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POISSON EQUATION, POINCARÉ-LELONG EQUATION AND CURVATURE DECAY ON COMPLETE KÄHLER MANIFOLDS

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Abstract

In the first part of this work, the Poisson equation on complete noncompact manifolds with nonnegative Ricci curvature is studied. Sufficient and necessary conditions for the existence of solutions with certain growth rates are obtained. Sharp estimates on the solutions are also derived. In the second part, these results are applied to the study of curvature decay on complete Kähler manifolds. In particular, the Poincaré-Lelong equation on complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature is studied. Several applications are then derived, which include the Steinness of the complete Kähler manifolds with nonnegative curvature and the flatness of a class of complete Kähler manifolds satisfying a curvature pinching condition. Liouville type results for plurisubharmonic functions are also obtained.

0. Introduction

In this paper, we will discuss the Poisson equation on complete noncompact manifolds and derive some applications on Kähler manifolds.

Let M^m be a complete noncompact Kähler manifold, where $m \geq 2$ is the complex dimension. Assume M has nonnegative holomorphic bisectional curvature and has maximal volume growth such that the scalar curvature decays like r^{-2} where r is the distance from a fixed point. Then it was proved in [20] by Mok-Siu-Yau that one can solve the following Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar{\partial}u = \rho \tag{0.1}$$

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by first solving the Poisson equation $1/2\Delta u = \text{trace}(\rho)$, where ρ is the Ricci form of M. They then applied the results to study the analytic and geometric properties of M. On the other hand, in [27] Yau discussed certain differential inequalities. Again, applications on Riemannian and Kähler manifolds were given. For example, some vanishing results for L^p holomorphic sections of holomorphic vector bundles over Kähler manifolds were obtained; see also [10], [18], [22]. In some cases, if one can solve the Poisson equation then it is rather easy to apply the methods in [27]. These motivate our study on the Poisson equation on complete noncompact manifolds.

We are mainly concerned with manifolds with nonnegative Ricci curvature. Let M^n be such a manifold and consider the Poisson equation:

$$\Delta u = f. \tag{0.2}$$

The first question is to find sufficient conditions for the existence of solutions of (0.2). If a solution u exists, it is also important for applications to estimate u together with its gradient and Hessian.

Our main result is that if f decays faster than r^{-1} in a certain sense, then (0.2) has a solution. More precisely, assume $f \geq 0$ and let $k(x,t) = k_f(x,t) = 1/V_x(t)\int_{B_x(t)} f$ be the average of f over the geodesic ball $B_x(t)$ with center at x and radius t, where $V_x(t)$ is the volume of $B_x(t)$. Let $o \in M$ be a fixed point and let $k(t) = k(o,t)$. We prove that if $\int_0^\infty k(t)dt < \infty$ and if there exist a constant $1 > \delta > 0$ and a nonnegative function $h(t) \geq 0$, $0 \leq t < \infty$ with $h(t) = o(t)$ as $t \to \infty$ such that

$$\int_0^t sk(x,s)ds \leq h(t)$$

for all x and for all $t \geq \delta r(x)$, then (0.2) has a solution u. Moreover, lower and upper estimates of u are obtained. In case that M is nonparabolic (that is, M supports a positive Green's function) such that the volume of geodesic balls satisfy certain assumptions then the condition $\int_0^\infty k(t)dt < \infty$ alone will be sufficient. This is the case if M has maximal volume growth with $n \geq 3$. In any case, pointwise and integral estimates for the gradient of the solution u and an integral estimate of the Hessian of u are obtained.

The above conditions on the average of f over geodesic balls for the existence of solution of (0.2) are reasonable, because in some cases they are also necessary. For example, we prove the following results. Let $f \geq 0$ be a function on a complete noncompact manifold with nonnegative Ricci curvature. Then:

(i) $\Delta u = f$ has a bounded solution if and only if there is a constant $C > 0$ such that

$$\int_0^\infty tk(x,t)dt \le C$$

for all x.

(ii) $\Delta u = f$ has a solution with $\sup_{B_o(r)} |u| \le C \log(2 + r)$ for some constants C for all r if any only if

$$\int_0^t sk(x,s)ds \le C' \log(2 + t)$$

for some constant C' for all $r = r(x)$ and for all $t \ge \frac{1}{5}r$.

(iii) $\Delta u = f$ has a solution with $\sup_{B_o(r)} |u| \le C(1 + r)^{1-\delta}$ for some constant C and $1 > \delta > 0$ for all r if any only if

$$\int_0^t sk(x,s)ds \le C'(1 + t)^{1-\delta}$$

for some constant C' for all $r = r(x)$ and for all $t \ge \frac{1}{5}r$.

In [12], Li proved that if u is a bounded subharmonic function on a complete noncompact manifold M with nonnegative Ricci curvature, then the average of u over a geodesic ball of radius r with a fixed center converges to $\sup_M u$ as $r \to \infty$. Using the result (i) in the above, we give another proof of Li's result. Furthermore, one can estimate the difference between $\sup_M u$ and the average of u over a geodesic ball of radius r in terms of Δu, r and the dimension of M. Using the pointwise estimate for the gradient of the solution of (0.2), we prove that if in addition that $f = \Delta u$ decays like r^{-2} then u will actually be asymptotically constant.

The rest of this work is to apply these results to Kähler manifolds. One of the applications is to study plurisubharmonic functions. It is easy to see that if $n \ge 3$, then there are nonconstant bounded subharmonic functions on $\mathbb{R}^n$ which are asymptotically constant. On the other hand, it was proved by Ni in [22] that if u is a plurisubharmonic function on a complete noncompact Kähler manifold M^m with nonnegative Ricci curvature and if u satisfies

$$\limsup_{x\to\infty} \frac{u(x)}{\log r(x)} = 0 \tag{0.3}$$

then $(\partial\bar{\partial}u)^m \equiv 0$. We prove that u is actually constant under some assumptions on M and Δu by using a minimum principle in [2], [1] or a method in [20].

The above mentioned result of Ni can be generalized to nonparabolic manifolds with scalar curvature $\mathcal{R}$ satisfying $\int_M \mathcal{R}_- < \infty$, where $\mathcal{R}_-$ is the negative part of $\mathcal{R}$. This generalization is a consequence of one of the vanishing results we obtain in this work. Consider a complete noncompact Kähler manifold M^m with nonnegative Ricci curvature and a Hermitian holomorphic line bundle L over M. We prove that given any $\tau > 0$ and $0 < \epsilon < 1$ there is a constant a depending only on τ, $\epsilon > 0$ and m such that if the average of the trace of positive part of the curvature of L over $B(r)$ is less than ar^{-2}, then any holomorphic $(p, 0)$ form ϕ with value in L is trivial if

$$\frac{1}{V_o(r)}\int_{B(r)} |\phi|^\tau = O(r^{-\epsilon})$$

as $r \to \infty$, where $o \in M$ is a fixed point and $V_o(r)$ is the volume of the ball of radius r centered at o. The proof is a combination of our results on the Poisson equation and the mean value inequality in [13]. If M is nonparabolic then a similar result is true. In this case, we assume that the negative part of the scalar curvature of M and the positive part of the trace of the curvature of L are both integrable. The vanishing theorems are similar to some results in [27], [10], [22].

Using the vanishing results and the L^2 estimate in [11], [7] one can prove the following: Let M be a complete noncompact Käher manifold with nonnegative Ricci curvature. Suppose the scalar curvature $\mathcal{R}$ satisfies

$$\limsup_{r\to\infty} \frac{r^2}{V_o(r)}\int_{B_o(r)} \mathcal{R} = 0, \tag{0.4}$$

where $o \in M$ is a fixed point, then the Ricci form ρ of M must satisfy $\rho^m \equiv 0$. Observe that if M is nonparabolic and $\mathcal{R}$ is integrable, then (0.4) is true. Hence this generalizes a result in [22, Theorem 3.6].

From the arguments in [23], under the additional assumption that M has nonnegative holomorphic bisectional curvature which is also bounded, one can conclude that if (0.4) is true for all base point o so that the convergence is uniform then M is flat as observed in [4]. In our case, we only assume that the Ricci curvature is nonnegative and we do not assume the scalar curvature being bounded. The result is weaker and it is interesting to see whether M is actually Ricci flat in this case. In fact, for Riemannian case, it is proved by Chen and Zhu [3] that if (0.4) is true uniformly and if the Riemannian manifold is locally conformally flat then the manifold is flat.

Finally we solve the Poincaré-Lelong equation (0.1). Let M^m be a complete Kähler manifold with nonnegative bisectional curvature and let ρ be a real closed $(1,1)$ form with trace f. We prove that if $f \geq 0$ and ρ satisfies the following conditions:

$$\int_0^\infty \frac{1}{V_o(t)} \int_{B_o(t)} ||\rho|| dt < \infty, \tag{0.5}$$

and that

$$\liminf_{r\to\infty} \frac{1}{V_o(r)} \int_{B_o(r)} ||\rho||^2 = 0, \tag{0.6}$$

then (0.1) has a solution u. It is easy to see that if M^m has maximal volume growth with $m \geq 2$ and $||\rho||$ decays like r^{-2}, then the above conditions will be satisfied. In fact in this case we have

$$\frac{1}{V_o(r)} \int_{B_o(r)} ||\rho|| \leq Cr^{-2} \tag{0.7}$$

for some C for all r. Hence our result is a generalization of a related result in [20], see also [19]. Note that we do not assume $||\rho||$ to be bounded.

