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HYPERELLIPTIC SZPIRO INEQUALITY

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Abstract

We generalize the classical Szpiro inequality to the case of a semistable family of hyperelliptic curves. We show that for a semistable symplectic Lefschetz fibration of hyperelliptic curves of genus g, the number N of non-separating vanishing cycles and the number D of singular fibers satisfy the inequality $N \leq (4g+2)D$.

1. Introduction

The classical Szpiro inequality [19] asserts that for any semistable algebraic family of genus one curves $f : X \to \mathbb{CP}^1$, the number of components of the singular fibers of f is bounded from above by 6 times the number of singular fibers. A symplectic generalization of Szpiro's result was proven in [1] by a purely group-theoretic technique. Unfortunately the analogous bounds for fibrations of higher genus curves are extremely hard to obtain (or even guess).

In this note we generalize the techniques developed in [1] to obtain a proof of a Szpiro type bound for symplectic families of hyperelliptic curves. For hyperelliptic curves over number fields such a bound was conjectured by P. Lockhart in [12]. Our goal is to prove the following symplectic version of Lockhart's conjecture:

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Theorem A. *Let $f : X \to S^2$ be a symplectic fibration of hyperelliptic curves of genus g with only semistable fibers. Assume further that f admits a topological section and that all the vanishing cycles of f are nonseparating. Let D be the number of singular fibers of f and let N be the number of vanishing cycles. Then $N \leq (4g + 2)D$.*

Note that as a special case of this theorem one obtains a Szpiro inequality for algebraic families of hyperelliptic curves over $\mathbb{CP}^1$. Surprisingly enough, even this purely algebro-geometric statement was unknown before the present work. In fact the preprint version of our paper had prompted several research projects exploring the inequalities governing the structure of Lefschetz fibrations both in the algebraic and in the symplectic setting. One work, particularly relevant to the current discussion, is a recent theorem of Beauville [4] who found a proof of the algebraic-geometric version of the Szpiro inequality for (not necessarily hyperelliptic) fibers of arbitrary genus. Beauville's theorem settles the Szpiro question completely in the algebraic situation but his method of proof relies heavily on some deep results from the theory of algebraic surfaces, notably the Xiao inequality [24] and Tan's strict canonical class inequality [20]. Unfortunately none of these generalizes to the symplectic situation and so our theorem above remains the optimal result which is valid for symplectic Lefschetz fibrations.

It is worth pointing out that Theorem A is applicable to a strictly larger class of examples than the ones that can be handled by Beauville's result. Indeed, there are many hyperelliptic non-Kähler symplectic Lefschetz fibrations which can be constructed e.g., by the method of [1]. Other examples in the existing literature include an interesting genus two example of Ozbagci-Stipsicz [14] and a genus three example of Fuller and Smith [18] which can easily be realized by a hyperelliptic pencil with one reducible fiber. In fact, Smith [16] has constructed an infinite series of non-Kähler examples of genus two symplectic Lefschetz fibrations by taking fiber sums of Kähler ones along nonalgebraic diffeomorphisms of the fiber surface.

An interesting special feature of the symplectic hyperelliptic fibrations is that they can often be realized as symplectic double covers of rational surfaces [15]. The relationship between the existence of double cover realizations and the Szpiro inequality is not clear at the moment but is certainly an interesting question which has to be investigated in detail. In particular the structure theory of Seibert-Tian suggests that the closer a hyperelliptic symplectic Lefschetz fibration is to saturating the Szpiro bound, the further it is from being holomorphic [15, Remark 2.2].

Finally, from a slightly different perspective the Szpiro inequality can be viewed as an obstruction for the existence of a symplectic structure on a the total space of a topological Lefschetz fibration. Indeed, Theorem A implies that a topological Lefschetz fibration which violates the inequality $N \leq (4g+2)D$ can not be symplectic or, equivalently, orientable (see e.g., [1], [9]).

The paper is organized as follows. In Section 2 we recall some (mostly standard) material about hyperelliptic symplectic fibrations, the hyperelliptic mapping class group and its relation with the braid group. In Section 3 we describe a criterion for the triviality of a central extension of an Artin braid group. This criterion plays an important role in the proof of Theorem A — it provides an efficient way of controlling the ambiguity in lifting relations from the hyperelliptic mapping class group to the braid group. In Section 4 we introduce our main technical tool — the displacement angle of an element in the universal cover of the symplectic group. Finally in Section 5 we compare the values of the degree character and the displacement angle character on the braid group and use this comparison to deduce the hyperelliptic Szpiro inequality. We conclude in Section 6 with a brief discussion of some ideas concerning the general Szpiro inequality.

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2. Hyperelliptic symplectic fibrations

First we recall some basic definitions and results and describe the precise setting in which the Szpiro inequality will be considered in this paper. More details can be found in the papers [1], [9], [15], [16].

Let (X, ω) be a smooth compact symplectic 4-fold. A *differentiable fibration* on X is a surjective C^∞ map $f : X \to S^2$ with finitely many critical points $Q_1, Q_2, \ldots, Q_N$ (not necessarily in distinct fibers) such that locally near each $Q_i \in X$ and $f(Q_i) \in S^2$, there exist complex analytic coordinates x, y on X and t on S^2, so that $t = f(x, y) = x^2 + y^2$. A differentiable fibration $f : X \to S^2$ is called *symplectic* if the smooth fibers of f are symplectic submanifolds with respect to ω and if for every Q_i the symplectic form $\omega_{Q_i} \in \wedge^2 T^*_{Q_i} X$ is nondegenerate on each of the two planes contained in the tangent cone of $f^{-1}(f(Q_i))$ at Q_i. In particular, for a symplectic fibration the local complex analytic coordinates around each Q_i can be chosen to be compatible with a global orientation on X.

For a point $p \in S^2$ we will denote the fiber $f^{-1}(p)$ by X_p. Since by definition the rank of df drops only at the points Q_i, it follows that for each p the fiber X_p is singular only at the points $X_p \cap \{Q_1, \ldots, Q_N\}$. Let $p \in S^2$ and let $X_p^\sharp = X_p - \{Q_1, \ldots, Q_N\}$ be the smooth locus of X_p. A compact surface $Z \subset X$ which is the closure of some connected component of some $X_p^\sharp$ is called a *fiber component* of $f : X \to S^2$. Note that for each p the homology class $[X_p] \in H_2(X, \mathbb{Z})$ splits as a sum $[X_p] = \sum_{\Sigma \in \pi_0(X_p^\sharp)} n_\Sigma [\overline{\Sigma}]$, where $\overline{\Sigma}$ denotes the closure of Σ in X and n_Σ is a positive integer — the *multiplicity* of the fiber component $\overline{\Sigma}$. Again the assumption that the Q_i's are the only critical points of f implies that $n_\Sigma = 1$ for all possible fiber components.

Let $f : X \to S^2$ be a symplectic fibration of fiber genus $g \geq 1$. By analogy with the algebro-geometric case (see e.g., [10]) we will say that f is *semistable* if and only if for every $p \in S^2$ and every $\Sigma \in \pi_0(X_p^\sharp)$ of genus zero we have that Σ is homeomorphic to a sphere with at least two punctures.

Given a symplectic fibration $f : X \to S^2$, we denote by $p_1, \ldots, p_D \in S^2$ the critical values of f. The restriction of f to $S^2 - \{p_1, \ldots, p_D\}$ is a C^∞ fiber bundle with a fiber some closed oriented surface C_g of genus g. Choose a base point $o \in S^2 - \{p_1, \ldots, p_D\}$ and put

$$\mathrm{mon} : \pi_1 \left(S^2 - \{p_1, \ldots, p_D\}, o \right) \to \mathrm{Map}_g := \pi_0(\mathrm{Diff}^+(X_o))$$

for the corresponding geometric monodromy representation.

The hyperelliptic fibrations are singled out among all possible symplectic fibrations by a condition on the geometric monodromy. Fix a double cover $\nu : C_g \to S^2$ and let $\iota \in \mathrm{Map}_g$ denote the mapping class of the covering involution. The *hyperelliptic mapping class group* of genus g is the centralizer Δ_g of ι in Map_g:

$$\Delta_g := \{\phi \in \mathrm{Map}_g \,|\, \phi \iota \phi^{-1} = \iota\}.$$

Similarly we can consider versions of Δ_g that take into account punctures on C_g. Concretely, denote by $\mathrm{Diff}^+(C_g)^n_r$ the group of all orientation preserving diffeomorphisms of C_g preserving $n + r$ distinct points on C_g and inducing the identity on the tangent spaces at r of those points. Let $\mathrm{Map}^n_{g,r} := \pi_0(\mathrm{Diff}^+(C_g)^n_r)$ and define

$$\Delta^n_{g,r} := \Delta_g \times_{\mathrm{Map}_g} \mathrm{Map}^n_{g,r}\,.$$

With this notation we can now define:

Definition 2.1. A *hyperelliptic symplectic fibration* on a smooth symplectic 4-fold (X, ω) is a symplectic fibration $f : X \to S^2$, with a monodromy representation is conjugate to a representation taking values in Δ_g.

The fibration $f : X \to S^2$ is said to be a *hyperelliptic symplectic fibration with a section*, if f has a topological section and the corresponding monodromy representation in Map^1_g is conjugate to one taking values in Δ^1_g.

A classical theorem of Kas [11, Theorem 2.4] asserts that for a symplectic fibration $f : X \to S^2$ of genus $g \geq 2$ the diffeomorphism type of f is uniquely determined by the geometric monodromy of f. Therefore the fact that $f : X \to S^2$ is a hyperelliptic fibration with a section is equivalent to the existence of a topological section of f together with an involution of X which preserves the section and acts as a hyperelliptic involution on each fiber of f.

In the remainder of this paper we will consider only semistable hyperelliptic fibrations with a section. The geometric monodromy representation for such an $f : X \to S^2$ sends a small closed loop running once counterclockwise around one of the p_i into the product of right handed Dehn twists about the cycles vanishing at the points $\{Q_1, \dots, Q_N\} \cap X_{s_i}$. Thus the monodromy representation $\mathrm{mon} : \pi_1(S^2 - \{p_1, \dots, p_D\}) \to \Delta_g$ is encoded completely in the relation in Δ_g:

$$\tau_1 \tau_2 \dots \tau_N = 1,$$

where $\tau_i \in \Delta_g$ denotes the mapping class of the right-handed Dehn twist in $\mathrm{Diff}^+(C_g)$ about the loop vanishing at Q_i.

Similarly the monodromy representation $\pi_1(S^2 - \{p_1, \dots, p_D\}) \to \Delta^1_g$ is completely encoded in the relation in the group Δ^1_g:

$$\mathfrak{t}_1 \mathfrak{t}_2 \dots \mathfrak{t}_N = 1$$

where $\mathfrak{t}_i \in \Delta^1_g$ denotes the right-handed Dehn twist in $\mathrm{Diff}^+(C_g)^1$ about the loop vanishing at Q_i.

3. Central extensions of Artin braid groups

In this section we recall some standard facts about Artin braid groups and study an important class of central extensions of such groups.

Let Γ be a graph with a vertex set I. Assume that Γ has no loops and that any two vertices of Γ are connected by at most finitely many edges.

Definition 3.1. The *Artin braid group* associated with Γ is the group Art_Γ generated by elements $\{t_i | i \in I\}$, so that if $i, j \in I$ are two distinct vertices connected by k_{ij} edges, then t_i and t_j satisfy the relation

$$t_i t_j t_i t_j \cdots = t_j t_i t_j t_i \ldots,$$

where both sides are words of length $k_{ij} + 2$.

Remark 3.2.

