set consists of all points (m,z) with $\phi(m) = |z|^2 > 0$ with s^1 acting freely on the second factor together with $\phi^{-1}(0) \times \{0\}$ where S^1 acts freely by hypothesis. So M^+ as a topological space can be identified with the manifold $\widetilde{\phi}_+^{-1}(0)/S^1$. q.e.d.

We define M^- similarly.

Proposition 11.3. *M is cobordant to the disjoint union of M^+ and M^-.*

Proof. Since 0 is a regular value of ϕ the spaces M^+ and M^- are diffeomorphic to

$$M_\epsilon^+ := \widetilde{\phi}^{-1}(\epsilon) \text{ and } M_\epsilon^-; = \widetilde{\phi}^{-1}(-\epsilon)$$

respectively for sufficiently small $\epsilon > 0$. Let

$$\psi : M \times \mathbb{C} \times [0,1] \to \mathbb{R}, \quad \psi(m,z,t) := t\phi(m)^2 + |z|^2.$$

Consider

$$\psi^{-1}(\epsilon^2)/S^1.$$

This is a manifold whose boundary at $t = 0$ is M and whose boundary at $t = 1$ is the disjoint union of M_ϵ^+ and M_ϵ^-. q.e.d.

As we mentioned, in Section 7 the operation

$$M \mapsto M^+ \sqcup M^-$$

is called "cutting". The dual operation

$$M^+ \sqcup M^- \to M$$

is called **gluing**. For details see [**13**] and [**26**].

Here are some properties of cutting:

1) If M is oriented, the standard orientation of $\mathbb{C}$ gives an orientation of $M \times \mathbb{C}$ and hence an orientation of $\widetilde{\phi}_\pm^{-1}(0)$ which then descends to give an orientation of $M^\pm$.
2) Let S^1 act on $M \otimes \mathbb{C}$ by the product of its action on M and the trivial action on $\mathbb{C}$. This commutes with the previous action of S^1 on $M \times \mathbb{C}$ and hence descends to give an action of S^1 on $M^\pm$.
3) M^+ is the disjoint union of the open subset

$$\phi^{-1}((0,\infty)) \subset M$$

and the **cut divisor**

$$F^+ = M_{\text{red}} = \phi^{-1}(0)/S^1.$$

Similarly M^- is the disjoint union of the open subset

$$\phi^{-1}((-\infty,0)) \subset M$$

and

$$F^- = M_{\text{red}} = \phi^{-1}(0)/S^1.$$

4) If $\phi^{-1}(0)$ is connected, the cut divisor $F^\pm$ is a component of the fixed point set of the action of S^1 on $M^\pm$.

5) The function $\pm|z|^2$ on $M\times\mathbb{C}$ descends to a an S^1 invariant function on $M^\pm$ which is Bott-Morse, and which determines an orientation of the normal bundle of $F^\pm$ in $M^\pm$ and the normal weight $\alpha_{F^\pm}$ fo the action of S^1 on this normal bundle is ± 1.

Now let $\mathbb{L}\to M$ be a Hermitian line bundle over M, ∇ a connection on $\mathbb{L}$ and τ an action of S^1 on $\mathbb{L}$ which is compatible with the action of S^1 on M and preserves ∇. Let the function ϕ figuring in the discussion above be the moment map associated with τ. Then, if s is a section of $\mathbb{L}$,

$$\left(\delta\tau\left(\frac{\partial}{\partial\theta}\right)-\nabla_{\frac{\partial}{\partial\theta}}\right)s=2\pi i\phi s.$$

In particular, on the hypersurface $\phi^{-1}(0)$,

$$\delta\tau=\nabla. \tag{51}$$

Let π be the projection

$$\phi^{-1}(0)\to\phi^{-1}(0)/S^1=M_{\mathrm{red}},$$

and let $\mathbb{L}_{\mathrm{red}}$ be the line bundle over M_{red} whose fiber at p is the one-dimensional vector space of S^1-invariant sections of the line bundle $\mathbb{L}|_{\pi^{-1}(p)}$. By (51), the connection ∇ descends to a connection ∇_{red} on $\mathbb{L}_{\mathrm{red}}$; and hence from the line bundle with connection $(\mathbb{L},\nabla)$ one gets a line bundle with connection $(\mathbb{L}_{\mathrm{red}},\nabla_{\mathrm{red}})$ on M_{red}.

In particular we can apply this result to the space $\widetilde{M}=M\times\mathbb{C}$. with the line bundle $\widetilde{\mathbb{L}}=\mathbb{L}\otimes\widetilde{\mathbb{C}}$, $\widetilde{\mathbb{C}}$ being the trivial line bundle on $\mathbb{C}$. From the given connection on $\mathbb{L}$ and the Bargmann connection on $\widetilde{\mathbb{C}}$ one gets an S^1 invariant connection $\widetilde{\nabla}$ on $\widetilde{\mathbb{L}}$, and the moment map associated with this connection is the function

$$\widetilde{\phi}_+(m,z)=\phi(m)+|z|^2.$$

Thus $\widetilde{M}_{\mathrm{red}}=M_-$ and, by the construction above, one gets from the pair $(\widetilde{\mathbb{L}},\widetilde{\nabla})$ a line bundle $\mathbb{L}^-$ and connection ∇^- on M^-. Similarly one gets from $\widetilde{\mathbb{L}}$ and $\widetilde{\nabla}$ a line bundle $\mathbb{L}^+$ and connection ∇^+ on M^+.

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