Using solutions of (0.1), we can discuss properties of Kähler manifolds with nonnegative holomorphic bisectional curvature. For example, we prove that if in addition M has positive Ricci curvature which satisfies (0.5) and (0.6), then M is Stein, provided the sectional curvature is nonnegative. This is related to the works of [8] and [19], [20]. In [8], it was proved that M is Stein under the assumption that M has positive biholomorphic bisectional curvature and nonnegative sectional curvature. In [19], it was proved that M is Stein under the assumptions that M has positive Ricci curvature and nonnegative holomorphic bisectional curvature, has maximal volume growth and the scalar curvature decays likes r^{-2}. Recently, it is proved by Chen and Zhu [4] that a complete noncompact Kähler manifold M^m with nonnegative holomorphic bisectional curvature and with maximal volume growth is Stein if the scalar curvature decays like $r^{-1-\epsilon}$ for some $\epsilon > 0$.

In [24], it was proved that if M^m is a complete noncompact Kähler manifold of complex dimension $m \geq 3$ with nonnegative holomorphic bisectional curvature and with a pinching condition, then (0.7) is satisfied with a constant independent of the point o. Using solution of (0.1), we prove that if in addition the scalar curvature has a pointwise decay like r^{-2} or the volume of geodesic ball of radius r is no greater than r^m, then M is actually flat. Here m is the complex dimension of M.

Using solutions of (0.1), we can also study relations between the decay of the scalar curvature and volume growth of a complete Kähler manifold with nonnegative biholomorphic bisectional curvature. For example, we prove that if the Ricci form ρ is positive at some point, ρ satisfies (0.7) and $||\rho||$ decays like r^{-2}, then M must have maximal volume growth. In this case, the scalar curvature cannot decay too fast in the sense that we have a reverse inequality of (0.7):

$$\frac{1}{V_o(r)}\int_{B_o(r)} ||\rho|| \geq Cr^{-2}$$

for some positive constant C for all r. If we only assume that M is not Ricci flat in the above, then one can prove that $V_o(r) \geq Cr^2$ for some positive constant C.

The arrangement of the paper is as follows. In Section 1 and Section 2 we study the Poisson equation. Section 3 contains some vanishing theorems. Section 4 is a discussion of Liouville property of plurisubharmonic functions. Section 5 gives a solution to the Poincaré-Lelong equation with applications on manifolds with nonnegative holomorphic bisectional curvature.

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1. The Poisson equation (I)

Let M^n be a complete noncompact manifold. Given any function $f \geq 0$ on M, define

$$k_f(x,t) = \frac{1}{V_x(t)}\int_{B_x(t)} f.$$

In the following $C(a, b, \dots)$ will denote a constant depending only on a, $b, \dots$. We also denote $r(x, y)$ to be the distance between x and y, and $r(x) = r(x, o)$ where $o \in M$ is a fixed point. In this section, we will discuss the conditions on f so that $\Delta u = f$ has a solution u and we will also discuss the properties of u.

Theorem 1.1. *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature. Assume M is nonparabolic and there is a constant $\sigma > 0$ such that the minimal positive Green's function $G(x,y)$ satisfies*

$$\sigma^{-1}\frac{r^2(x,y)}{V_x(r(x,y))} \le G(x,y) \le \sigma\frac{r^2(x,y)}{V_x(r(x,y))} \tag{1.1}$$

for all $x \neq y$ in M. Let $f \geq 0$ be a locally Hölder continuous function and let $k(x,t) = k_f(x,t)$ and $k(t) = k(o,t)$, where $o \in M$ is a fixed point. Suppose that $\int_0^\infty k(t)dt < \infty$. Then the Poisson equation $\Delta u = f$ has a solution u such that for all $1 > \epsilon > 0$

$$\begin{aligned}\alpha_1 r\int_{2r}^\infty k(t)dt + \beta_1\int_0^{2r} tk(t)dt \geq u(x)&\\ \geq\ -\alpha_2 r\int_{2r}^\infty k(t)dt&\\ -\beta_2\int_0^{\epsilon r} tk(x,t)dt + \beta_3\int_0^{2r} tk(t)dt&\end{aligned}$$

for some positive constants $\alpha_1(n,\sigma)$, $\alpha_2(n,\sigma,\epsilon)$ and $\beta_i(n)$, $1 \leq i \leq 3$, where $r = r(x)$. Moreover $u(o) = 0$.

Proof. By the estimate of Green's function in [17, Theorem 5.2], (1.1) implies

$$C^{-1}\frac{r^2(x,y)}{V_x(r(x,y))} \le \int_{r(x,y)}^\infty \frac{t}{V_x(t)}\,dt \le C\frac{r^2(x,y)}{V_x(r(x,y))} \tag{1.2}$$

for some constant $C = C(n,\sigma) > 0$. For all $R > 0$, let G_R be the positive Green's function on $B_o(R)$ with zero boundary value and let

$$u_R(x) = \int_{B_o(R)} (G_R(o,y) - G_R(x,y))\,f(y)dy.$$

Then $\Delta u_R = f$ in $B_o(R)$ and $u_R(o) = 0$. For any x with $r(x) = r$, suppose $R \gg r$, then

$$\begin{aligned}u_R(x) &= \left\{\int_{B_o(R)\setminus B_o(2r)} + \int_{B_o(2r)}\right\}(G_R(o,y) - G_R(x,y))\,f(y)dy\\ &= I + II.\end{aligned} \tag{1.3}$$

To estimate I, let y be any point in $B_o(R) \setminus B_o(2r)$, then $r_1 = r(y) \geq 2r = 2r(x)$ and so $r(z,y) \geq \frac{1}{2}r_1$ if $z \in B_o(r)$. Also $B_z(\frac{1}{4}r) \subset B_o(2r)$. Hence by the gradient estimate [5, Theorem 6],

$$\begin{aligned}
|G_R(o,y) - G_R(x,y)| &\leq r \sup_{z \in B_o(r)} |\nabla_z G_R(z,y)| \\
&\leq C_1 \frac{r}{r_1} \sup_{z \in B_o(r)} G_R(z,y) \\
&\leq C_2 \frac{r}{r_1} G(o,y) \\
&\leq C_3 \frac{r}{r_1} \int_{r_1}^{\infty} \frac{t}{V_o(t)}\, dt,
\end{aligned}$$

where $C_1 - C_3$ are constants depending only on n by [5], [17]. Here we have used the Harnack inequality for $G_R(\cdot, y)$, the fact that $G_R(o,y) \leq G(o,y)$ [17, Theorem 5.2].

$$\begin{aligned}
(1.4) \qquad |I| &\leq C_3 r \int_{B_o(R)\setminus B_o(2r)} r^{-1}(y) \left(\int_{r(y)}^{\infty} \frac{t\, dt}{V_o(t)} \right) f(y) dy \\
&= C_3 r \int_{2r}^{R} t^{-1} \left(\int_t^{\infty} \frac{s}{V_o(s)}\, ds \right) \left(\int_{\partial B_o(t)} f \right) dt \\
&\leq C_3 r \left[R^{-1} \left(\int_R^{\infty} \frac{s}{V_o(s)}\, ds \right) \left(\int_{B_o(R)} f \right) \right. \\
&\qquad \left. + \int_{2r}^{R} \left(\frac{1}{t^2} \int_t^{\infty} \frac{s}{V_o(s)} ds + \frac{1}{V(t)} \right) \left(\int_{B_o(t)} f \right) dt \right] \\
&\leq C_4 r \left(R k(R) + \int_{2r}^{R} k(t) dt \right)
\end{aligned}$$

for some constant $C_4(n, \sigma)$, where we have used (1.2).

$$
\begin{aligned}
(1.5)\quad \int_{B_o(2r)} G_R(o,y)f(y)dy &\le \int_{B_o(2r)} G(o,y)f(y)dy \\
&\le C_5 \int_0^{2r} \left(\int_t^\infty \frac{s}{V_o(s)}\,ds\right)\left(\int_{\partial B_o(t)} f\right) dt \\
&= C_5 \left[\left(\int_{2r}^\infty \frac{t}{V_o(t)}\,dt\right)\left(\int_{B_o(2r)} f\right)\right. \\
&\qquad \left. + \int_0^{2r} \frac{t}{V_o(t)}\Big(\int_{B_o(t)} f\Big)\,dt\right] \\
&\le C_5 \left[C_6 r^2 k(2r) + \int_0^{2r} tk(t)dt\right]
\end{aligned}
$$

for some constants $C_5(n)$ and $C_6(n,\sigma)$. Combining (1.3), (1.4) and (1.5), we have

$$
(1.6)\quad u_R(x) \le C_7 \left(rRk(R) + r^2 k(2r) + r\int_{2r}^R k(t)dt)\right) + \beta_1 \int_0^{2r} tk(t)dt
$$

for some constants $C_7(n,\sigma)$ and $\beta_1(n)$.