(i) Note that by specifying a graph Γ one specifies not only the Artin braid group Art_Γ but also a presentation of Art_Γ. The pair $(\mathrm{Art}_\Gamma, \{t_i\}_{i\in I})$ consisting of an abstract Artin braid group together with a set of standard generators is called an *Artin system.*

(ii) Given an Artin system $(\mathrm{Art}_\Gamma, \{t_i | i \in I\})$, one obtains a natural character

$$\deg : \mathrm{Art}_\Gamma \to \mathbb{Z},$$

which sends each generator to $1 \in \mathbb{Z}$.

(iii) If $\Lambda \subset \Gamma$ is a full subgraph (i.e., $I(\Lambda) \subset I(\Gamma)$ and if $i, j \in I(\Lambda)$, then $k_{ij}(\Lambda) = k_{ij}(\Gamma)$, then the natural homomorphism $\mathrm{Art}_\Lambda \subset \mathrm{Art}_\Gamma$ is known to be injective [21].

Let now $(\mathrm{Art}_\Gamma, \{t_i\}_{i\in I})$ be the Artin system corresponding to a graph Γ. We are interested in the central extensions of Art_Γ by $\mathbb{Z}$ or equivalently in the group cohomology $H^2(\mathrm{Art}_\Gamma, \mathbb{Z})$, where $\mathbb{Z}$ is taken with the trivial Art_Γ-action. The group $H^2(\mathrm{Art}_\Gamma, \mathbb{Z})$ can be quite complicated. For example, due to the work of Arnold [3], Cohen [7] and Fuks [8] it is known that when Γ is a Dynkin graph of type A_n, the group $H^2(\mathrm{Art}_\Gamma, \mathbb{Z})$ is torsion and that when $n \geq 3$ it does contain a nontrivial two torsion.

Let $\mathcal{A}(\Gamma)$ be the set of all abelian subgroups Art_Γ which are generated by $\{t_i\}_{i\in J}$ for some $J \subset I$. Equivalently $\mathcal{A}(\Gamma)$ can be identified with the set of all subgroups $G \subset \mathrm{Art}_\Gamma$ of the form $G = \mathrm{Art}_\Lambda$, where $\Lambda \subset \Gamma$ is a full subgraph with no edges.

Let $\gamma \in H^2(\mathrm{Art}_\Gamma, \mathbb{Z})$ and let

$$(\gamma) \qquad 0 \to \mathbb{Z} \to \Phi_\gamma \to \mathrm{Art}_\Gamma \to 1,$$

be the corresponding central extension of Art_Γ.

Consider the subgroup

$$E(\Gamma) := \bigcap_{G\in\mathcal{A}(\Gamma)} \ker\left[H^2(\mathrm{Art}_\Gamma, \mathbb{Z}) \to H^2(G, \mathbb{Z})\right] \subset H^2(\mathrm{Art}_\Gamma, \mathbb{Z}).$$

Explicitly $E(\Gamma)$ consists of all $\gamma \in H^2(\mathrm{Art}_\Gamma, \mathbb{Z})$ for which the natural pullback sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] & \Phi_\gamma \arrow[r] & \mathrm{Art}_\Gamma \arrow[r] & 1 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & \Phi_\gamma \times_{\mathrm{Art}_\Gamma} G \arrow[r] \arrow[u, hook] & G \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

is split.

We have the following simple:

Lemma 3.3. *Assume that the graph Γ is simply-laced* (*and hence simply connected*)*. Then $E(\Gamma) = 0$.*

Proof. Fix $\gamma \in E(\Gamma)$. Consider the central extension (γ) and let $\{a_i\}_{i\in I} \subset \Phi_\gamma$ be lifts of $t_i \in \mathrm{Art}_\Gamma$.

Since Γ is assumed to be simply-laced, it follows that all relations defining Art_Γ are:

- $t_i t_j = t_j t_i$ if i and j are not connected by an edge;
- $t_i t_j t_i = t_j t_i t_j$ if i and j are connected by an edge.

Let $i \neq j$ be two vertices of Γ which are not connected by an edge. Consider the subgroup $G = \langle t_i, t_j\rangle \subset \mathrm{Art}_\Gamma$. Then $G \in \mathcal{A}(\Gamma)$ and so by our hypothesis this implies that the sequence

$$0 \to \mathbb{Z} \to \Phi_\gamma \times_{\mathrm{Art}_\Gamma} G \to G \to 1$$

is split. In particular this means that $\Phi_\gamma \times_{\mathrm{Art}_\Gamma} G$ is abelian and so for $a_i, a_j \in \Phi_\gamma \times_{\mathrm{Art}_\Gamma} G \subset \Phi_\gamma$ we get $a_i a_j = a_j a_i$.

Let now $c \in \Phi_\gamma$ be the generator of $\mathbb{Z} \subset \Phi_\gamma$ and let $i, j \in I$ be two vertices of Γ which are connected by an edge. Since $t_i t_j t_i = t_j t_i t_j$ we have that

$$a_i a_j a_i = a_j a_i a_j c^{n_{[ij]}}, \tag{3.1}$$

for some integer $n_{[ij]}$.

Consider the one dimensional complex Γ. Let $C_1'(\Gamma, \mathbb{Z})$ be the free abelian group generated by the oriented edges of Γ. In particular, for every edge of Γ we have two generators of $C_1'(\Gamma, \mathbb{Z})$. Introduce the relation that the two generators corresponding to an edge are negative of each other. Let $C_1(\Gamma, \mathbb{Z})$ denote the quotient group. It has one generator for each edge of Γ. Note that these generators can be denoted by their end points. We put $[ij]$ for the edge connecting i and j, with the orientation 'from i to j'. In particular $[ji] = -[ij]$ in $C_1(\Gamma, \mathbb{Z})$.

The group of 1-cochains of Γ with coefficients in $\mathbb{Z}$ is the group $\mathrm{Hom}_{\mathbb{Z}}(C_1(\Gamma, \mathbb{Z}), \mathbb{Z})$. In other words, a 1-cochain of Γ is given by a collection of integers

$$\{f_{[ij]} \in \mathbb{Z}\}_{[ij] \text{ is an oriented edge of } \Gamma},$$

such that $f_{[ij]} = -f_{[ji]}$.

Note that $n_{[ij]} = -n_{[ji]}$ due to the defining relation (3.1) and so $n := \{n_{[ij]}\} \in C^1(\Gamma, \mathbb{Z})$. However $\dim \Gamma = 1$ and so $C^1(\Gamma, \mathbb{Z}) = Z^1(\Gamma, \mathbb{Z})$. Furthermore Γ is simply connected and so $Z^1(\Gamma, \mathbb{Z}) = \delta C^0(\Gamma, \mathbb{Z})$. Hence we can find a zero cochain $m := \{m_i\}_{i \in I}$ of the simplicial complex Γ, so that $n = \delta m$.

Consider the elements $b_i = a_i c^{m_i} \in \Phi_\Gamma$. Clearly the b_i's also lift the t_i's and we have $b_i c = c b_i$ and $b_i b_j = b_j b_i$ for $i, j \in I$ which are not connected by an edge in Γ. Finally, for $i, j \in I$ which are connected by an edge, we calculate

$$\begin{aligned} b_i b_j b_i &= a_i a_j a_i c^{2m_i + m_j} = a_j a_i a_j c^{n_{[ij]} + 2m_i + m_j} \\ &= b_j b_i b_j c^{n_{[ij]} + 2m_i + m_j - 2m_j + m_i} = b_j b_i b_j c^{n_{[ij]} + m_i - m_j} \\ &= b_j b_i b_j c^{(n - \delta m)_{[ij]}} = b_j b_i b_j. \end{aligned}$$

This implies that the subgroup of Φ_γ generated by the b_i's is isomorphic to Art_Γ and splits off as a direct summand in Φ_γ. Hence $\gamma = 0$ in $H^2(\mathrm{Art}_\Gamma, \mathbb{Z})$ and so the lemma is proven. q.e.d.

4. Displacement angles

Let H be a free abelian group of rank $2g$ and let $\theta : H \otimes H \to \mathbb{Z}$ be a symplectic unimodular pairing on H. Consider the $2g$ dimensional vector space $H_{\mathbb{R}} := H \otimes \mathbb{R}$. The real symplectic group $\mathrm{Sp}(H_{\mathbb{R}}, \theta)$ is homotopy equivalent to its maximal compact subgroup which in turn is isomorphic to the unitary group $\mathrm{U}(g)$. In particular $\pi_1(\mathrm{Sp}(H_{\mathbb{R}}, \theta)) \cong \pi_1(\mathrm{U}(g)) \cong \mathbb{Z}$ and so the universal cover $\widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$ of $\mathrm{Sp}(H_{\mathbb{R}}, \theta)$ is naturally a central extension

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta) \to \mathrm{Sp}(H_{\mathbb{R}}, \theta) \to 1. \tag{4.1}$$

Let $\Lambda(H_{\mathbb{R}}, \theta)$ be the Lagrangian Grassmanian of the symplectic vector space $(H_{\mathbb{R}}, \theta)$. The Grassmanian $\Lambda(H_{\mathbb{R}}, \theta)$ can be identified with the homogeneous space $\mathrm{U}(g)/\mathrm{O}(g)$ as follows. Choose a complex structure $I : H_{\mathbb{R}} \to H_{\mathbb{R}}$ which is θ-tamed. This simply means that $\gamma(x, y) := \theta(I(x), y)$ is a positive definite symmetric form and so $\eta = \gamma + \sqrt{-1}\,\theta$ is a positive definite Hermitian form on the g-dimensional complex vector space $H^{\mathbb{C}} := (H_{\mathbb{R}}, I)$. Now every element in the unitary group $\mathrm{U}(H^{\mathbb{C}}, \eta)$ necessarily preserves θ and so we get an inclusion $\mathrm{U}(g) \cong \mathrm{U}(H^{\mathbb{C}}, \eta) \subset \mathrm{Sp}(H_{\mathbb{R}}, \theta)$ which is a homotopy equivalence. If $\lambda \subset H_{\mathbb{R}}$ is a Lagrangian subspace, then every basis of λ which is orthonormal w.r.t. $\gamma_{|\lambda}$ will also be a $\mathbb{C}$-basis of $H^{\mathbb{C}}$ which is orthonormal w.r.t. η. In particular if $\lambda, \mu \in \Lambda(H_{\mathbb{R}}, \theta)$ are two Lagrangian subspaces and we choose γ-orthonormal bases in λ and μ respectively, then there will be unique element $u \in \mathrm{U}(H^{\mathbb{C}}, \eta)$ which sends the basis for λ to the basis for μ and so $u(\lambda) = \mu$. This shows that $\mathrm{U}(H^{\mathbb{C}}, \eta)$ will act transitively on $\Lambda(H_{\mathbb{R}}, \theta)$ and that the stabilizer of a point $\lambda \in \Lambda(H_{\mathbb{R}}, \theta)$ in $\mathrm{U}(H^{\mathbb{C}}, \eta)$ can be identified with the orthogonal group $\mathrm{O}(\lambda, \gamma_{|\lambda})$. Thus $\Lambda(H_{\mathbb{R}}, \theta) = \mathrm{U}(H^{\mathbb{C}}, \eta)/\mathrm{O}(\lambda, \gamma_{|\lambda}) \cong \mathrm{U}(g)/\mathrm{O}(g)$.

This homogeneous space interpretation can be used to show [2] that the fundamental group of the Lagrangian Grassmanian is isomorphic to $\mathbb{Z}$. Indeed the natural determinant homomorphism $\det : \mathrm{U}(g) \to S^1$ restricts to $\det : \mathrm{O}(g) \to \{\pm 1\}$ on $\mathrm{O}(g)$ and so descends to a well-defined map $d : \mathrm{U}(g)/\mathrm{O}(g) \to S^1/\{\pm 1\} \cong S^1$. The fiber of d is diffeomorphic to the homogeneous space $\mathrm{SU}(g)/\mathrm{SO}(g)$. But $\mathrm{SU}(g)$ is simply connected and $\mathrm{SO}(g)$ is connected and so $\pi_1(\mathrm{SU}(g)/\mathrm{SO}(g)) = \{1\}$ from the long exact sequence of homotopy groups for the fibration $\mathrm{SU}(g) \to \mathrm{SU}(g)/\mathrm{SO}(g)$. Therefore by the long exact sequence of homotopy groups for the fibration $d : \mathrm{U}(g)/\mathrm{O}(g) \to S^1$ we conclude that d induces an isomorphism on fundamental groups, i.e., $\pi_1(\Lambda(H_{\mathbb{R}}, \theta)) =$

$\pi_1(\mathrm{U}(g)/\mathrm{O}(g)) = \mathbb{Z}$.

Note that the group $\mathrm{Sp}(H_{\mathbb{R}}, \theta)$ also acts transitively on $\Lambda(H_{\mathbb{R}}, \theta)$ and that $\widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$ acts transitively on the universal cover $\widetilde{\Lambda}(H_{\mathbb{R}}, \theta)$ of $\Lambda(H_{\mathbb{R}}, \theta)$.

Recall that every vector $a \in H_{\mathbb{R}}$ generates a one parameter unipotent subgroup in $\mathrm{Sp}(H_{\mathbb{R}}, \theta)$ by the formula

$$\begin{aligned} T_a : \mathbb{R} &\longrightarrow \mathrm{Sp}(H_{\mathbb{R}}, \theta) \\ s &\mapsto (x \mapsto x + s\theta(a, x)a). \end{aligned}$$

Every element of the form $T_a(s) = T_{\sqrt{s}\,a}(1)$ is called a *symplectic transvection*. In the case when $H = H_1(C, \mathbb{Z})$ for some smooth surface C, the element $T_a(1)$ is the image of the oriented Dehn twist along a simple closed curve representing the homology class $a \in H_1(C, \mathbb{Z})$.

We begin with the following lemma.

Lemma 4.1. *Let $t \in \mathrm{Sp}(H_{\mathbb{R}}, \theta)$ be a symplectic transvection. Then there exists a unique lift $\widetilde{t} \in \widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$ of t which acts with fixed points on $\widetilde{\Lambda}(H_{\mathbb{R}}, \theta)$.*

Proof. To check that $\widetilde{t}$ exists write $t = T_a(1)$ for some $a \in H_{\mathbb{R}}$. The vector a can be included in a symplectic basis $a_1 = a, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g$ of $H_{\mathbb{R}}$, and so t preserves the Lagrangian subspace $\lambda := \mathrm{Span}_{\mathbb{R}}(a_1, \ldots, a_g)$. Let $\widetilde{\lambda} \in \widetilde{\Lambda}(H_{\mathbb{R}}, \theta)$ be a preimage of $\lambda \in \Lambda(H_{\mathbb{R}}, \theta)$ and let $p \in \widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$ be a preimage of $t \in \mathrm{Sp}(H_{\mathbb{R}}, \theta)$. Then $p \cdot \widetilde{\lambda}$ maps to $t \cdot \lambda = \lambda$ and so we can find a deck transformation $c \in \pi_1(\Lambda(H_{\mathbb{R}}, \theta), \lambda) = \mathbb{Z}$ satisfying $c(p \cdot \widetilde{\lambda}) = \widetilde{\lambda}$. On the other hand, the continuous map $m : \mathrm{Sp}(H_{\mathbb{R}}, \theta) \to \Lambda(H_{\mathbb{R}}, \theta)$, given by $m(g) = g \cdot \lambda$ induces a homomorphism $m_* : \pi_1(\mathrm{Sp}(H_{\mathbb{R}}, \theta), e) \to \pi_1(\Lambda(H_{\mathbb{R}}, \theta), \lambda)$. If $c = m_*(\widetilde{c})$ for some $\widetilde{c} \in \pi_1(\mathrm{Sp}(H_{\mathbb{R}}, \theta), e) \subset Z(\widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta))$, we get

$$\widetilde{\lambda} = c(p \cdot \widetilde{\lambda}) = m_*(\widetilde{c})(p \cdot \widetilde{\lambda}) = (\widetilde{c}p) \cdot \widetilde{\lambda},$$

where by $(\widetilde{c}p)$ we mean the product of $\widetilde{c} \in Z(\widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)) \subset \widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$ and $p \in \widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$ in the group $\widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$. However multiplication by elements in $\pi_1(\mathrm{Sp}(H_{\mathbb{R}}, \theta), e) \subset Z(\widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta))$ preserves the fibers of the covering map $\widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta) \to \mathrm{Sp}(H_{\mathbb{R}}, \theta)$. and so we may take $\widetilde{t} := \widetilde{c}p$.

Therefore, in order to finish the proof of the existence of $\widetilde{t}$ we have to show that $c \in \mathrm{im}(m_*)$. As we explained above the identification $\Lambda(H_{\mathbb{R}}, \theta) = \mathrm{U}(H^{\mathbb{C}}, \eta)/\mathrm{O}(\lambda, \gamma_{|}\lambda)$ implies that the map m_* fits in the following commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & \pi_1(\mathrm{Sp}(H_{\mathbb{R}},\theta),e) \arrow[r, "m_*"] \arrow[d, equal] & \pi_1(\Lambda(H_{\mathbb{R}},\theta),\lambda) \arrow[r] \arrow[d, equal] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{mult}_2"'] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0.
\end{tikzcd}
$$

In particular if $\Lambda_{(2)}(H_{\mathbb{R}},\theta) \to \Lambda(H_{\mathbb{R}},\theta)$ denotes the unramified double cover corresponding to the surjection $\pi_1(\Lambda(H_{\mathbb{R}},\theta),\lambda) \to \mathbb{Z}/2$, then m factors as $\mathrm{Sp}(H_{\mathbb{R}},\theta) \to \Lambda_{(2)}(H_{\mathbb{R}},\theta) \to \Lambda(H_{\mathbb{R}},\theta)$. Furthermore, by the definition of m we have $m(tg) = t \cdot m(g)$ for all $g \in \mathrm{Sp}(H_{\mathbb{R}},\theta)$ and so we have a natural lift $t_{(2)}$ of the action of t on $\Lambda(H_{\mathbb{R}},\theta)$ to an automorphism of the double cover $\Lambda_{(2)}(H_{\mathbb{R}},\theta)$. Since the map $\mathrm{Sp}(H_{\mathbb{R}},\theta) \to \Lambda_{(2)}(H_{\mathbb{R}},\theta)$ has connected fibers, it induces an isomorphism on fundamental groups and so $c \in \mathrm{im}(m_*)$ if and only if the automorphism $t_{(2)}$ acts trivially on the fiber of $\Lambda_{(2)}(H_{\mathbb{R}},\theta) \to \Lambda(H_{\mathbb{R}},\theta)$ over the point $\lambda \in \Lambda(H_{\mathbb{R}},\theta)$. But by construction this fiber can be identified with the quotient $\mathbb{Z}/2$ of $\pi_1(\Lambda(H_{\mathbb{R}},\theta),\lambda)$ and the action $t_{(2)}$ on the fiber can be identified with the action on $\mathbb{Z}/2$ induced from $t_* : \pi_1(\Lambda(H_{\mathbb{R}},\theta),\lambda) \to \pi_1(\Lambda(H_{\mathbb{R}},\theta),\lambda)$. Finally note that $\mathrm{Sp}(H_{\mathbb{R}},\theta)$ is connected and so the action of t on $\Lambda(H_{\mathbb{R}},\theta)$ is homotopic to the identity. Hence t_* acts trivially on $\pi_1(\Lambda(H_{\mathbb{R}},\theta),\lambda)$ and $c \in \mathrm{im}(m_*)$.

Next we prove the uniqueness of $\widetilde{t}$. To that end we calculate the fixed locus of $\widetilde{t}$ on $\widetilde{\Lambda}(H_{\mathbb{R}},\theta)$ explicitly. Clearly the image of $\mathrm{Fix}_{\widetilde{t}}(\widetilde{\Lambda}(H_{\mathbb{R}},\theta))$ in $\Lambda(H_{\mathbb{R}},\theta)$ is contained in the fixed locus $\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}},\theta))$. Since $\pi_1(\mathrm{Sp}(H_{\mathbb{R}},\theta))$ is central in $\widetilde{\mathrm{Sp}}(H_{\mathbb{R}},\theta)$, it follows that the element $\widetilde{t} \in \widetilde{\mathrm{Sp}}(H_{\mathbb{R}},\theta)$ commutes with all $c \in \pi_1(\mathrm{Sp}(H_{\mathbb{R}},\theta))$. Since $\pi_1(\mathrm{Sp}(H_{\mathbb{R}},\theta))$ acts transitively on the fibers of $\pi : \widetilde{\Lambda}(H_{\mathbb{R}},\theta) \to \Lambda(H_{\mathbb{R}},\theta)$ and $\widetilde{t} \cdot \widetilde{\lambda} = \lambda$, it follows that $\widetilde{t}$ fixes all points in $\widetilde{\Lambda}(H_{\mathbb{R}},\theta)$ that map to $\lambda \in \Lambda(H_{\mathbb{R}},\theta)$. In particular $\pi^{-1}(\lambda) \subset \mathrm{Fix}_{\widetilde{t}}(\widetilde{\Lambda}(H_{\mathbb{R}},\theta))$. In fact we have

$$
\mathrm{Fix}_{\widetilde{t}}(\widetilde{\Lambda}(H_{\mathbb{R}},\theta)) = \pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}},\theta))). \tag{4.2}
$$

Indeed, the locus $\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}},\theta))$ consists of all Lagrangian subspaces $\mu \in \Lambda(H_{\mathbb{R}},\theta)$ that contain the vector $a \in H_{\mathbb{R}}$ and so is isomorphic to the Lagrangian Grassmanian of a symplectic vector space of dimension $2g-2$. To see this consider the θ-orthogonal complement $a^{\perp}$ of a in $H_{\mathbb{R}}$. The $2g-1$ dimensional subspace $a^{\perp} \subset H_{\mathbb{R}}$ contains a and inherits a skew-symmetric form $\theta_{|a^{\perp}}$ whose kernel is spanned by a. The θ-lagrangian subspaces in $H_{\mathbb{R}}$ which contain the vector a are all contained in $a^{\perp}$. Therefore $\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}},\theta))$ can be identified with the set of all g-dimensional subspaces in $a^\perp$ which are θ-isotropic and contain the vector a. Since $\ker(\theta_{|a^\perp}) = \mathbb{R}\cdot a$, the form $\theta_{|a^\perp}$ descends to a symplectic form $\bar{\theta}$ on the quotient space $\overline{H}_{\mathbb{R}} := a^\perp/\mathbb{R}\cdot a$. Now every g-dimensional θ-isotropic subspace in $a^\perp$ which contains a will map onto a $\bar{\theta}$-Lagrangian subspace of $\overline{H}_{\mathbb{R}}$ and conversely — the preimage of a $\bar{\theta}$-Lagrangian subspace of $\bar{\mu} \subset \overline{H}_{\mathbb{R}}$ will be a g-dimensional θ-isotropic subspace $\mu \subset a^\perp$ which contains a. In other words, we have constructed an inclusion $\Lambda(\overline{H}_{\mathbb{R}}, \bar{\theta}) \hookrightarrow \Lambda(H_{\mathbb{R}}, \theta)$, whose image is precisely $\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))$. In particular this shows that $\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))$ is connected and that the inclusion map $\Lambda(\overline{H}_{\mathbb{R}}, \bar{\theta}) = \mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta)) \hookrightarrow \Lambda(H_{\mathbb{R}}, \theta)$ induces an isomorphism on fundamental groups. This implies that $\pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta)))$ is and simply connected and that the map $\pi : \pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))) \to \mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))$ is the universal covering map for $\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))$. But by definition $\pi \circ \widetilde{t} = t \circ \pi$ and that t acts trivially on $\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))$. This shows that $\widetilde{t}$ is an automorphism of the universal covering map $\pi : \pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))) \to \mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))$ and so $\widetilde{t}|\pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta)))$ must be a deck transformation. However by construction $\widetilde{t}$ fixes $\widetilde{\lambda} \in \pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta)))$ and so $\widetilde{t}|\pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))) = \mathrm{id}$. Thus we have established the validity of (4.2).