As in the proof of (1.5), using the lower bound of the Green's function, we have

$$
(1.7)\qquad \int_{B_o(2r)} G(o,y)f(y)dy \ge C_8 \left[C_9 r^2 k(2r) + \int_0^{2r} tk(t)dt\right]
$$

for some constants $C_8(n) > 0$ and $C_9(n,\sigma) > 0$. For $1 > \epsilon > 0$,

$$
\begin{aligned}
(1.8)\quad \int_{B_x(\epsilon r)} G_R(x,y)f(y)dy &\le \int_{B_x(\epsilon r)} G(x,y)f(y)dy \\
&\le \beta_2 \left[\left(\int_{\epsilon r}^\infty \frac{t}{V_x(t)}\,dt\right)\left(\int_{B_x(\epsilon r)} f\right)\right. \\
&\qquad \left. + \int_0^{\epsilon r} \frac{t}{V_x(t)}\left(\int_{B_x(t)} f\right) dt\right] \\
&\le C_{10}(\epsilon r)^2 k(x,\epsilon r) + \beta_2 \int_0^{\epsilon r} tk(x,t)dt \\
&\le C_{11} r^2 k\left((1+\epsilon)r)\right) + \beta_2 \int_0^{\epsilon r} tk(x,t)dt
\end{aligned}
$$

for some constants $C_{10}(n,\sigma)$, $C_{11}(n,\sigma,\epsilon)$ and $\beta_2(n)$. Here we have used volume comparison and the fact that $B_x(\epsilon r)\subset B_o((1+\epsilon)r))$.

$$\begin{aligned}\int_{B_o(2r)\setminus B_x(\epsilon r)} G_R(x,y)f(y)dy &\le \int_{B_o(2r)\setminus B_x(\epsilon r)} G(x,y)f(y)dy\\ &\le \sigma\cdot\frac{16r^2}{V_x(\epsilon r)}\int_{B_o(2r)} f(y)dy\\ &\le C_{12}r^2k(2r)\end{aligned}\tag{1.9}$$

for some constant $C_{12}(n,\sigma,\epsilon)$. By (1.3), (1.4), (1.8) and (1.9), if $R\ge 4r$, we have

$$\begin{aligned}u_R(x)\ge & -C_{13}r\left(Rk(R)+\int_{2r}^{R}k(t)dt\right)\\ & -\beta_2\int_0^{\epsilon r}tk(x,t)dt\\ & +\int_{B_o(2r)}G_R(o,y)f(y)dy\end{aligned}\tag{1.10}$$

where $\beta_3(n)$ and $C_{13}(n,\sigma,\epsilon)$ are positive constants. Here we have used the fact that for any $\alpha>1$, $k(\alpha R)\ge Ck(R)$ for some positive constant $C(n,\alpha)$ for all R. Since $\int_0^\infty k(t)dt<\infty$, $\lim_{R\to\infty}Rk(R)=0$. Hence from (1.6) and (1.10), u_R is bounded on compact sets and there exists $R_i\to\infty$ such that u_{R_i} converges uniformly on compact sets to a function u which satisfies $\Delta u=f$. By (1.6), (1.7) and (1.10), let $R_i\to\infty$ we can conclude that u satisfies the estimates in the theorem. q.e.d.

Note that if $n\ge 3$ and M has maximal volume growth, then M satisfies the assumptions in Theorem 1.1. In general, if M is a complete noncompact manifold with nonnegative Ricci curvature, then $M\times\mathbb{R}^4$ with the standard metric on $\mathbb{R}^4$ satisfies the condition of the theorem as observed in [23]. Using this, in some cases one can remove the assumptions that M is nonparabolic and that its Green's function satisfies (1.1).

Lemma 1.1. *Let $M=M_1\times M_2$, where M_1 and M_2 are complete noncompact manifolds with nonnegative Ricci curvature. Let $f\ge 0$ be a function on M_1, and be considered also as a function on M, which is independent of the second variable. Let $x=(x_1,x_2)\in M$ and $r>0$. Then*

$$\frac{C^{-1}}{V^{(1)}_{x_1}\left(\frac{1}{\sqrt{2}}r\right)}\int_{B^{(1)}_{x_1}\left(\frac{1}{\sqrt{2}}r\right)} f\le\frac{1}{V_x(r)}\int_{B_x(r)}f\le\frac{C}{V^{(1)}_{x_1}(r)}\int_{B^{(1)}_{x_1}(r)}f$$

for some constant $C > 0$ depending only on the dimensions of M_1 and M_2. Here $B_x(t)$, $B^{(1)}_{x_1}(t)$ are geodesic balls with radius t in M, M_1, with centers at x, x_1 respectively, and $V_x(t)$, $V^{(1)}_{x_1}(t)$ are the respective volumes.

Proof. Denote the geodesic ball with center x_2 and radius t in M_2 by $B^{(2)}_{x_2}(t)$ and its volume by $V^{(2)}_{x_2}(t)$. Then

$$B^{(1)}_{x_1}\left(\frac{1}{\sqrt{2}}r\right) \times B^{(2)}_{x_2}\left(\frac{1}{\sqrt{2}}r\right) \subset B_x(r) \subset B^{(1)}_{x_1}(r) \times B^{(2)}_{x_2}(r).$$

$$\begin{aligned}\int_{B_x(r)} f &\le \int_{B^{(1)}_{x_1}(r)\times B^{(2)}_{x_2}(r)} f \\ &= V^{(2)}_{x_2}(r)\int_{B^{(1)}_{x_1}(r)} f\end{aligned}$$

since f is independent of x_2. On the other hand,

$$\begin{aligned}V_x(r) &\ge V^{(1)}_{x_1}\left(\frac{1}{\sqrt{2}}r\right) V^{(2)}_{x_2}\left(\frac{1}{\sqrt{2}}r\right) \\ &\ge C V^{(1)}_{x_1}(r) V^{(2)}_{x_2}(r)\end{aligned}$$

for some constant $C > 0$ depending only on the dimensions of M_1 and M_2 by volume comparison. Hence

$$\frac{1}{V_x(r)}\int_{B_x(r)} f \le \frac{C}{V^{(1)}_{x_1}(r)}\int_{B^{(1)}_{x_1}(r)} f.$$

The other inequality can be proved similarly. q.e.d.

Theorem 1.2. *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature. Let $f \ge 0$ be a locally Hölder continuous function and let $k(x,t) = k_f(x,t)$ and $k(t) = k(o,t)$, where $o \in M$ is a fixed point. Suppose that $\int_0^\infty k(t)dt < \infty$ and suppose that there exist $1 > \delta > 0$, $h(t) \ge 0$, $0 \le t < \infty$ with $h(t) = o(t)$ as $t \to \infty$ such that*

$$\int_0^t sk(x,s)ds \le h(t)$$

for all x and for all $t \ge \delta r(x)$. Then the Poisson equation $\Delta u = f$ has *a solution u such that for all $1 > \epsilon > 0$*

$$\alpha_1 r\int_{2r}^{\infty} k(t)dt + \beta_1\int_0^{2r} tk(t)dt \geq u(x)$$
$$\geq -\alpha_2 r\int_{2r}^{\infty} k(t)dt - \beta_2\int_0^{\epsilon r} tk(x,t)dt + \beta_3\int_0^{2r} tk(t)dt$$

for some positive constants $\alpha_1(n)$, $\alpha_2(n,\epsilon)$ and $\beta_i(n)$, $1 \leq i \leq 3$. In particular, $|u(x)| = o\,(r(x))$ as $x \to \infty$.

Proof. Let $\widetilde{M} = M \times \mathbb{R}^4$ with the flat metric on $\mathbb{R}^4$. Then $\widetilde{M}$ is nonparabolic by [17, Theorem 5.2]. By the volume comparison, it is easy to see that (1.2) is satisfied by $\widetilde{M}$ and hence (1.1) is also satisfied by $\widetilde{M}$ with σ depending only on n. Denote a point on $\widetilde{M}$ by $\widetilde{x} = (x, x')$, and let $\widetilde{o} = (o, 0)$. Let $\widetilde{r}(\widetilde{x})$ and $r(x)$ be the distance functions on $\widetilde{M}$ and M from the $\widetilde{o}$ and o respectively. Let $\widetilde{f}(\widetilde{x}) = f(x)$ for $\widetilde{x} = (x, x')$ and let $\widetilde{k}(\widetilde{x}, t)$ be the average of $\widetilde{f}$ over the geodesic ball of radius t with center at $\widetilde{x}$. By Lemma 1.1, for any $t > 0$ and $\widetilde{x} = (x, x')$ in $\widetilde{M}$,

$$C_1^{-1}k(x, 1/\sqrt{2}t) \leq \widetilde{k}(\widetilde{x}, t) \leq C_1 k(x,t)$$

for some constant $C_1(n)$. Since $\int_0^\infty k(t)dt < \infty$ and

$$\int_0^t sk(x,s)ds \leq h(t)$$

for all $t \geq \delta r(x)$ with $h(t) = o(t)$ as $t \to \infty$, we have

$$\int_0^\infty \widetilde{k}(t)dt < \infty \text{ and } \int_0^{\delta\widetilde{r}} t\widetilde{k}(\widetilde{x},t)dt = o(\widetilde{r}) \quad \text{as } \widetilde{r} = \widetilde{r}(x) \to \infty \tag{1.11}$$

where we have used the fact that $\widetilde{r} \geq r(x)$. Let $\widetilde{\Delta}$ be the Laplacian of $\widetilde{M}$. By Theorem 1.1, there is a solution $\widetilde{u}$ of $\widetilde{\Delta}\widetilde{u} = \widetilde{f}$ on $\widetilde{M}$ such that for all $1 > \epsilon > 0$

$$\alpha_1\widetilde{r}\int_{2\widetilde{r}}^{\infty}\widetilde{k}(t)dt+\beta_1\int_0^{2\widetilde{r}}t\widetilde{k}(t)dt \geq \widetilde{u}(x) \geq -\alpha_2\widetilde{r}\int_{2\widetilde{r}}^{\infty}\widetilde{k}(t)dt - \beta_2\int_0^{\epsilon\widetilde{r}}t\widetilde{k}(\widetilde{x},t)dt + \beta_3\int_0^{2\widetilde{r}}t\widetilde{k}(t)dt \tag{1.12}$$

for some positive constants $\alpha_1(n)$, $\alpha_2(n,\epsilon)$, and $\beta_i(n)$, $1 \leq i \leq 3$. Moreover, $\widetilde{u}(\widetilde{o}) = 0$. By (1.11) and (1.12), it is easy to see that

$$|\widetilde{u}(\widetilde{x})| = o\left(\widetilde{r}(x)\right)$$

as $\widetilde{x} \to \infty$. Let $x_0' \in \mathbb{R}^4$ be fixed, then $\widetilde{v}(x,x') = \widetilde{u}(x,x'+x_0')$ is also a solution of $\widetilde{\Delta}\widetilde{v} = \widetilde{f}$. Hence $\widetilde{v}-\widetilde{u}$ is harmonic and is of sublinear growth. By the result of [5], $\widetilde{u}-\widetilde{v}$ must be a constant. Hence

$$\widetilde{u}(x,x'+x_0') - \widetilde{u}(x,x') = \widetilde{u}(o,x_0').$$

Let $x = o$, we conclude that

$$\widetilde{u}(o,x'+x_0') = \widetilde{u}(o,x_0') + \widetilde{u}(o,x')$$

for all x', $x_0' \in \mathbb{R}^4$. Since $\widetilde{u}(o,x')$ is continuous, it must be a linear function. Using the fact that u is of sublinear growth, we conclude that $\widetilde{u}(o,x')$ is a constant which is zero because $\widetilde{u}(\widetilde{o}) = 0$. Hence $\widetilde{u}$ is independent of x' and if we let $u(x) = \widetilde{u}(x,0)$, then $\Delta u = f$ in M and u satisfies the estimates in the theorem by (1.12), the fact that $C_1^{-1}k(x,1/\sqrt{2}t) \leq \widetilde{k}(\widetilde{x},t) \leq C_1k(x,t)$ and the fact that if $\widetilde{x} = (x,0)$ then $\widetilde{r}(\widetilde{x}) = r(x)$.