Consider now some other lift q of t. Then $q = c\widetilde{t}$ for some $c \in \pi_1(\mathrm{Sp}(H_{\mathbb{R}}, \theta))$. Now again $\mathrm{Fix}_q(\widetilde{\Lambda}(H_{\mathbb{R}}, \theta)) \subset \pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))$ and since $\widetilde{t}$ acts trivially on $\pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))$ we see that $q_{|\pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))}$ $= c_{|\pi^{-1}(\mathrm{Fix}_t(\Lambda(H_{\mathbb{R}}, \theta))}$. But c is a deck transformation and so acts without fixed points. Thus $\mathrm{Fix}_q(\widetilde{\Lambda}(H_{\mathbb{R}}, \theta)) = \varnothing$ and the uniqueness of the lift $\widetilde{t}$ is proven. q.e.d.

Consider next the standard Artin braid group B_{2g+2} on $2g+2$ strands. In other words $B_{2g+2} := \mathrm{Art}_{\boldsymbol{A}_{2g+1}}$ is the Artin braid group corresponding to the Dynkin graph $\boldsymbol{A}_{2g+1}$. Explicitly

$$B_{2g+2} = \Big\langle t_1, t_2, \ldots, t_{2g+1} \mid t_i t_j = t_j t_i \quad \text{for} \quad |i-j| \geq 2, t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \Big\rangle.$$

Fix a closed oriented surface C_g of genus g and a hyperelliptic involution ι on C_g. If we choose a sequence of loops $c_1, c_2, \ldots, c_{2g+1}$ on C_g as depicted on Figure 4.1, we can realize the generators t_i geometrically as the right-handed Dehn twists t_{c_i}. The assignment $t_i \to t_{c_i} \in \mathrm{Map}_{g,1}$ induces a homomorphism $\kappa_{g,1} : B_{2g+2} \to \mathrm{Map}_{g,1}$ and after compositions with the natural projections induces homomorphisms

$$\begin{aligned}\kappa_g^1 &: B_{2g+2} \to \mathrm{Map}_g^1,\\ \kappa_g &: B_{2g+2} \to \mathrm{Map}_g,\\ \sigma_g &: B_{2g+2} \to \mathrm{Sp}(H_1(C_g, \mathbb{Z})).\end{aligned}$$

Furthermore, if one is careful enough to chose the c_i so that they are invariant under the hyperelliptic involution ι, then the image of $\kappa_{g,1}$ will be the hyperelliptic mapping class group $\Delta_{g,1}$.

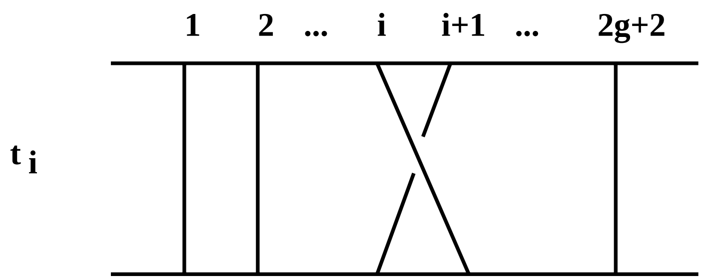

Figure 4.1: Loops generating B_{2g+2}.

It is known [5] that $\kappa_{g,1} : B_{2g+2} \to \Delta_{g,1}$ is surjective with a kernel normally generated by the element

$$(t_1 \dots t_{2g+1})^{2g+1}(t_1 t_2 \dots t_{2g+1} t_{2g+1} \dots t_2 t_1)^{-1},$$

and that $\kappa_g^1 : B_{2g+2} \to \Delta_g^1$ is surjective with a kernel normally generated by the elements $(t_1 \dots t_{2g+1})^{2g+1}(t_1 t_2 \dots t_{2g+1} t_{2g+1} \dots t_2 t_1)^{-1}$ and $(t_1 \dots t_{2g+1})^{2g+2}$. It is also known [22] that the map σ_g is surjective, and so we have a sequence of surjective group homomorphisms

$$B_{2g+2} \twoheadrightarrow \Delta_{g,1} \twoheadrightarrow \Delta_g^1 \twoheadrightarrow \Delta_g \twoheadrightarrow \mathrm{Sp}(H_1(C_g, \mathbb{Z})).$$

Consider the lattice $H := H_1(C_g, \mathbb{Z})$ together with symplectic unimodular pairing $\theta : H \otimes H \to \mathbb{Z}$ corresponding to the intersection of cycles. Let now

$$0 \to \mathbb{Z} \to \widetilde{B}_{2g+2} \to B_{2g+2} \to 0 \tag{4.3}$$

be the pullback of the central extension (4.1) via the homomorphism

$$\sigma_g : B_{2g+2} \to \mathrm{Sp}(H, \theta) \subset \mathrm{Sp}(H_{\mathbb{R}}, \theta).$$

we have the following important result:

Proposition 4.2. *The extension* (4.3) *is a split extension.*

Proof. Let $\gamma \in H^2(B_{2g+2}, \mathbb{Z}) = H^2(\mathrm{Art}_{\boldsymbol{A}_{2g+2}}, \mathbb{Z})$ be the extension class of (4.3). In view of Lemma 3.3 it suffices to show that $\gamma \in E(\boldsymbol{A}_{2g+2})$ or equivalently that γ splits when restricted on every $G \in \mathcal{A}(\boldsymbol{A}_{2g+2})$. Since by definition (4.3) is a pullback of a central extension of the group $\mathrm{Sp}(H, \theta)$, it suffices to check that the extension (4.1) splits when restricted on any abelian subgroup of $\mathrm{Sp}(H, \theta)$ which is generated by the Dehn twists about any finite collection of nonintersecting c_i's.

Let $\{a_1, \dots, a_k\} \subset \{c_1, \dots, c_{2g+1}\}$ be such that $\theta(a_i, a_j) = 0$ for all i, j. Let $t_i := T_{a_i}(1) \in \mathrm{Sp}(H, \theta)$ be the corresponding symplectic transvections and let $S \subset \mathrm{Sp}(H, \theta)$ be the subgroup generated by the t_i's. Consider the pullback of the extension (4.1) via the inclusion map $S \hookrightarrow \mathrm{Sp}(H, \theta) \subset \mathrm{Sp}(H_{\mathbb{R}}, \theta)$:

$$0 \to \mathbb{Z} \to \widetilde{S} \to S \to 0. \tag{4.4}$$

As explained above, it suffices to show that (4.4) is split in order to prove the proposition. To achieve this consider the subspace $\mathrm{Span}_{\mathbb{R}}(a_1, \dots, a_k) \subset H_{\mathbb{R}}$. Since by assumption this subspace is θ-isotropic we can find a Lagrangian subspace $\lambda \in \Lambda(H_{\mathbb{R}}, \theta)$ so that $\lambda \supset \mathrm{Span}_{\mathbb{R}}(a_1, \dots, a_k)$. In particular we will have $t_i \cdot \lambda = \lambda$ for all $i = 1, \dots, k$ and hence $\lambda \in \mathrm{Fix}_S(\Lambda(H_{\mathbb{R}}, \theta)) \neq \varnothing$. We now have the following:

Lemma 4.3. *For every element $g \in S$ there exists a lift $\widetilde{g} \in \widetilde{S}$ which is uniquely characterized by the property*

$$\mathrm{Fix}_{\widetilde{g}}(\widetilde{\Lambda}(H_{\mathbb{R}}, \theta)) \supset \pi^{-1}(\mathrm{Fix}_S(\Lambda(H_{\mathbb{R}}, \theta))).$$

Proof. We will prove the lemma by induction on k. For $k = 1$ this is precisely the statement of Lemma 4.1. Let $k > 1$. By the inductive hypothesis the elements of the subgroup generated by $t_1, \dots, t_{k-1}$ lift uniquely to elements in $\widetilde{S}$ which fix all points in the set $\pi^{-1}(\mathrm{Fix}_{\langle t_1, \dots, t_{k-1}\rangle}(\Lambda(H_{\mathbb{R}}, \theta)))$. By applying Lemma 4.1 again to the image of the element a_k in the symplectic vector space

$$\mathrm{Span}(a_1, \dots, a_{k-1})^{\perp} / \mathrm{Span}(a_1, \dots, a_{k-1})$$

we get the required lifts for all elements in S. q.e.d.

By Lemma 4.3 we get a set theoretic map $S \to \widetilde{S}$, $g \mapsto \widetilde{g}$, which splits the exact sequence (4.4). However if $g, h \in S$ are two elements, then $\widetilde{g} \cdot \widetilde{h}$

is a lift of $g \cdot h$ which necessarily fixes all points in $\pi^{-1}(\mathrm{Fix}_S(\Lambda(H_{\mathbb{R}}, \theta))$ since $\widetilde{g}$ and $\widetilde{h}$ fix those points individually. Thus $\widetilde{g \cdot h} = \widetilde{g} \cdot \widetilde{h}$ and hence the assignment $g \mapsto \widetilde{g}$ is a group-theoretic splitting of (4.4). This finishes the proof of Proposition 4.2. q.e.d.

The reasoning in the proof of the previous proposition gives as an immediate corollary the following statement which we record here for future use.

Corollary 4.4. *There exists a homomorphism* $\widetilde{\sigma}_g : B_{2g+2} \to \widetilde{\mathrm{Sp}}(H, \theta)$ *so that* $\pi \circ \widetilde{\sigma}_g = \sigma_g$ *and for which* $\widetilde{\sigma}_g(t_i) = \widetilde{\sigma_g(t_i)}$ *is the unique lift of* $\sigma_g(t_i)$ *from Lemma* 4.1.

For the remainder of the paper we fix once and for all a base point $\lambda_0 \in \Lambda(H_{\mathbb{R}}, \theta)$. Let $\mathrm{U}(g) \cong K \subset \mathrm{Sp}(H_1(C_g, \mathbb{R}))$ be a maximal compact subgroup. The choice of λ_0 determines a K-equivariant surjection $K \to \Lambda(H_{\mathbb{R}}, \theta)$, which as explained at the beginning of this section combines with the determinant homomorphism $\det : \mathrm{U}(g) \to S^1$ into a well defined map $d_K : \Lambda(H_{\mathbb{R}}, \theta) \to S^1$. Let $\widetilde{d}_K : \widetilde{\Lambda}(H_{\mathbb{R}}, \theta) \to \mathbb{R}$ be a lift of d_K to the universal cover.

Consider the subset

$$^{K}\widetilde{\mathrm{Sp}}(-) := \{s \in \widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta) \mid \widetilde{d}_K(s \cdot \widetilde{\lambda}) \leq \widetilde{d}_K(\widetilde{\lambda}) \text{ for all } \widetilde{\lambda} \in \widetilde{\Lambda}(H_{\mathbb{R}}, \theta)\},$$

and let $\widetilde{\mathrm{Sp}}(-) := \cap_K {}^{K}\widetilde{\mathrm{Sp}}(-)$. Clearly ${}^{K}\widetilde{\mathrm{Sp}}(-)$ and $\widetilde{\mathrm{Sp}}(-)$ are subsemigroups in $\widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$ and for every simple right Dehn twist $t \in \mathrm{Sp}(H_{\mathbb{R}}, \theta)$ we have that $\widetilde{t} \in \widetilde{\mathrm{Sp}}(-)$ for the lift $\widetilde{t}$ from Lemma 4.1.

For future reference we give an alternative description of the elements in the subsemigroup $\widetilde{\mathrm{Sp}}(-)$ in terms of linear algebraic data. First we have the following easy lemma.

Lemma 4.5. *Let* $\widetilde{K}$ *be the universal cover of the maximal compact subgroup* K. *Let* $\widetilde{\det} : \widetilde{K} \to \mathbb{R}$ *denote the lift of* $\det : K \cong \mathrm{U}(g) \to S^1$ *as a group homomorphism. Then an element* $\mathfrak{u} \in \widetilde{K}$ *will belong to the subsemigroup* ${}^{K}\widetilde{\mathrm{Sp}}(-)$ *if and only if* $\widetilde{\det}(\mathfrak{u}) \leq 0$. *In other words*

$$\widetilde{K} \cap {}^{K}\widetilde{\mathrm{Sp}}(-) = \widetilde{\det}^{-1}(\mathbb{R}_{\leq 0}).$$

Proof. Let $\widetilde{\lambda} \in \widetilde{\Lambda}(H_{\mathbb{R}}, \theta)$ and let $\mathfrak{v} \in \widetilde{K}$ be an element which maps to $\widetilde{\lambda}$ under the natural map $\widetilde{K} \to \widetilde{\Lambda}(H_{\mathbb{R}}, \theta)$. Then by the definition of the lifts $\widetilde{d}_K$ and $\widetilde{\det}$ we have $\widetilde{\det}(\mathfrak{v}) = \widetilde{d}_K(\widetilde{\lambda}) + c$ where $c \in \mathbb{R}$ is a fixed constant depending on the choice of the lift $\widetilde{d}$ only, and not on the choices of $\widetilde{\lambda}$ or $\mathfrak{v}$. Similarly $\widetilde{\det}(\mathfrak{u}\cdot\mathfrak{v}) = \widetilde{d}_K(\mathfrak{u}\cdot\widetilde{\lambda}) + c$ for all $\mathfrak{u} \in \widetilde{K}$. Therefore

$$\begin{aligned}\widetilde{d}_K(\widetilde{\lambda}) - \widetilde{d}_K(\mathfrak{u}\cdot\widetilde{\lambda}) &= (\widetilde{\det}(\mathfrak{v}) - c) - (\widetilde{\det}(\mathfrak{u}\cdot\mathfrak{v}) - c)\\ &= \widetilde{\det}(\mathfrak{v}) - (\widetilde{\det}(\mathfrak{u}) + \widetilde{\det}(\mathfrak{v})) = -\widetilde{\det}(\mathfrak{u}).\end{aligned}$$

The lemma is proven. q.e.d.

The previous lemma gives a description of the semigroups $K \cap {}^K\widetilde{\mathrm{Sp}}(-)$ and ${}^K\widetilde{\mathrm{Sp}}(-)$ in terms of the natural maps $\widetilde{\det}$ and $\widetilde{d}_K$. We would like to have a similar intrinsic description for the smaller semigroups $K \cap \widetilde{\mathrm{Sp}}(-)$ and $\widetilde{\mathrm{Sp}}(-)$ respectively.

To that end consider the Lie algebra $\mathfrak{sp}(H_{\mathbb{R}}, \theta) \subset \mathrm{End}(H_{\mathbb{R}})$. Every element $x \in \mathfrak{sp}(H_{\mathbb{R}}, \theta)$ defines a real valued symmetric bilinear form γ_x on $H_{\mathbb{R}}$ via the formula $\gamma_x(\bullet, \bullet) := \theta(x(\bullet), \bullet)$. Using the assignment $x \mapsto \gamma_x$ we can now define the *nonpositive subcone* in the Lie algebra $\mathfrak{sp}(H_{\mathbb{R}}, \theta)$ as the cone

$$\mathfrak{sp}(-) := \{x \in \mathfrak{sp}(H_{\mathbb{R}}, \theta) \mid \gamma_x \text{ is nonpositive definite}\}.$$

Similarly, for the Lie algebra $\mathfrak{k} \subset \mathfrak{sp}(H_{\mathbb{R}}, \theta)$ of the compact group K we have a nonpositive cone $\mathfrak{k}(-) := \mathfrak{k} \cap \mathfrak{sp}(-)$. The elements of this subcone admit a particularly simple characterization in the fundamental representation of $\mathfrak{k} \cong \mathfrak{u}(g)$:

Lemma 4.6.

(i) *An element $x \in \mathfrak{k} \subset \mathfrak{sp}(H_{\mathbb{R}}, \theta)$ belongs to the nonpositive subcone $\mathfrak{k}(-)$ if and only if all the eigenvalues of x in the fundamental representation of $\mathfrak{k} \cong \mathfrak{u}(g)$ are contained in $\sqrt{-1}\cdot\mathbb{R}_{\leq 0}$.*

(ii) *The interior of the nonpositive cone $\mathfrak{sp}(-)$ is the union of the interiors of all cones of the form $\mathfrak{k}(-)$.*

Proof. To prove part (i) recall that the choice of a maximal compact subgroup $K \subset \mathrm{Sp}(H_{\mathbb{R}}, \theta)$ corresponds to the choice of a complex structure $I : H_{\mathbb{R}} \to H_{\mathbb{R}}$ which is θ-tamed. Once such an I is chosen we can identify K with the unitary group $\mathrm{U}(H^{\mathbb{C}}, \eta)$, where $H^{\mathbb{C}} = (H_{\mathbb{R}}, I)$ and $\eta(x, y) = \theta(I(x), y) + \sqrt{-1}\cdot\theta(x, y)$. Under this identification $H^{\mathbb{C}}$ becomes the fundamental representation of $\mathfrak{k}$. Furthermore if $a_1, \ldots, a_g$ is a basis of $\lambda_0 \subset H_{\mathbb{R}}$, which is orthonormal w.r.t. to the form $\theta(I(x), y)$, it follows that $a_1, \ldots, a_g$ is an orthonormal basis of

$(H^{\mathbb{C}}, \eta)$ and $a_1, \ldots, a_g, I(a_1), \ldots, I(a_g)$ is a symplectic basis of $(H_{\mathbb{R}}, \theta)$. In particular in the basis $a_1, \ldots, a_g, I(a_1), \ldots, I(a_g)$ the Gram matrix of the skew-symmetric form θ equals the matrix of the linear operator I equals the standard matrix

$$\begin{pmatrix} 0 & \mathbb{I}_g \\ -\mathbb{I}_g & 0 \end{pmatrix}.$$

(Here as usual $\mathbb{I}_g$ denotes the identity $g \times g$ matrix.) Let now $x \in \mathfrak{k}$ be any element. If we view x as a linear operator on the complex vector space $H^{\mathbb{C}}$, then in the basis $a_1, \ldots, a_g$ this linear operator is given by some skew-hermitian $g \times g$ matrix A. Since every skew-hermitian matrix can be diagonalized in an orthonormal basis we may assume without a loss of generality that the basis $a_1, \ldots, a_g$ is chosen so that $A = \sqrt{-1} \cdot D$ where D is a real diagonal $g \times g$ matrix. Therefore the Gram matrix of the symmetric form γ_x in the basis $a_1, \ldots, a_g, I(a_1), \ldots, I(a_g)$ is the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}^t \cdot \begin{pmatrix} 0 & \mathbb{I}_g \\ -\mathbb{I}_g & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}.$$

This proves part (i) of the lemma. Part (ii) is straightforward and is left to the reader. q.e.d.

Remark 4.7.

(i) Since every unipotent element in the symplectic group $\mathrm{Sp}(H_{\mathbb{R}}, \theta)$ can be written as a limit of conjugates of elements in a fixed maximal compact subgroup, part (ii) of the previous lemma implies that the boundary $\partial\mathfrak{sp}(-)$ of the nonpositive cone contains the intersection $\mathcal{N} \cap \mathfrak{sp}(-)$ of the nilpotent cone $\mathcal{N} \subset \mathfrak{sp}(H_{\mathbb{R}}, \theta)$ and the nonpositive cone.

(ii) The same reasoning as in the proof of part (i) of the previous lemma shows that $\partial\mathfrak{sp}(-)$ consists of all elements $x \in \mathfrak{sp}(-)$ for which we can find a decomposition $H_{\mathbb{R}} = W' \oplus W''$ so that:

- $W', W'' \subset H_{\mathbb{R}}$ are nontrivial symplectic subspaces;
- $x(W') \subset W'$ and $x(W'') \subset W''$;
- $x_{|W'} \in \mathcal{N}(\mathfrak{sp}(W')) \cap \mathfrak{sp}(W')(-)$ and $x_{|W''} \in \mathfrak{k}''(-)$ for some maximal compact subgroup $K'' \subset \mathrm{Sp}(W'')$.

To finish our linear-algebraic description of the semigroups $K \cap \widetilde{\mathrm{Sp}}(-)$ and $\widetilde{\mathrm{Sp}}(-)$ it remains only to observe that by Lemmas 4.5 and 4.6 the exponential map $\mathrm{Exp} : \mathfrak{sp}(H_{\mathbb{R}}, \theta) \to \widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$ maps the cone $\mathfrak{k}(-)$ to the semigroup $K \cap \widetilde{\mathrm{Sp}}(-)$. Since the exponential map for $\mathrm{U}(g)$ is surjective this implies that the map $\mathrm{Exp} : \mathfrak{sp}(-) \to \widetilde{\mathrm{Sp}}(-)$ is surjective and so $\widetilde{\mathrm{Sp}}(-)$ should be simply thought of as the exponentiation of $\mathfrak{sp}(-)$. Therefore the restriction of Exp on the domain

$$\mathfrak{sp}(-2\pi, 0] := \{x \in \mathfrak{sp}(H_{\mathbb{R}}, \theta) |\, \mathrm{spectrum}(\gamma_x) \subset (-2\pi, 0]\} \subset \mathfrak{sp}(-),$$

induces a homeomorphism between $\mathfrak{sp}(-2\pi, 0]$ and $\widetilde{\mathrm{Sp}}$.

Keeping this in mind we can make the following general definition:

Definition 4.8. Let $s \in \widetilde{\mathrm{Sp}}(H_{\mathbb{R}}, \theta)$.

(a) We will say that s has a nonpositive displacement angle if $s \in \widetilde{\mathrm{Sp}}(-)$.

(b) If s has a nonpositive displacement angle define the displacement angle of s to be the number

$$\mathrm{da}(s) := \frac{\mathrm{tr}(\gamma_x)}{2g} \in (-2\pi, 0],$$

where $x \in \mathfrak{sp}(-2\pi, 0]$ is the unique element satisfying $\mathrm{Exp}(x) = s$.

(c) For an element $t \in B_{2g+2}$ of the braid group we will say that t has a nonpositive displacement angle (respectively has displacement angle $\mathrm{da}(t)$ equal to ϕ) if its image $\widetilde{\sigma}_g(t) \in \widetilde{\mathrm{Sp}}(H, \theta)$ is contained in the subsemigroup $\widetilde{\mathrm{Sp}}(-)$ (respectively if $\mathrm{da}(\widetilde{\sigma}_g(t)) = \phi$).