The last assertion of the theorem follows easily from the assumptions that $\int_0^\infty k(t)dt < \infty$ and $\int_0^t sk(x,s)ds = o(t)$ as $t \to \infty$ uniformly on x. q.e.d.

Observe that if $\int_0^\infty k(o,t)dt < \infty$ then $\int_0^\infty k(x,t)dt < \infty$ for all x. If u is the solution obtained in Theorem 1.1 or 1.2, then for any $x_0 \in M$

$$u(x) - u(x_0) = \int_M \left(G(x_0,y) - G(x,y)\right) f(y)dy.$$

From the proof of the theorem, it is easy to see that

$$\begin{aligned}(1.13)\quad \alpha_1 r\int_{2r}^{\infty} k(x_0,t)dt + \beta_1\int_0^{2r} tk(x_0,t)dt &\geq u(x)-u(x_0)\\ &\geq -\alpha_2 r\int_{2r}^{\infty} k(x_0,t)dt\\ &\quad -\beta_2\int_0^{\epsilon r} tk(x,t)dt\\ &\quad +\beta_3\int_0^{2r} tk(x_0,t)dt\end{aligned}$$

where α_i and β_j are the constants in Theorem 1.1 or 1.2 and $r = r(x, x_0)$.

In the following proposition, we will give a criteria for f to satisfy the assumptions in Theorem 1.2.

Proposition 1.1. *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature and let $f \geq 0$ be a function on M. Define $k(x,t) = k_f(x,t)$ and $k(t) = k(o,t)$ as before. Suppose $\int_0^\infty k(t)dt < \infty$ and $\sup_{\partial B_o(r)} f = o(r^{-1})$ as $r \to \infty$. Then*

$$\int_0^t sk(x,s)ds = o(t)$$

as $t \to \infty$ uniformly on x. In particular f satisfies the assumptions in Theorem 1.2.

Proof. Given $\epsilon > 0$, there exists $r_0 > 0$ such that if $r \geq r_0$, then

$$\sup_{\partial B_o(r)} f \leq \epsilon r^{-1},$$

and there exists $t_0 > 0$ such that

$$\int_0^t sk(s)ds < \epsilon t$$

for $t \geq t_0$ because $\int_0^\infty k(t)dt < \infty$. Let $t_1 = \max\{r_0, \frac{1}{3}t_0\}$. Let $x \in M$ be such that $r(x) = r \geq 2r_0$. If $\frac{r}{2} \geq t \geq t_1$, then $r \geq 2r_0$ and

$$\int_0^t sk(x,s)ds \leq C_1\epsilon r^{-1}t^2 \leq \frac{1}{2}C_1\epsilon t$$

for some absolute constant C_1. If $t \geq \frac{r}{2}$, then

$$\begin{aligned}\int_0^t sk(x,s)ds &= \int_0^{\frac{r}{2}} sk(x,s)ds + \int_{\frac{r}{2}}^t sk(x,s)ds \\ &\leq \frac{1}{2}C_1\epsilon t + C_2\int_0^{3t} sk(s)ds \\ &\leq C_3\epsilon t\end{aligned}$$

for some constants $C_2(n)$, $C_3(n)$. This completes the proof of the first part of the proposition. As for the second part, we just take $h(t) = \sup_{x\in M}\int_0^t sk(x,s)ds$ which is well-defined because f is bounded.

q.e.d.

Suppose $f(x) \leq Cr^{-2}(x)$ and $k(t) = 1/V_o(t)\int_{B_o(t)} f(y)\,dy \leq Ct^{-2}$, then the solution u obtained in Theorem 1.2 is bounded if and only if $\int_0^\infty tk(t)dt < \infty$. We will discuss bounded subharmonic functions in the next section. Here we consider the case when $\int_0^\infty tk(t)dt = \infty$.

Corollary 1.1. *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature and let $f \geq 0$ be a locally Hölder continuous function on M. Assume that $f(x) \leq Cr^{-2}(x)$ and that $k(t) = 1/V_o(t)\int_{B_o(t)} f(y)\,dy \leq Ct^{-2}$ for some constant C for all $x \in M$ and $t > 0$. Let u be the solution of the Poisson equation $\Delta u = f$ which is obtained in Theorem* 1.2. *Suppose $\int_0^\infty tk(t)dt = \infty$. We have:*

(i)
$$\beta_1 \geq \limsup_{r\to\infty}\frac{\sup_{\partial B_o(r)} u}{\int_0^r tk(t)dt} \geq \liminf_{r\to\infty}\frac{\inf_{\partial B_o(r)} u}{\int_0^r tk(t)dt} \geq \beta_3$$

where β_1 and β_3 are the positive constants in Theorem 1.2.

(ii) *If M^n has maximal volume growth with $n \geq 3$ then*

$$\lim_{x\to\infty}\frac{u(x)}{\int_0^{r(x)} tk(t)dt} = \frac{1}{n}.$$

Proof. Since f satisfies the conditions in Proposition 1.1, one can solve $\Delta u = f$ by Theorem 1.2. The solution satisfies the estimates in the theorem. By the assumptions on f, we can prove that

$$C_1 + \beta_1\int_0^{2r} tk(t)dt \geq u(x) \geq -C_2 + \beta_3\int_0^{2r} tk(t)dt$$

for some positive constants C_1 and C_2 independent of x, where $r = r(x)$. From this, (i) follows easily.

If $n \geq 3$, M has maximal volume growth and u is unbounded, then by (i) and the proof of Theorem 1.1, it is easy to see that

$$\lim_{x\to\infty} \frac{u(x)}{\int_{B_o(2r)} G(o,y)f(y)dy} = 1.$$

(ii) follows from the sharp bound of the Green's function in [6], see also [16]. q.e.d.

Next we will estimate the gradient and the Hessian of the solution u obtained in Theorem 1.1 or 1.2.

Theorem 1.3. *With the same assumptions and notations as in Theorem* 1.1 *or* 1.2 *and let u be the solution of $\Delta u = f$ obtained in Theorem* 1.1 *or* 1.2. *We have the following:*

(i)

$$|\nabla u(x)| \leq C(n,\sigma) \int_0^\infty k(x,t)dt.$$

(ii) *For any $p \geq 1$ and $\alpha \geq 2$, if u is the solution obtained in Theorem* 1.1, *then*

$$\frac{1}{V_o(R)} \int_{B_o(R)} |\nabla u|^p \leq C' \left(\int_{\alpha R}^\infty k(t)dt \right)^p + \frac{C'' R^p}{V_o(R)} \int_{B_o(\alpha R)} f^p$$

for some constants $C'(n,\sigma,p)$ and $C''(n,\sigma,p,\alpha)$, and if u is the solution obtained in Theorem 1.2, *then*

$$\frac{1}{V_o\left(\frac{1}{\sqrt{2}}R\right)} \int_{B_o\left(\frac{1}{\sqrt{2}}R\right)} |\nabla u|^p \leq C' \left(\int_{\alpha R}^\infty k(t)dt \right)^p + \frac{C'' R^p}{V_o(R)} \int_{B_o(\alpha R)} f^p$$

for some constants $C'(n,p)$ and $C''(n,p,\alpha)$.

(iii) *If u is the solution obtained in Theorem* 1.1, *then*

$$\frac{1}{V_o(R)} \int_{B_o(R)} |\nabla^2 u|^2 \leq C \left[R^{-2} \left(\int_{4R}^\infty k(t)dt \right)^2 + \frac{1}{V_o(4R)} \int_{B_o(4R)} f^2 \right]$$

for some constant $C(n)$, and if u is the solution obtained in Theorem 1.2, *then*

$$\frac{1}{V_o\left(\frac{1}{\sqrt{2}}R\right)}\int_{B_o\left(\frac{1}{\sqrt{2}}R\right)}|\nabla^2 u|^2 \le C\left[R^{-2}\left(\int_{4R}^{\infty}k(t)dt\right)^2+\frac{1}{V_o(4R)}\int_{B_o(4R)}f^2\right]$$

for some constant $C(n,\sigma)$.

Proof. Let us first consider the solution u obtained in Theorem 1.1. Divide (1.13) by r, choose $\epsilon=\frac{1}{4}$ and let $r\to 0$, it is easy to see (i) is true. To prove (ii), by Theorem 1.1 for any $x\in B_o(R)$, we have

$$\begin{aligned}|\nabla u(x)| &\le \int_M |\nabla_x G(x,y)|f(y)dy\\ &\le C_1\int_M r^{-1}(x,y)G(x,y)f(y)dy\\ &\le C_2\int_{M\setminus B_o(\alpha R)}\frac{r(x,y)}{V_x(r(x,y))}f(y)dy\\ &\quad + C_1\int_{B_o(\alpha R)} r^{-1}(x,y)G(x,y)f(y)dy\end{aligned}\tag{1.15}$$

for some constants $C_1(n)$, $C_2(n,\sigma)$. Here we have used (1.1) and (1.2) and the gradient estimate in [5]. Now

$$\begin{aligned}&\int_{M\setminus B_o(\alpha R)}\frac{r(x,y)}{V_x(r(x,y))}f(y)dy\\ &\le C_3\int_{M\setminus B_o(\alpha R)}\frac{r(y)}{V_o(r(y))}f(y)dy\\ &= C_3\int_{\alpha R}^{\infty}\frac{t}{V_o(t)}\left(\int_{\partial B_o(t)}f\right)dt\\ &= C_3\left[\frac{t}{V_o(t)}\int_{B_o(t)}f\Big|_{\alpha R}^{\infty}\right.\\ &\qquad\left.-\int_{\alpha R}^{\infty}\left(\frac{1}{V_o(t)}-\frac{tA_o(t)}{V_o^2(t)}\right)\left(\int_{B_o(t)}f\right)dt\right]\\ &\le C_4\int_{\alpha R}^{\infty}\frac{1}{V_o(t)}\left(\int_{B_o(t)}f\right)dt\\ &= C_4\int_{\alpha R}^{\infty}k(t)dt\end{aligned}\tag{1.16}$$

for some constants $C_3(n)$, $C_4(n)$, where we have used the fact that $\alpha \geq 2$, volume comparison, $tA_o(t) \leq nV_o(t)$ and the fact that $tk(t) \to 0$ as $t \to \infty$. Note that for any $z \in M$ and for any $\rho > 0$

$$\int_{B_z(\rho)} \frac{r(z,y)}{V_z(r(z,y))} dy = \int_0^\rho \frac{tA_z(t)}{V_z(t)} dt \leq n\rho. \tag{1.17}$$

Let us first assume $p > 1$ and let $q = p/(p-1)$. By (1.15) and (1.16), we have

$$\begin{aligned} \int_{B_o(R)} |\nabla u|^p \leq{}& C_5 V_o(R) \left(\int_{\alpha R}^\infty k(t)dt \right)^p \\ &+ C_6 \int_{B_o(R)} \left(\int_{B_o(\alpha R)} r^{-1}(x,y)G(x,y)f(y)dy \right)^p dx \end{aligned} \tag{1.18}$$

for some constants $C_5(n,\sigma,p)$ and $C_6(n,p)$.

(1.19)

$$\begin{aligned} &\int_{B_o(R)} \left(\int_{B_o(\alpha R)} r^{-1}(x,y)G(x,y)f(y)dy \right)^p dx \\ &\qquad \leq \int_{B_o(R)} \left(\int_{B_o(\alpha R)} r^{-1}(x,y)G(x,y)dy \right)^{\frac{p}{q}} \\ &\qquad\qquad\qquad \cdot \left(\int_{B_o(\alpha R)} r^{-1}(x,y)G(x,y)f^p(y)dy \right) dx \\ &\qquad \leq C_7(n,\sigma,p)R^{\frac{p}{q}} \int_{B_o(R)} \left(\int_{B_o(\alpha R)} r^{-1}(x,y)G(x,y)f^p(y)dy \right) dx \\ &\qquad = C_7 R^{\frac{p}{q}} \int_{B_o(\alpha R)} \left(\int_{B_o(R)} r^{-1}(x,y)G(x,y)dx \right) f^p(y)dy \\ &\qquad \leq C_8 R^p \int_{B_o(\alpha R)} f^p(y)dy \end{aligned}$$

for some constants $C_7(n,\sigma,p,\alpha)$ and $C_8(n,\sigma,p,\alpha)$, where we have used (1.1) and (1.17). Combine this with (1.18), (ii) follows if $p > 1$. The case that $p = 1$ can be proved similarly.

To prove (iii), in terms of a local orthonormal frame,

$$\begin{aligned} \frac{1}{2}\Delta|\nabla u|^2 &= \sum_{k,l} u_{kl}^2 + \sum_k u_k(\Delta u)_k + \sum_{k,l} R_{kl}u_k u_l \\ &\geq |\nabla^2 u|^2 + \langle \nabla u, \nabla f \rangle \end{aligned}$$

where R_{kl} is the Ricci curvature tensor of M which is positive semi-definite. Let $\varphi \geq 0$ be a smooth function with compact support in $B_o(2R)$. Multiplying the above inequality by φ^2 and integrating by parts, we have

$$\begin{aligned}
\int_{B_o(2R)} \varphi^2|\nabla^2 u|^2 & \\
\leq & \int_{B_o(2R)} \varphi^2 f^2 + \int_{B_o(2R)} \varphi|\nabla\varphi|\ |\nabla u|\ |f| \\
& + 2\int_{B_o(2R)} \varphi|\nabla\varphi|\ \left|\nabla(|\nabla u|^2)\right| \\
\leq & C_9 \left[\int_{B_o(2R)} \varphi^2 f^2 \right.\\
& \left. + \int_{B_o(2R)} |\nabla\varphi|^2|\nabla u|^2 + \int_{B_o(2R)} \varphi|\nabla\varphi|\ |\nabla u||\nabla^2 u|\right] \\
\leq & C_9 \left[\int_{B_o(2R)} \varphi^2 f^2 + (1+\frac{1}{\epsilon})\int_{B_o(2R)} |\nabla\varphi|^2|\nabla u|^2 \right.\\
& \left. +\epsilon\int_{B_o(2R)} \varphi^2|\nabla^2 u|^2\right]
\end{aligned}$$

for any $\epsilon > 0$, for some absolute constant C_9. Hence choose $\epsilon = (2C_9)^{-1}$, we have

$$\int_{B_o(2R)} \varphi^2|\nabla^2 u|^2 \leq 2C_9 \left(\int_{B_o(2R)} \varphi^2 f^2 + \int_{B_o(2R)} |\nabla\varphi|^2|\nabla u|^2\right)$$

for all $\varphi \geq 0$ with compact support in $B_o(R)$. Choose a suitable φ we obtain

$$\int_{B_o(R)} |\nabla^2 u|^2 \leq C_{10} \left(\int_{B_o(2R)} f^2 + R^{-2}\int_{B_o(2R)} |\nabla u|^2\right)$$

for some absolute constant C_{10}. Combining this with (ii) the results follows.

Suppose the assumptions of Theorem 1.2 are satisfied. Using the same notations as in the proof of Theorem 1.2. Then the gradient and the Hessian of $\widetilde{u}$ can be estimated as before. Since $\widetilde{u}$ is independent of $x' \in \mathbb{R}^4$, the results then follow easily from Lemma 1.1. q.e.d.

Remark 1.1. The assumption that $f \geq 0$ in Theorem 1.1, 1.2 and 1.3 can be relaxed. For general f, let $k(x,t) = 1/V_x(t)\int_{B_x(t)}|f|$ instead. Under similar assumptions as in Theorem 1.1 or 1.2, we can solve $\Delta u_1 = \max\{f, 0\}$ and $\Delta u_2 = \max\{-f, 0\}$ using these theorems. Then $u = u_1 - u_2$ satisfies $\Delta u = f$. Even though the estimates for u in Theorem 1.1 or 1.2 will no longer be true, however the estimates for $|\nabla u|$ and $|\nabla^2 u|$ in Theorem 1.3 still hold.

Corollary 1.2. *With the same assumptions on M and f and with the same notations as in Theorem* 1.1 *or* 1.2*. Let u be the solution of $\Delta u = f$ obtained in Theorem* 1.1 *or* 1.2*. We have the following:*

(i) *Suppose there is a constant $C > 0$ such that $\int_0^\infty k(x,t)dt \leq C$ for all x. Then $\sup_M |\nabla u| < \infty$.*

(ii) *Suppose $f(x) \leq Cr^{-2}(x)$ and $k(t) \leq Ct^{-2}$ for some constant C for all $x \in M$ and $t > 0$. Then*

$$|\nabla u(x)| \leq C' r^{-1}(x)$$

for some constant C' for all x.

(iii) *Suppose there is a constant $C > 0$ such that $f(x) \leq Cr^{-2}(x)$ and $k(t) \leq t^{-2}h(t)$ with $\lim_{t\to\infty} h(t) = 0$. Then*

$$|\nabla u(x)| = o\left(r^{-1}(x)\right)$$

as $x \to \infty$.

Proof. (i) follows easily from Theorem 1.3(i). To prove (iii), by Theorem 1.3(i), it is sufficient to estimate $\int_0^\infty k(x,t)dt$. For any $\frac{1}{2} > \epsilon > 0$, let $x \in M$ and let $r = r(x)$, then

$$\int_0^{\epsilon r} k(x,t)dt \leq C_1 \epsilon r^{-1} \tag{1.20}$$

for some constant C_1 independent of x and ϵ. For $t \geq \epsilon r$,

$$\begin{aligned} k(x,t) &= \frac{1}{V_x(t)}\int_{B_x(t)} f \\ &= \frac{V_x(t+r)}{V_x(t)} \cdot \frac{1}{V_x(t+r)}\int_{B_o(t+r)} f \\ &\leq C_2(n)\left(1+\epsilon^{-1}\right)^n \frac{1}{V_o(t)}\int_{B_o((1+\epsilon^{-1}))t} f \\ &\leq C_1\left(1+\epsilon^{-1}\right)^{2n} k\left((1+\epsilon^{-1})t\right) \end{aligned} \tag{1.21}$$

for some constant $C_2(n)$. By (1.20) and (1.21), we have

$$\begin{aligned}\int_0^\infty k(x,t)dt &\le C_1\epsilon r^{-1} + \int_{\epsilon r}^\infty \left(1+\epsilon^{-1}\right)^{2n} k\left((1+\epsilon^{-1})t\right) \\ &\le \beta\epsilon r^{-1} + C_2\left(\sup_{t\ge(1+\epsilon)r} h(t)\right)\cdot r^{-1}\end{aligned}$$

for some constant $C_2(n,\epsilon)$. Since $h(t)\to 0$ as $t\to\infty$ (iii) follows.