Remark 4.9. The notion of a displacement angle that we have just introduced specializes to the one considered in [1] for the case when $g = 1$. Similarly to the genus one case, the displacement angle of an element $t \in B_{2g+2}$ makes sense 'on the nose' only if the image of t in $\widetilde{\mathrm{Sp}}(H, \theta)$ is contained in $\widetilde{\mathrm{U}}(g)$. For arbitrary elements t we can talk only about the direction or the amplitude of a displacement via t, but the actual value of the displacement depends on the point in the Lagrangian Grassmanian on which t acts. Note also that in contrast with the $g = 1$ case the displacement angle can not be defined directly for hyperelliptic mapping classes $\tau \in \Delta_g^1$ but only for their lifts in B_{2g+2}. In other words, rather than working directly with τ we need to chose an element

$t \in B_{2g+2}$ such that $\kappa_g^1(t) = \tau$ and work with t instead. One does not see the necessity of such a choice in the genus one case where the natural map $\kappa_{1,1} : B_4 = \widetilde{SL}(2, \mathbb{Z}) \to \Delta_{1,1}$ is an isomorphism and so we have canonical lifts for Dehn twists.

Remark 4.10. In order to define the subsemigroup $\widetilde{\mathrm{Sp}}(-)$ and to characterize the elements in $\widetilde{K} \cap \widetilde{\mathrm{Sp}}(-)$ in terms of the character $\widetilde{\det}$ in Lemma 4.5, we had to make some (rather mild) choices. Namely we had to choose a maximal compact subgroup $K \cong \mathrm{U}(g) \subset \mathrm{Sp}(H_1(C_g, \mathbb{R}))$ and a $\mathrm{U}(g)$-equivariant surjection $\mathrm{U}(g) \to \Lambda(H_{\mathbb{R}}, \theta)$. It is instructive to examine the geometric meaning of these choices.

As explained at the beginning of the section, the choice of K is equivalent to choosing a θ-tamed complex structure I on the real vector space $H_{\mathbb{R}} := H_1(C_g, \mathbb{R})$. A natural choice for I will be the Hodge $*$ operator corresponding to a (conformal class of a) Riemannian metric on C. In other words, every choice of a complex structure on C corresponds to a choice of $\mathrm{U}(g)$. However not every I comes from a choice of a complex structure on C_g. Indeed, specifying the complex structure I on $H_{\mathbb{R}}$ is equivalent to specifying a splitting of the complex vector space $H_{\mathbb{C}} := H_{\mathbb{R}} \otimes \mathbb{C}$ as $H_{\mathbb{C}} = H^{\mathbb{C}} \oplus \overline{H^{\mathbb{C}}}$, i.e., to viewing the triple $(H_{\mathbb{Z}}, H_{\mathbb{C}}, \theta)$ as a pure polarized Hodge structure of weight one. Thus the choice of $\mathrm{U}(g)$ is equivalent to endowing the torus $H_{\mathbb{R}}/H_{\mathbb{Z}}$ with the structure of a principally polarized abelian variety and the choice of K will correspond to a complex structure on C_g if and only if the corresponding period matrix satisfies the Schottky relations.

The ambiguity in the choice of the surjection $\mathrm{U}(g) \to \Lambda(H_{\mathbb{R}}, \theta)$ also has a transparent geometric meaning. In order to map $\mathrm{U}(g)$ equivariantly to $\Lambda(H_{\mathbb{R}}, \theta)$ we only need to choose a base point $\lambda_0 \in \Lambda(H_{\mathbb{R}}, \theta)$, i.e., a Lagrangian subspace in $H_{\mathbb{R}}$. This choice can be rigidified somewhat if we choose a Lagrangian subspace in $H_{\mathbb{R}}$ which is defined over $\mathbb{Z}$. A standard way to make such a choice will be to choose a collection of a and b cycles on C_g and then take $\lambda_0 = \mathrm{Span}(a_1, \dots, a_g)$.

We conclude this section with an estimate for the amplitude of the displacement angle of some special elements of B_{2g+2} which act with fixed points on the Lagrangian Grassmanian $\Lambda(H_{\mathbb{R}}, \theta)$:

Lemma 4.11. *Let $t \in B_{2g+2}$ be such that $\kappa_g^1(t) \in \Delta_g^1$ is a product of commuting right Dehn twists. Then $-\pi \leq \mathrm{da}(t) \leq 0$.*

Proof. By hypothesis the element $\sigma_g(t)$ is unipotent and so the element $x \in \mathfrak{sp}(-2\pi, 0]$ for which $\mathrm{Exp}(x) = \widetilde{\sigma}_g(t)$ must be nilpotent. But the nilpotent elements in $\mathfrak{sp}(-)$ are contained in the boundary $\partial\mathfrak{sp}(-)$ which is in turn contained in the subdomain

$$\mathfrak{sp}[-\pi,0] := \{x \in \mathfrak{sp}(-) \mid \text{spectrum}(\gamma_x) \subset [-\pi,0]\}\,.$$

The lemma is proven. q.e.d.

Remark 4.12. Note that the hypothesis on t in the previous lemma implies that $\sigma_g(t)$ has a fixed point on $\Lambda(H_{\mathbb{R}},\theta)$. In fact by using the description of the boundary $\partial\mathfrak{sp}(-)$ in Remark 4.7 (ii) one can check that Lemma 4.11 holds for any nonpositive element t for which $\sigma_g(t)$ has a fixed point on $\Lambda(H_{\mathbb{R}},\theta)$.

5. The proof of Theorem A

Let $f : X \to S^2$ be a semistable hyperelliptic symplectic fibration with a topological section and general fiber C_g.

Consider the surjective homomorphisms $\kappa_g^1 : B_{2g+2} \to \Delta_g^1$ and $\sigma_g : B_{2g+2} \to \mathrm{Sp}(H,\theta)$ introduced in the previous section and let

$$\mathbb{K} := \ker\left[B_{2g+2} \overset{\kappa_g^1}{\to} \Delta_g^1\right].$$

Consider the following two elements in B_{2g+2}:

$$h := t_1 t_2 \dots t_{2g+1} \quad \text{and} \quad \bar{h} := t_{2g+1} \dots t_2 t_1.$$

It is known by [5] that the subgroup $\mathbb{K} \subset B_{2g+2}$ is generated by the elements $h^{2g+1}(h\bar{h})^{-1}$ and h^{2g+2} as a normal subgroup. Furthermore, since $\sigma_g(\mathbb{K}) = \{1\} \in \mathrm{Sp}(H_{\mathbb{Z}},\theta)$ it follows that

$$\widetilde{\sigma}_{g|\mathbb{K}} : \mathbb{K} \to \mathbb{Z} \subset \widetilde{\mathrm{Sp}}(H_{\mathbb{Z}},\theta),$$

where $\mathbb{Z} = \ker[\widetilde{\mathrm{Sp}}(H,\theta) \to \mathrm{Sp}(H,\theta)]$. In particular $\widetilde{\sigma}_{g|\mathbb{K}} : \mathbb{K} \to \mathbb{Z}$ is a character of $\mathbb{K}$. Our next goal is to compare the character $\widetilde{\sigma}_{g|\mathbb{K}}$ with the character $\deg_{|\mathbb{K}}$ and the angle character $\mathrm{da} : \widetilde{\mathrm{Sp}}(-) \to \mathbb{R}$. First we have the following:

Proposition 5.1.

(i) *The homomorphism $\sigma_g : B_{2g+2} \to \mathrm{Sp}(H,\theta)$ maps h and $\bar{h}$ to elements of order $2g+2$ in $\mathrm{Sp}(H,\theta)$. In particular $\sigma_g(h), \sigma_g(\bar{h}) \in \mathrm{U}(g)$ for a suitably chosen maximal compact subgroup $\mathrm{U}(g) \subset \mathrm{Sp}(H_{\mathbb{R}},\theta)$.*

(ii) *The elements* $\widetilde{\sigma}_g(h)$ *and* $\widetilde{\sigma}_g(\bar{h})$ *are conjugate in* $\widetilde{\mathrm{Sp}}(H,\theta)$.

(iii) *The displacement angles of* h *and* $\bar{h}$ *are equal to* $\left(-\dfrac{\pi}{2}\right)$.

Proof. For the proof of parts (i) and (ii) we will need the following standard geometric picture for h and $\bar{h}$. Choose a geometric realization for the double cover $\nu : C_g \to S^2$ in which the branch points of ν are the $2g+2$ roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_{2g+2}$ of order $2g+2$, labeled consecutively (in the counterclockwise direction) along the unit circle. Let $\mathrm{Diff}^+(S^2, \{\zeta_i\}_{i=1}^{2g+2})$ be the group of orientation preserving diffeomorphisms of S^2 which leave the set of points $\{\zeta_i\}_{i=1}^{2g+2}$ invariant and let $\Gamma_{2g+2} := \pi_0(\mathrm{Diff}^+(S^2, \{\zeta_i\}_{i=1}^{2g+2}))$ be the corresponding mapping class group. It is well-known [5, 6] that the hyperelliptic mapping class group Δ_g can be constructed as a central extension

$$0 \to \mathbb{Z}/2 \to \Delta_g \to \Gamma_{2g+2} \to 1,$$

where the central $\mathbb{Z}/2$ is generated by the mapping class of the hyperelliptic involution ι. In terms of this realization of $\nu : C_g \to S^2$ the surjective homomorphism

$$\begin{tikzcd} B_{2g+2} \arrow[r] \arrow[rr, bend right, "\rho_g"'] & \Delta_g \arrow[r] & \Gamma_{2g+2} \end{tikzcd}$$

can be described explicitly [6, p. 164]:

- $\rho_g : B_{2g+2} \to \Gamma_{2g+2}$ sends the positive half-twist $t_i \in B_{2g+2}$ to the mapping class x_i of a Dehn twist on S^2 which is the identity outside of a small neighborhood of the circle segment connecting ζ_i with ζ_{i+1} and which switches ζ_i with ζ_{i+1}.
- The kernel $\ker\left[B_{2g+2} \stackrel{\rho_g}{\to} \Gamma_{2g+2}\right]$ is generated as a normal subgroup by the elements $h\bar{h}$ and h^{2g+2}.

Note that h^{2g+2} is a full twist in B_{2g+2} and so in the above realization of $\nu : C_g \to S^2$ the mapping class $\rho_g(h^{2g+2})$ can be represented by a rotation on S^2 through angle 2π. In particular we see that $\rho_g(h) \in \Gamma_{2g+2}$ is the mapping class of the counterclockwise $\dfrac{\pi}{g+1}$-rotation on S^2. Similarly $\rho_g(\bar{h}) \in \Gamma_{2g+2}$ is the mapping class of the clockwise $\dfrac{\pi}{g+1}$-rotation. This proves part (i) of the proposition.

Next observe that $\rho_g(h)$ and $\rho_g(\bar{h})$ are manifestly conjugate in Γ_{2g+2}. Indeed we have $\rho_g(\bar{h}) = s\rho_g(h)s^{-1}$, where s is the antipodal involution $s(\zeta) = 1/\zeta$ on S^2.