The proof of (ii) is similar. q.e.d.

2. The Poisson equation (II)

In the previous section, we have obtained some conditions on f so that the Poisson equation $\Delta u = f$ has a solution. In this section, we will study the problem from another perspective. Namely, suppose a solution u of the Poisson equation $\Delta u = f$ exists, we want to discuss the properties of f. We have the following general result.

Theorem 2.1. *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature. Suppose u is a solution of $\Delta u = f$ on M, where $f \ge 0$ is a nonnegative function. Suppose that there exist nondecreasing functions h and g such that*

$$-g(r) \le \inf_{B_o(r)} u \le \sup_{B_o(r)} u \le h(r)$$

for all r, where $o\in M$ is a fixed point. Then for any $R>0$ and $x\in M$

$$\begin{aligned}C(n)\left[R^2k(x,\,R) + \int_0^R tk(x,t)dt\right] &\le -u(x) + h(5R+r) \\ &\le g(r) + h(5R+r)\end{aligned}$$

for some positive constant $C(n)$, where $r=r(x)$,

$$k(x,t) = 1/V_x(t)\int_{B_x(t)} f.$$

In particular, if we let $k(t)=k(o,t)$, then

$$C(n)\left[R^2k(R) + \int_0^R tk(t)dt\right] \le -u(o) + h(5R)$$

for all R.

Proof. Let $x \in M$ and let $r = r(x)$. For any $R > 0$, let G_R be the positive Green's function on $B_x(R)$ with Dirichlet boundary data. Then

$$\begin{aligned}\int_{B_x(R)} G_R(x,y)f(y)dy &= \int_{B_x(R)} G_R(x,y)\Delta u(y)dy \\ &= -u(x) - \int_{\partial B_x(R)} u\frac{\partial G_R}{\partial \nu} \\ &\leq -u(x) + h(R+r)\end{aligned}$$

where we have used the fact that $\frac{\partial G_R}{\partial \nu} < 0$ on $\partial B_x(R)$ and that $\int_{\partial B_x(R)} \frac{\partial G_R}{\partial \nu} = -1$. By Lemma 1.1 in [25]

$$G_R(x,y) \geq C_1 \int_{r(x,y)}^{R} \frac{t}{V_x(t)} dt$$

for all $y \in B_x(\frac{1}{5}R)$ for some constant $C_1(n) > 0$. Hence

$$\begin{aligned}g(r) + h(R+r) &\geq -u(x) + h(R+r) \\ &\geq C_1 \int_0^{\frac{R}{5}} \left(\int_t^R \frac{s}{V_x(s)}\, ds\right)\left(\int_{\partial B_x(t)} f\right) dt \\ &\geq C_1 \left[\left(\int_{\frac{R}{5}}^R \frac{t}{V_o(t)} dt\right)\left(\int_{B_x(\frac{R}{5})} f\right)\right. \\ &\qquad\qquad \left. + \int_0^{\frac{R}{5}} \left(\frac{t}{V_x(t)} \int_{B_x(t)} f\right) dt\right] \\ &\geq C_2 \left[R^2 k(x, \frac{R}{5}) + \int_0^{\frac{R}{5}} tk(x,t)dt\right]\end{aligned}$$

for some positive constant $C_2(n)$, where we have used volume comparison. From this the theorem follows. q.e.d.

Note that similar estimate has been obtained in [15, Lemma 2.1], where no curvature assumption was made and hence the result was weaker.

Using Theorem 2.1 and the results in Section 1, we can obtain necessary and sufficient conditions for a function f so that $\Delta u = f$ has a solution with certain growth rate.

Theorem 2.2. *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature and let $m(t)$, $0 \le t < \infty$, be a nonnegative nondecreasing function such that for any $A > 1$ there exists $C > 0$ with*

$$m(At) \le Cm(t) \tag{2.1}$$

for all t, and

$$\int_1^\infty t^{-2} m(t)dt < \infty. \tag{2.2}$$

Let $f \ge 0$ be a locally Hölder continuous function on M and let $k(x,t)$ as in Theorem 2.1. Then the Poisson equation $\Delta u = f$ has a solution on M with $\sup_{B_o(r)} |u| \le Cm(r)$ for some constant C for all r if and only if

$$\int_0^t sk(x,s)ds \le C'm(t) \tag{2.3}$$

for some constant C' for all x and for all $t \ge \frac{1}{5}r(x)$.

Proof. By (2.2) and the fact that m is nondecreasing, $m(t) = o(t)$ as $t \to \infty$. Suppose f satisfies (2.3). Then it is easy to see that f satisfies the assumptions in Theorem 2.1. Hence $\Delta u = f$ has a solution u such that

$$\begin{aligned}\alpha_1 r \int_{2r}^\infty k(t)dt + \beta_1 \int_0^{2r} tk(t)dt \ge u(x)& \\ \ge -\alpha_2 r \int_{2r}^\infty k(t)dt& \\ - \beta_2 \int_0^{\frac{1}{5}r} tk(x,t)dt& \\ + \beta_3 \int_0^{2r} tk(t)dt&\end{aligned}$$

for some positive constants $\alpha_1(n)$, $\alpha_2(n)$ and $\beta_i(n)$, $1 \le i \le 3$, where $r = r(x)$, $k(t) = k(o,t)$. Combine this with (2.1)–(2.3), we have

$$\sup_{B_o(r)} |u| \le C_1 m(r) \tag{2.4}$$

for some constant C_1 for all r.

Conversely, suppose $\Delta u = f$ has a solution satisfying (2.4). Then by Theorem 2.1 and (2.1), it is easy to see that (2.3) is true. q.e.d.

Remark 2.1. (i) We assume condition (2.1) so that we can state the theorem more simply. Otherwise, we may replace some of the $m(t)$ in the theorem by $m(At)$ for some constant A. (ii) From the proof it is easy to see that if $\Delta u = f$ has a solution satisfying $\sup_{B_o(r)} |u| \leq Cm(r)$, then u is the solution obtained in Theorem 1.2. Because in this case, both u and the solution in Theorem 1.2 are of sub-linear growth.

If we take $m(t)$ =constant, $m(t) = \log(2+t)$ or $m(t) = (1+t)^{1-\delta}$ for some $1 > \delta > 0$, then we have the following.

Corollary 2.1. *Let M^n be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let $f \geq 0$ be a locally Hölder continuous function on M. Let $k(x,t)$ and $k(t)$ be as in Theorem 2.2. Then:*

(i) *$\Delta u = f$ has a bounded solution if and only if there is a constant $C > 0$ such that*

$$\int_0^\infty tk(x,t)dt \leq C$$

for all x.

(ii) *$\Delta u = f$ has a solution with $\sup_{B_o(r)} |u| \leq C\log(2+r)$ for some constant C for all r if any only if*

$$\int_0^t sk(x,s)ds \leq C' \log(2+t)$$

for some constant C' for all $r = r(x)$ and for all $t \geq \frac{1}{5}r$.

(iii) *$\Delta u = f$ has a solution with $\sup_{B_o(r)} |u| \leq C(1+r)^{1-\delta}$ for some constants C and $0 < \delta < 1$ for all r if any only if*

$$\int_0^t sk(x,s)ds \leq C'(1+t)^{1-\delta}$$

for some constant C' for all $r = r(x)$ and for all $t \geq \frac{1}{5}r$.

Suppose $\int_0^\infty tk(t)dt < \infty$ and if f is not identically zero, then $\int_1^\infty t/V_o(t)dt < \infty$ and M must be nonparabolic. In this case, it is easy to see that $u(x) = -\int_M G(x,y)f(y)dy + C$ for some constant C. As an application of this remark and Corollary 2.1, we will give another proof of a result of Li [12, Theorem 4] on bounded subharmonic functions. The method is not simpler, but we obtain some estimates that may be useful.

Theorem 2.3. *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature. Let u be a bounded subharmonic function, and let $\alpha = \sup_M u$, then:*

(i)

$$\frac{1}{V_o(R)}\int_{B_o(R)}(\alpha - u) \le C\left(\frac{R^2}{V_o(2R)}\int_{B_o(2R)} f + \int_{2R}^{\infty}\left(\frac{t}{V_o(t)}\int_{B_o(t)} f\right)dt\right)$$

for some constant $C(n)$ for all $R > 0$, where $f = \Delta u$.

(ii) (Li [12])

$$\lim_{R\to\infty}\frac{1}{V_o(R)}\int_{B_o(R)} u = \sup_M u.$$

Proof. First we assume that M is nonparabolic and satisfies (1.1) and (1.2). Then $f \ge 0$ because u is subharmonic. By Corollary 2.1, we have

$$\int_0^{\infty}\left(\frac{t}{V_x(t)}\int_{B_x(t)} f\right)dt \le C_1 \tag{2.5}$$

for some constant C_1 for all $x \in M$. Moreover, by adding a constant to u, we may assume that

$$u(x) = -\int_M G(x,y)f(y)dy.$$

Note that u is nonpositive. Using similar methods as in the proof of Theorem 1.3, let $R > 0$

$$\begin{aligned}\int_{B_o(R)}(-u) &= \int_{x\in B_o(R)}\left(\int_{y\in M} G(x,y)f(y)dy\right)dx \\ &= \int_{x\in B_o(R)}\left(\int_{y\in B_o(2R)} G(x,y)f(y)dy\right)dx \\ &\quad + \int_{x\in B_o(R)}\left(\int_{y\in M\setminus B_o(2R)} G(x,y)f(y)dy\right)dx.\end{aligned} \tag{2.6}$$

For any $y \in M$,

$$\begin{aligned}\int_{x\in B_y(3R)} G(x,y)dx &\le C_2\int_0^{3R} A_y(t)\frac{t^2}{V_y(t)}dt\\ &= C_3\int_0^{3R} tdt\\ &\le C_4R^2\end{aligned}$$

for some constants C_2, C_3 and C_4 depending only on n and σ, where we have used (1.1) and (1.2). Hence

$$\begin{aligned}\int_{x\in B_o(R)}&\left(\int_{y\in B_o(2R)} G(x,y)f(y)dy\right)dx\\ &= \int_{y\in B_o(2R)} f(y)\left(\int_{x\in B_o(R)} G(x,y)dx\right)dy\\ &\le \int_{y\in B_o(2R)} f(y)\left(\int_{x\in B_y(3R)} G(x,y)dx\right)dy\\ &\le C_4R^2\int_{B_o(2R)} f.\end{aligned} \tag{2.7}$$

For any $x \in B_o(R)$, using (2.5)

$$\begin{aligned}\int_{y\in M\setminus B_o(2R)} G(x,y)f(y)dy &\le C_5\int_{y\in M\setminus B_o(2R)} G(o,y)f(y)dy\\ &\le C_6\int_{2R}^{\infty}\left(\int_t^{\infty}\frac{s}{V_o(s)}ds\right)\left(\int_{\partial B_o(t)} f\right)dt\\ &\le C_6\int_{2R}^{\infty}\left(\frac{t}{V_o(t)}\int_{B_o(t)} f\right)dt\end{aligned} \tag{2.8}$$

for some constants C_5, C_6 depending only on n and σ. By (2.6)–(2.8), we have

$$\begin{aligned}\frac{1}{V_o(R)}&\int_{B_o(R)}(-u)\\ &\le (C_4+C_6)\left(\frac{R^2}{V_o(R)}\int_{B_o(2R)} f + \int_{2R}^{\infty}\left(\frac{t}{V_o(t)}\int_{B_o(t)} f\right)dt\right).\end{aligned}$$

This implies (i) because $\alpha = \sup_M u \le 0$. By (2.5), the right side of the above inequality will tend to 0 as $R \to \infty$. This implies (ii) by noting that

$$\frac{1}{V_o(R)} \int_{B_o(R)} (-u) \ge \inf_M (-u) \ge 0.$$

For general cases, we just take $M \times \mathbb{R}^4$ with flat metric on $\mathbb{R}^4$ and consider u as a subharmonic function on $M \times \mathbb{R}^4$ and use Lemma 1.1. Note that in this case, we can choose σ to depend only on n. q.e.d.

Consider the following example: let u be a nonconstant bounded subharmonic function on $\mathbb{R}^3$ and consider u as a bounded subharmonic function on $\mathbb{R}^4$. Then the average of u over $B_o(r)$ will tends to $\sup_M u$ as $r \to \infty$. However, it is obvious that u will not be asymptotically constant at infinity. In this respect, we have:

Theorem 2.4. *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature and let u be a smooth bounded subharmonic function on M. Suppose $f = \Delta u$ is such that $f(x) \le Cr^{-2}(x)$. Then*

$$\lim_{x\to\infty} u(x) = \sup_M u$$

where $x \to \infty$ means that $r(x) \to \infty$.

Proof. Since u is bounded and subharmonic, by Corollary 2.1 we can conclude that

$$t^{-2}k(t) \to 0$$

as $t \to \infty$, where $k(t) = 1/V_o(t) \int_{B_o(t)} f$. By Corollary 1.2(iii) and the assumption that $f(x) \le Cr^{-2}(x)$, we have

$$|\nabla u(x)| = o(r^{-1}(x)) \tag{2.9}$$

as $r(x) \to \infty$. We may assume that $\sup_M u = 0$. By Li's result Theorem 2.3(ii)

$$\lim_{r\to\infty} \frac{1}{V_o(r)} \int_{B_o(r)} u = 0.$$

Since $u \le 0$, for any $\epsilon > 0$, let $D_r = \{x \in B_o(r) |\ u(x) \ge -\epsilon\}$, then it is easy to see that

$$V(D_r) \ge (1-\epsilon)V(r)$$

if r is large enough. Hence if x is such that $r(x) = R$ and if R is large enough, $V_x(\frac{1}{2}R) \cap D_{2R} \ne \emptyset$. By (2.9) we conclude that $u(x) \to 0$ as $x \to \infty$. q.e.d.

3. Some vanishing results

In this section, we will apply the results in Section 1 and Section 2 to obtain some vanishing theorems on holomorphic line bundles over complete noncompact Kähler manifolds. The results are related to those in [22] and [10]. We need the following Kodaira-Bochner formula [21, Chapter 3, §6]:

Lemma 3.1. *Let M be a Kähler manifold, let L be a Hermitian holomorphic line bundle over M and let ϕ be a holomorphic $(p,0)$ form with value in L. Denote $|\phi|$ to be the norm of ϕ with respect to the Kähler metric on M and the Hermitian metric h on L. Then*

$$
\begin{aligned}
&|\phi|^2\Delta|\phi|^2 - |\nabla|\phi|^2|^2 \\
&\qquad \geq 4\left(-\Omega + \min_{1\leq i_1<i_2<\cdots<i_p\leq m}(\gamma_{i_1}+\gamma_{i_2}+\cdots+\gamma_{i_p})\right)|\phi|^4.
\end{aligned}
$$

where γ_i are the eigenvalues of the Ricci form $R_{i\bar{j}}$ of M, Ω is the trace of the curvature form $\Omega_{i\bar{j}}$ of h.

Theorem 3.1. *Let M^m be a complete noncompact Kähler manifold of complex dimension m with nonnegative Ricci curvature. Let L be a Hermitian holomorphic line bundle over M and let Ω be the trace of the curvature form of L. For any $0<\epsilon<1$ and $\tau>0$, there exists a constant $a(m,\tau,\epsilon)$ such that if*

$$
\limsup_{r\to\infty}\frac{r^2}{V_o(r)}\int_{B_o(r)}\Omega_+ \leq a, \tag{3.1}
$$

where Ω_+ is the positive part of Ω, then any holomorphic $(p,0)$ form ϕ (valued in L) is trivial if the norm of ϕ satisfies

$$
\frac{1}{V_o(r)}\int_{B_o(r)}|\phi|^{2\tau} = O(r^{-\epsilon}) \tag{3.2}
$$

as $r\to\infty$.

Proof. Suppose Ω_+ satisfies the condition (3.1) with a to be determined later. Let $\widetilde{M} = M\times\mathbb{R}^4$ and denote $\widetilde{x}=(x,x')$. Consider Ω_+ as a function on $\widetilde{M}$. By Theorem 1.1 and Lemma 1.1, we can find a solution $u(\widetilde{x})$ of $\widetilde{\Delta}u = 4\tau\Omega_+$ such that

$$
u(\widetilde{x}) \leq \alpha_1\widetilde{r}\int_{\widetilde{r}}^{\infty}k(t)dt + \beta_1\tau\int_0^{2\widetilde{r}}tk(t)dt
$$

where α_1 and β_1 are the constants depending only on m, $\widetilde{r} = \widetilde{r}(\widetilde{x})$ is the distance from $\widetilde{x}$ to $\widetilde{o} = (o, 0)$ and $k(t) = 1/V_o(t) \int_{B_o(t)} \Omega_+$. Here $\widetilde{\Delta}$ is the Laplacian on $\widetilde{M}$. Choose $a > 0$ such that $a\beta_1\tau < \frac{1}{2}\epsilon$, then a depends only on m, τ, ϵ. By (3.1), we have

$$\limsup_{r(\widetilde{x})\to\infty} \frac{u(\widetilde{x})}{a\beta_1\tau \log \widetilde{r}(x)} \leq 1$$

and hence

$$e^{u(\widetilde{x})} \leq C_1(1+\widetilde{r})^{\frac{1}{2}\epsilon}(\widetilde{x}) \tag{3.3}$$

for some constant C_1 for all $\widetilde{x}$. Let ϕ be a holomorphic $(p, 0)$ form such that

$$\frac{1}{V_o(r)} \int_{B_o(r)} |\phi|^\tau = O(r^{-\epsilon}), \tag{3.4}$$

and let $f = |\phi|^2$. By Lemma 3.1, we have

$$f\Delta f - |\nabla f|^2 \geq -4\Omega_+ f^2$$

and if we consider f as a function on $\widetilde{M}$, then

$$f\widetilde{\Delta} f - |\widetilde{\nabla} f|^2 \geq -4\Omega_+ f^2 \tag{3.5}$$

where $\widetilde{\nabla}$ is the gradient on $\widetilde{M}$. For any $\delta > 0$, let $g = (f+\delta)^\tau$. At a point $\widetilde{x}$ where $f(\widetilde{x}) > 0$, we have

$$\begin{aligned} g\widetilde{\Delta} g - |\widetilde{\nabla} g|^2 &= g^2 \widetilde{\Delta} \log g \\ &= \tau g^2 \left(\frac{\widetilde{\Delta} f}{f+\delta} - \frac{|\widetilde{\nabla} f|^2}{(f+\delta)^2} \right) \\ &\geq \tau g^2 \left(-\frac{4\Omega_+ f}{f+\delta} + \frac{|\widetilde{\nabla} f|^2}{f(f+\delta)} - \frac{|\widetilde{\nabla} f|^2}{(f+\delta)^2} \right) \\ &\geq -4\tau\Omega_+ g^2. \end{aligned}$$