The elements $\rho_g(h)$ and s generate a dihedral subgroup $D_{2g+2} \subset \Gamma_{2g+2}$ of Γ_{2g+2}. The preimage $\widetilde{D}_{2g+2} := \Delta_g \times_{\Gamma_{2g+2}} D_{2g+2}$ of D_{2g+2} in Δ_g is a central extension of D_{2g+2} by $\mathbb{Z}/2$, which is the pull back of the standard Heisenberg extension

$$0 \to \mathbb{Z}/2 \to H_2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0,$$

via the canonical quotient map $D_{2g+2} \twoheadrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Since the Heisenberg extension splits over any cyclic subgroup, it follows that we can find a natural lift of s to an element in $\widetilde{D}_{2g+2} \subset \Delta_g$ which conjugates $\kappa_g(h)$ into $\kappa_g(\bar{h})$. Combined with the fact that $\Delta_g \to \mathrm{Sp}(H,\theta)$ is a group homomorphism this implies that $\sigma_g(\bar{h}) = s'\sigma_g(h)s'^{-1}$ for a suitably chosen $s' \in \mathrm{Sp}(H,\theta)$. Let $\widetilde{s} \in \widetilde{\mathrm{Sp}}(H,\theta)$ be an element which maps to $s' \in \mathrm{Sp}(H,\theta)$. We will check that $\widetilde{s}\widetilde{\sigma}_g(h)\widetilde{s}^{-1} = \widetilde{\sigma}_g(\bar{h})$. First observe that for any symplectic transvection $t \in \mathrm{Sp}(H_{\mathbb{R}},\theta)$ the conjugate element $s'ts'^{-1} \in \mathrm{Sp}(H_{\mathbb{R}},\theta)$ is also a symplectic transvection. Let $\widetilde{t} \in \widetilde{\mathrm{Sp}}(H_{\mathbb{R}},\theta)$ be the standard lift of t described in Lemma 4.1. By definition $\widetilde{t}$ has fixed points on $\widetilde{\Lambda}(H_{\mathbb{R}},\theta)$ and so $\widetilde{s}\widetilde{t}\widetilde{s}^{-1}$ will also have fixed points. Hence by Lemma 4.1 we conclude that $\widetilde{s}\widetilde{t}\widetilde{s}^{-1}$ is the standard lift of the transvection $s'ts'^{-1}$. Consider now the element $\sigma_g(h) = \prod_{i=1}^{2g+1}\sigma_g(t_i)$. By Corollary 4.4 we know that $\widetilde{\sigma}_g(t_i) = \widetilde{\sigma_g(t_i)}$ is the standard lift of the transvection $\sigma_g(t_i)$ and that $\widetilde{\sigma}_g(h) = \prod_{i=1}^{2g+1}\widetilde{\sigma_g}(t_i)$. In particular we have

$$\begin{aligned}
\widetilde{s}\widetilde{\sigma}_g(h)\widetilde{s}^{-1} &= \prod_{i=1}^{2g+1}(\widetilde{s}\widetilde{\sigma_g}(t_i)\widetilde{s}^{-1}) = \prod_{i=1}^{2g+1}\widetilde{s't_is'^{-1}} = \prod_{i=1}^{2g+1}\widetilde{\sigma}_g(s't_is'^{-1}) \\
&= \widetilde{\sigma}_g\left(\prod_{i=1}^{2g+1}(s't_is'^{-1})\right) = \widetilde{\sigma}_g(s'hs'^{-1}) = \widetilde{\sigma}_g(\bar{h}).
\end{aligned}$$

This completes the proof of part (ii) of the proposition.

We are now ready to prove part (iii). The elements $h, \bar{h} \in \Delta_g^1$ are products of right-handed Dehn twists and so the elements $\widetilde{\sigma}_g(h), \widetilde{\sigma}_g(\bar{h})$ both belong to $\widetilde{\mathrm{Sp}}(-)$. Since by part (i) of the proposition we know that $\widetilde{\sigma}_g(h), \widetilde{\sigma}_g(\bar{h})$ also belong to $\widetilde{\mathrm{U}}(g)$ it follows that h and $\bar{h}$ have well-defined negative displacement angles. Furthermore, note that $h, \bar{h}$ must have the same displacement angle since $\widetilde{\sigma}_g(h)$ and $\widetilde{\sigma}_g(\bar{h})$ belong to the same conjugacy class in $\widetilde{\mathrm{U}}(g)$. In view of this and the fact that $\widetilde{\det}$ is additive on $\widetilde{\mathrm{U}}(g)$, it suffices to show that the displacement angle of $h\bar{h}$ is equal to $-\pi$. Since the element $h\bar{h}$ maps to the mapping class in Δ_g^1 represented by the hyperelliptic involution $\iota : C_g \to C_g$, it follows that $\sigma_g(h\bar{h}) = -1 \in \mathrm{Sp}(H, \theta) \subset SL(H)$. In particular $\widetilde{\sigma}_g(h\bar{h}) \in \widetilde{\mathrm{Sp}}(H, \theta)$ will be a lift of -1. But we already have one distinguished lift of -1, namely the negative generator $\mathfrak{c}$ of the center of $\widetilde{\mathrm{Sp}}(H, \theta)$. Indeed the center $Z(\widetilde{\mathrm{Sp}}(H, \theta))$ of $\widetilde{\mathrm{Sp}}(H, \theta)$ is an infinite cyclic group which maps onto the center $Z(\mathrm{Sp}(H, \theta)) \cong \mathbb{Z}/2$ of $\mathrm{Sp}(H, \theta)$. Since the latter is generated by -1 we conclude that $\widetilde{\sigma}_g(h\bar{h})$ must be some odd power of $\mathfrak{c}$. On the other hand the element -1 considered as an element in $\mathrm{U}(g)$ has eigenvalues $e^{\pi i}$ and so $\widetilde{\det}(\mathfrak{c}) = -g\pi$, i.e., $\mathrm{da}(\mathfrak{c}) = -\pi$. Finally, if we write $h_g = t_1 t_2 \dots t_{2g+1}$ and $\bar{h}_g = t_{2g+1} \dots t_2 t_1$ we have $h\bar{h} = h_g \bar{h}_g = t_1 t_2 h_{g-1} \bar{h}_{g-1} t_2 t_1$ and so $\mathrm{da}(h\bar{h}) \geq \mathrm{da}(t_1 t_2) + \mathrm{da}(h_{g-1}\bar{h}_{g-1}) + \mathrm{da}(t_2 t_1)$. By induction on g and the obvious compatibility of displacement angles for linear embedding of symplectic groups we have that $\mathrm{da}(h_{g-1}\bar{h}_{g-1}) \geq -\pi$. Moreover by writing the elements t_1 and t_2 as homological Dehn twists in the standard basis of a and b cycles on C_g we see that $t_1 t_2$ and $t_2 t_1$ are represented by matrices with eigenvalues 1 and $e^{-i\pi/3}$. Thus $\mathrm{da}(t_1 t_2) > -\pi$ and $\mathrm{da}(t_2 t_1) > -\pi$ and so $\mathrm{da}(h\bar{h}) > -3\pi$. Since as we saw $\mathrm{da}(h\bar{h})$ must be an odd multiple of $\mathrm{da}(\mathfrak{c})$ we must have $\mathrm{da}(h\bar{h}) = \mathrm{da}(\mathfrak{c}) = -\pi$. The proposition is proven. q.e.d.

We can now finish the proof of Theorem A. Let $\mathfrak{t}_1 \dots \mathfrak{t}_N = 1$ be the relation in Δ_g^1 corresponding to the monodromy representation of the pencil $f : X \to S^2$. Choose elements $\nu_1, \dots, \nu_N \in B_{2g+2}$ satisfying $\kappa_g^1(\mathfrak{t}_i) = \nu_i$. Then the element $\mu := \nu_1 \dots \nu_N \in B_{2g+2}$ belongs to the subgroup $\mathbb{K} \subset B_{2g+2}$. But the subgroup $\mathbb{K}$ is normally generated by the elements $h^{2g+1}(h\bar{h})^{-1}$ and h^{2g+2} and hence is contained in the subgroup $\mathbb{L}$ of B_{2g+2} which is normally generated by the elements h and $\bar{h}$. Let now $\mathbb{L}(-) = \mathbb{L} \cap \widetilde{\sigma}_g^{-1}(\widetilde{\mathrm{Sp}}(-))$ be the nonpositive subsemigroup of $\mathbb{L}$. Then we have two natural $\mathbb{R}_{\leq 0}$-valued characters on $\mathbb{L}(-)$:

$$-\deg : \mathbb{L}(-) \to \mathbb{Z}_{\leq 0} \subset \mathbb{R}_{\leq 0} \quad \text{and} \quad \chi := \mathrm{da} \circ \widetilde{\sigma}_g : \mathbb{L}(-) \to \mathbb{R}_{\leq 0}.$$

By the previous proposition we have $\chi(h) = \chi(\bar{h}) = -\pi/2$. On the other hand $\deg(h) = \deg(\bar{h}) = 2g + 1$ and since $\mathbb{L}$ is normally generated by h and $\bar{h}$ we have

$$\chi = -\frac{\pi}{4g+2} \cdot \deg$$

on $\mathbb{L}(-)$.

Since $\mu \in \mathbb{L}(-)$ this implies

$$\mathrm{da}(s_1 s_2 \dots s_N) = -\frac{\pi N}{4g+2},$$

where $s_i \in \widetilde{\mathrm{Sp}}(-)$ is the canonical lift (see Lemma 4.1) of the simple homological Dehn twist $\sigma_g(\mathfrak{t}_i)$. Finally, the product $\prod_{i=1}^{N} s_i$ can be rewritten as $\prod_{j=1}^{D} p_j$ where p_j is the product of those Dehn twists among the s_i's which correspond to the all the cycles vanishing at the j-th singular fiber of $f : X \to S^2$. Since each p_j is a product of commuting Dehn twists Lemma 4.11 applies and so $\mathrm{da}(p_j) \geq -\pi$. Thus we get

$$\mathrm{da}\left(\prod_{j=1}^{D} p_j\right) = \sum_{j=1}^{D} \mathrm{da}(p_j) \geq -\pi,$$

and so

$$-D\pi \leq -\frac{\pi N}{4g+2}.$$

Theorem A is proven. q.e.d.

6. Concluding remarks

The method of proof of Theorem A can be generalized in several directions and we intend to pursue such generalizations in a forthcoming paper. We conclude the present discussion by indicating some of the possible venues of generalization:

(i) Obtain analogues of the Szpiro inequality for a general semistable family of curves of genus g.

The inequality which our method produces in this case is similar to the one in Theorem A but instead of N — the length of the word of the Dehn twists defining the fibration we have additional characteristics — the number of lantern relations in the monodromy word. In fact, the only ingredient missing here is the exact coefficient for the number of lantern relations in the word.

In order to explain this in more detail recall first that the main obstruction to generalizing the genus one proof from [1] to higher genus is the existence of two different natural central extensions of $\mathrm{Map}_{g,1}$ when $g \geq 2$. This leads to two different and rather mild geometric constraints on the monodromy representation of a Lefschetz family, which when taken separately are not stringent enough to produce an inequality of Szpiro type.

The idea therefore is to look for a different group which surjects onto $\mathrm{Map}_{g,1}$ and for which the pullback of the two central extensions of $\mathrm{Map}_{g,1}$ become easier to compare, i.e., coincide or at least become proportional. The fact that in the hyperelliptic case one can successfully implement this idea by using the braid group suggests that for a general family of genus g curves one may try to work with the Artin braid group Art_Γ corresponding to the T-shaped graph Γ of Wajnryb's presentation of Map_g^1 [23]. Since Γ is a tree we can again apply Lemma 3.3 and conclude that pullback of the central extension (4.1) will be zero in $H^2(\mathrm{Art}_\Gamma, \mathbb{Z})$.

Next observe that in Wajnryb's presentation of Map_g^1 an extra relation — the *lantern relation* — appears when $g \geq 3$. The kernel $\mathbb{K}_\Gamma := \ker[\mathrm{Art}_\Gamma \to \mathrm{Map}_g^1]$ is now normally generated by the hyperelliptic relations $h^{2g+1}(h\bar{h})^{-1}$ and h^{2g+2} in a subgroup $B_{2g+2} \subset \mathrm{Art}_\Gamma$ and by the lantern relation. In this setup one can argue that the analogue $\mathbb{K}_\Gamma \to \mathbb{Z}$ of the character $\widetilde{\sigma}_{g|\mathbb{K}}$ can be expressed as $a \deg + b\ell$ where ℓ is a character which is nontrivial on the lantern relations in $\mathbb{K}_\Gamma$.

This reasoning leads to a formula for the left-hand side of the Szpiro inequality which instead of one parameter N involves two parameters N and L where the latter corresponds to the number of lantern relations in the monodromy word in Map_g^1. We do not know the exact coefficient for L yet but it is clear that this coefficient can not be trivial. Indeed otherwise the character $\mathbb{K}_\Gamma \to \mathbb{Z}$ will be (rationally) proportional deg which contradicts the fact that the central extension (4.1) pull back to a nontorsion element in $H^2(\mathrm{Map}_g^1, \mathbb{Z})$.

(ii) Work out the hyperelliptic Szpiro inequality in the case of a general base curve.

In this setup one needs to change the right-hand side of the inequality by an appropriate multiple of the genus of the base curve (in the case of elliptic fibrations the correct modification was worked out by S.-W. Zhang [25]).

To carry out the argument in this case one needs to analyze more carefully the monodromy word corresponding to a Lefschetz family of surfaces over a closed base surface C of arbitrary genus. The product of local monodromy transformations $\mathfrak{t}_i$ is not 1 but a product of at most $g(C)$ commutators $aba^{-1}b^{-1}$. Now the set of commutators in $\widetilde{\mathrm{Sp}}(2g,\mathbb{R})$ maps isomorphically to the corresponding set in $\mathrm{Sp}(2g,\mathbb{R})$ and in fact consists of the elements $h \in \widetilde{\mathrm{Sp}}(2g,\mathbb{R})$ with bounded displacement angle. Thus there is a number A so that for any λ in the Lagrangian Grassmanian, we have that the angle between $h \cdot \lambda$ and λ is not less than $-A\pi$. This results in a correction term $Ag(C)$ in the right-hand side of the Szpiro formula.

(iii) Find a monodromy proof of the negativity of self intersections for sections in a symplectic Lefschetz fibration.

Until recently the existing proofs of this fact used either the theory of pseudo-holomorphic curves or properties of the Seiberg-Witten Floer homology. Both approaches are quite technical and require the existence of global solutions of special partial differential equations. In contrast with these 'high-brow' methods Ivan Smith has recently found an alternative proof [17, Proposition 3.3], which uses only the geometry of the hyperbolic disk and no heavy machinery.

In the same spirit, the proof suggested by our method is completely algebraic and relies only on the combinatorial properties of the monodromy group. The idea is to look for a 'displacement angle' description of the subsemigroup in Map_g^1 consisting of elements which are negative with respect to the Morita move on the boundary circle for the uniformizing disk of the fiber (see e.g., [1]). Here is a brief sketch of such a proof.

Let $\rho : \mathrm{Map}_g^1 \to \mathrm{Homeo}^+(S^1)$ be the Morita homomorphism and let $e \in H^2(\mathrm{Map}_g^1, \mathbb{Z})$ be the pullback of the natural central extension

$$0 \to \mathbb{Z} \to \mathrm{Homeo}^{\mathrm{per}}(\mathbb{R}) \to \mathrm{Homeo}^+(S^1) \to 1$$

via ρ. Recall that the middle term of the extension e can be naturally identified with the mapping class group $\mathrm{Map}_{g,1}$ [13].

Using Lemma 3.3 one can again argue that the pullback of the extension e to Art_Γ is trivial and so there is a natural homomorphism $\mathrm{Art}_\Gamma \to \mathrm{Homeo}^{\mathrm{per}}(\mathbb{R})$. Now by shearing the hyperbolic disk one can show (see Proposition 2.1 and remark 2.2 in [17]) that the generators in Art_Γ covering right-handed Dehn twists in Map_g^1 get mapped to elements in $\mathrm{Homeo}^{\mathrm{per}}(\mathbb{R})$ which fix a countable set of points in $\mathbb{R}$ and move all other points in $\mathbb{R}$ to the right. In particular any positive word x in the generators of Art_Γ which is in $\ker[\mathrm{Art}_\Gamma \to \mathrm{Map}_g^1]$ defines a nontrvial right-shifting periodic homeomorphism of $\mathbb{R}$.

Given a symplectic Lefschetz family over S^2 with a fixed symplectic section s, we get a lifting of the monodromy representation into the group Map_g^1 and the selfintersetion of s corresponds to the image of (the lifting in Art_Γ of) the monodromy word in $\mathrm{Homeo}^{\mathrm{per}}(\mathbb{R})$. Thus the fact that the resulting element is right-shifting corresponds precisely to the negativity of the selfintersection of s.

We can make this more precise (in the spirit of Lemma 4.11) if we manage to bound the amount of the shifting in terms of the numbers N and L mentioned in (i) above. This will give an explicit bound on the selfintersection of s in terms of N and L. Some preliminary computations we have made show that the N-L bounds one gets imply also effective bounds on s^2 which depend on N only. Estimates of this type are of independent interest since they provide a simple way to show finiteness of types of symplectic Lefshetz pencils ($s^2 = -1$) for a given genus $g > 1$. (The finiteness for symplectic four manifolds admitting genus two pencils is proven in [18].)

(iv) Find arithmetic analogues of the symplectic Szpiro inequality.

It is very tempting to try to apply our method to the arithmetic situation. In this case the analogue of the global displacement angle is clearly the height of a curve with a point. The monodromy representation corresponds to the Galois representation. There are several problems with this approach. One of them is that there are no direct analogs of the $\mathbb{Z}$-central extension in this case and the other one is that we do not yet understand what should the local inequalities be near singular fibers.

(v) Find generalizations of the Szpiro inequality for symplectic fibrations of higher fiber dimension.

It was pointed out to us by A. Tyurin that our method for proving inequalities of Szpiro type may be applicable to tackling higher dimensional symplectic pencils and especially families of higher dimensional Calabi-Yau manifolds. In particular one expects that for an algebraic Lefschetz pencil of Calabi-Yau threefolds whose singular fibers have only ordinary double points, there will be a Szpiro inequality relating the number of singular fibers in the pencil to the number of vanishing cycles. In fact, the monodromy action on the third cohomology of the general fiber is by symplectic transformations. If this monodromy homomorphism is liftable to the universal central extension of the symplectic group, then one can easily employ the displacement angle invariant to obtain the Szpiro inequality. It is not hard to construct such a lift in specific examples by examining the mixed Hodge structure on the relative third cohomology of the general fiber modulo the base surface of the pencil. We will treat this question in detail in a future paper.

References

[1] J. Amorós, F. Bogomolov, L. Katzarkov, & T. Pantev, *Symplectic Lefschetz fibrations with arbitrary fundamental groups*, J. Differential Geom. **54**(**3**) (2000) 489–545, MR 2002g:57051.

[2] V.I. Arnold, *On a characteristic class entering into conditions of quantization*, Funkcional. Anal. i Priložen. **1** (1967) 1–14, MR 35 #2296.

[3] V.I. Arnold, *Braids of algebraic functions and cohomologies of swallowtails*, Uspehi Mat. Nauk, 23 **142**(**4**) (1968) 247–248, MR 38 #156.

[4] A. Beauville, *The Szpiro inequality for higher genus fibrations*, math.AG/0109080.

[5] J. Birman & H. Hilden, *On the mapping class groups of closed surfaces as covering spaces*, in 'Advances in the theory of Riemann surfaces' (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies, **66**, Princeton Univ. Press, Princeton, N.J., 1971, 81–115, MR 45 #1169, Zbl 0217.48602.

[6] J. Birman, *Braids, links, and mapping class groups*, Annals of Mathematics Studies, **82**, Princeton University Press, Princeton, N.J., 1974, MR 51 #11477, Zbl 0305.57013.

[7] F.R. Cohen, *Artin's braid groups, classical homotopy theory, and sundry other curiosities*, in 'Braids' (Santa Cruz, CA, 1986), Amer. Math. Soc., Providence, RI, 1988, 167–206, MR 90a:20077, Zbl 0682.55011.

[8] D.B. Fuks, *Cohomology of the braid group* mod 2, Funkcional. Anal. i Priložen. **4**(**2**) (1970) 62–73, MR 43 #226.

[9] R. Gompf & A. Stipsicz, 4-*Manifolds and Kirby Calculus*, American Mathematical Society, Providence, RI, 1999, MR 2000h:57038, Zbl 0933.57020.

[10] J. Harris & I. Morrison, *Moduli of Curves*, Graduate Texts in Mathematics, **187**, Springer-Verlag, New York, 1998, MR 99g:14031, Zbl 0913.14005.

[11] A. Kas, *On the handlebody decomposition associated to a Lefschetz fibration*, Pacific J. Math. **89**(**1**) (1980) 89–104, MR 82f:57012, Zbl 0457.14011.

[12] P. Lockhart, *On the discriminant of a hyperelliptic curve*, Trans. Amer. Math. Soc. **342**(**2**) (1994) 729–752, MR 94f:11054, Zbl 0815.11031.

[13] S. Morita, *Characteristic classes of surface bundles and bounded cohomology*, in 'A fête of topology', 233–257, Academic Press, Boston, MA, 1988, MR 89c:57030, Zbl 0892.57019.

[14] B. Ozbagci & A. Stipsicz, *Noncomplex smooth* 4*-manifolds with genus-*2 *Lefschetz fibrations*, Proc. Amer. Math. Soc. **128**(**10**) (2000) 3125–3128, MR 2000m:57036, Zbl 0951.57015.

[15] B. Siebert & G. Tian, *On hyperelliptic C^∞-Lefschetz fibrations of four-manifolds*, Commun. Contemp. Math. **1**(**2**) (1999) 255–280, MR 2001g:57053, Zbl 0948.57018.

[16] I. Smith, *Symplectic geometry of Lefschetz fibrations*, Ph.D. thesis, New College, University of Oxford, 1998.

[17] I. Smith, *Geometric monodromy and the hyperbolic disc*, Quart. J. Math. **52** (2001) 217–228, math.SG/0011223, MR 2002c:57046, Zbl 0981.57013.

[18] I. Smith, *Lefschetz pencils and divisors in moduli space*, Geometry and Topology **5** (2001) 579–608, MR 2002f:57056.

[19] L. Szpiro, *Discriminant et conducteur des courbes elliptiques*, Séminaire sur les Pinceaux de Courbes Elliptiques (Paris, 1988), Astérisque **183** (1990) 7–18, MR 91g:11059, Zbl 0742.14026.

[20] S.T. Tan, *The minimal number of singular fibers of a semistable curve over $\mathbb{P}^1$* J. Algebraic Geometry **4** (1995) 591–596, MR 96e:14038, Zbl 0864.14003.

[21] H. van der Lek, *Extended Artin groups*, in 'Singularities', Part 2 (Arcata, Calif., 1981), Proc. Symp. Pure Math. 40, Part 2, Amer. Math. Soc., Providence, R.I., 1983, 117–121, MR 85b:14005, Zbl 0523.14005.

[22] A. Varchenko, *The branching of multiple integrals which depend on parameters*, Funkcional. Anal. i Priložen **3**(**4**) (1969) 79–80.

[23] B. Wajnryb, *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. **45**(**2-3**) (1983) 157–174, MR 85g:57007, Zbl 0533.57002.

[24] G. Xiao, *Fibered algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449–466, MR 88a:14046, Zbl 0596.14028.

[25] S.-W. Zhang, *Geometry of algebraic points*, in 'First International Congress of Chinese Mathematician' (Lo Yang and S.-T. Yau, editors), AMS/IP Studies in Advanced Mathematics, **20**, American Mathematical Society, International Press, 2001, 185–198, MR 1 830 174.

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