On the other hand, suppose $f(\widetilde{x}) = 0$, then g attains minimum at $\widetilde{x}$. Hence we still have

$$g\widetilde{\Delta} g - |\widetilde{\nabla} g|^2 \geq -4\tau\Omega_+ g^2.$$

Let $v = e^u g$, then

$$\begin{aligned} v\widetilde{\Delta} v &= e^u g\left(e^u \widetilde{\Delta} g + 2e^u < \widetilde{\nabla} u, \widetilde{\nabla} g > + e^u g \widetilde{\Delta} u + e^u g|\widetilde{\nabla} u|^2\right) \\ &\geq e^{2u}\left(-4\tau\Omega_+ g^2 + |\widetilde{\nabla} g|^2 - 2g|\widetilde{\nabla} u|\,|\widetilde{\nabla} g| + 4\tau\Omega_+ g^2 + g^2|\widetilde{\nabla} u|^2\right) \\ &\geq 0. \end{aligned}$$

By the mean value inequality of Li-Schoen ([13, Theorem 2.1]), for any $\widetilde{r} > 0$,

$$\sup_{B_{\widetilde{o}}(\widetilde{r})} v \leq \frac{C_2}{V_{\widetilde{o}}(2\widetilde{r})} \int_{B_{\widetilde{o}}(2\widetilde{r})} v$$

for some constant $C_2(m)$. Let $\delta \to 0$, we have

$$\begin{aligned} \sup_{B_{\widetilde{o}}(\widetilde{r})} e^u f^\tau &\leq \frac{C_2}{V_{\widetilde{o}}(2\widetilde{r})} \int_{B_{\widetilde{o}}(2\widetilde{r})} e^u f^\tau \\ &\leq C_3(1+\widetilde{r})^{\frac{1}{2}\epsilon} V_{\widetilde{o}}^{-1}(2\widetilde{r}) \int_{B_{\widetilde{o}}(2\widetilde{r})} f^\tau \\ &\leq C_4(1+\widetilde{r})^{\frac{1}{2}\epsilon} V_o^{-1}(2\widetilde{r}) \int_{B_o(2\widetilde{r})} f^\tau \end{aligned} \tag{3.6}$$

for some constants C_3 and C_4 independent of $\widetilde{r}$. Here we have used (3.3) and Lemma 1.1. For any $x \in M$ with $r = r(x)$, if we take $\widetilde{r} = R > 2r$ in (3.6), we have

$$\begin{aligned} e^{u(x,0)}|\phi(x)|^{2\tau} &= e^{u(x,0)} f^\tau(x) \\ &\leq \frac{C_5(1+R)^{\frac{1}{2}\epsilon}}{V_o(2R)} \int_{B_o(2R)} f^\tau \\ &= \frac{C_5(1+R)^{\frac{1}{2}\epsilon}}{V_o(2R)} \int_{B_o(2R)} |\phi|^{2\tau} \\ &= O\left(R^{-\frac{1}{2}\epsilon}\right) \end{aligned}$$

for some constant C_5 independent of r and R. Let $R \to \infty$, we conclude that $\phi \equiv 0$. q.e.d.

Remark 3.1. If $\int_M |\phi|^{2\tau} < \infty$, then obviously ϕ satisfies (3.2) because the volume growth of M is at least linear by [27].

As an application, we have:

Corollary 3.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative Ricci curvature and let L be a holomorphic line bundle* *over M with Hermitian metric h. Let $\rho = \sqrt{-1}\Omega_{i\bar{j}} dz^i \wedge d\bar{z}^j$ be the curvature form of L, h and let Ω be the trace of ρ. Suppose $\rho \geq 0$ and*

$$\limsup_{r\to\infty} \frac{r^2}{V_o(r)} \int_{B_o(r)} \Omega = 0.$$

Then $\rho^m \equiv 0$.

Proof. Suppose $\rho^m \neq 0$ at some point, then there exists a positive integer ℓ and a nontrivial holomorphic section ϕ of L^ℓ such that $|\phi| \in L^2(M)$ by Corollary 3.3 in [22]. Note that the trace of the curvature form of L^ℓ is $\ell\Omega$. By Theorem 3.1 and the assumption on Ω, we have a contradiction. q.e.d.

Later in Section 5, we will discuss conditions so that L is actually flat, see Proposition 5.2.

If we take L to be the anti-canonical bundle of M, then we have the following generalization of the first part of Corollary 3.5 in [22].

Corollary 3.2. *Let M^m be a complete noncompact Kähler manifold with nonnegative Ricci curvature and let $\mathcal{R}$ be the scalar curvature of M. Suppose*

$$\limsup_{r\to\infty} \frac{r^2}{V_o(r)} \int_{B_o(r)} \mathcal{R} = 0. \tag{3.7}$$

Then the Ricci form ρ of m satisfies $\rho^m = 0$.

Remark 3.2. (i) If M is nonparabolic and if $\mathcal{R}$ is integrable, then (3.7) is true. Hence Corollary 3.2 is a generalization of Corollary 3.5 and Theorem 3.6 in [22] for the case that M has nonnegative Ricci curvature. (ii) As observed in [4], from the arguments in [23], if (3.7) is true for all base point o so that the convergence is uniform and if the holomorphic bisectional curvature of M is bounded and nonnegative, then M is flat. In the above corollary, we only assume that the Ricci curvature is nonnegative and we do not assume that the scalar curvature is bounded. The result is weaker and it is interesting to see whether M is actually Ricci flat in this case. In fact, for Riemannian case, it is proved by Chen and Zhu [3] that if (3.7) is true uniformly and if the Riemannian manifold is locally conformally flat then the manifold is flat.

In the next result, we will relax the assumption that M has nonnegative Ricci curvature.

Theorem 3.2. *Let M^m be a complete noncompact nonparabolic Kähler manifold with complex dimension m. Let L be a Hermitian holomorphic line bundle and let Ω be the trace of the curvature form of L. For any $1 \le p \le m$, let*

$$\mathcal{S}(x) = \min_{1 \le i_1 < i_2 < \cdots < i_p \le m} (\gamma_{i_1}(x) + \gamma_{i_2}(x) + \cdots + \gamma_{i_p}(x)),$$

where γ_j are the eigenvalues of the Ricci form of M, and let $S \equiv 0$ if $p = 0$. Suppose

$$\int_M (-\Omega + \mathcal{S})_- < \infty,$$

in particular, suppose

$$\int_M (\Omega_+ + \mathcal{S}_-) < \infty$$

where Ω_+ is the positive part of Ω, $\mathcal{S}_-$ is the negative part of $\mathcal{S}$ and $(-\Omega + \mathcal{S})_-$ is the negative part of $-\Omega + S$. Then any holomorphic $(p, 0)$ form ϕ with value in L must be trivial if ϕ satisfies

$$\int_{B_o(r)} |\phi|^{2\tau} = o(r^2)$$

as $r \to \infty$ for some $\tau > 0$.

Proof. The result follows immediately from Lemma 3.1 and the result in [18, Corollary 2.2]. For the sake of completeness, we prove the result in the same spirit as the proof of Theorem 3.1.

First, we claim that there is a solution u of $\Delta u = (-\Omega + \mathcal{S})_-$ with $u \le 0$. To prove the claim, let $R > 0$ be fixed, and let G be the minimal positive Green's function of M. G exists because M is nonparabolic. Let $\sigma = 4\,(-\Omega + \mathcal{S})_-$, then for $x \in B_o(R)$,

$$\begin{aligned}\int_M G(x,y)\sigma(y)dy &= \int_{B_o(2R)} G(x,y)\sigma(y)dy + \int_{M \setminus B_o(2R)} G(x,y)\sigma(y)dy \\ &\le C_1 \left(1 + \int_{M \setminus B_o(2R)} G(o,y)\sigma(y)dy\right) \\ &\le C_2\end{aligned}$$

for some constants C_1, C_2 independent of x with $x \in B_o(R)$. Here we have used Harnack inequality, the fact that $\sup_{M \setminus B_o(2R)} G(o, y) < \infty$

(cf. [14, p.1138]), the fact that for $\sigma \in L^1(M)$, and that there exists a constant C_3 such that

$$\int_{B_o(2R)} G(x,y)\sigma(y)dy \le C_3$$

for all $x \in B_o(R)$. Hence $\int_M G(x,y)\sigma(y)dy$ is locally bounded and $u(x) = -\int_M G(x,y)\sigma(y)$ is well defined with $\Delta u = \sigma$. Obviously $u \le 0$.

To complete the proof of the theorem, let ϕ be a holomorphic $(p,0)$ form such that $\int_{B_o(r)} |\phi|^{2\tau} = o(r^2)$. As in the proof of Theorem 3.1, for any $\epsilon > 0$, let $f = |\phi|^2$ and let $g = (f+\epsilon)^\tau$. If $v = e^{\tau u} g$, then

$$v\Delta v \ge 0.$$

We can then apply the method in [27], [18]. Namely, multiplying the above inequality by a suitable cut off function, we have

$$\int_{B_o(r)\cap M_a} |\nabla v|^2 \le \int_{B_o(r)} |\nabla v|^2 \le \frac{C_4}{r^2}\int_{B_o(2r)} v^2$$

where $M_a = \{x \in M|\ f(x) > a\}$ and C_4 is a constant independent of r and a. Let $\epsilon \to 0$, we have

$$\int_{B_o(r)\cap M_a} |\nabla w|^2 \le \frac{C_4}{r^2}\int_{B_o(2r)} w^2 \tag{3.8}$$

where $w = e^{\tau u} f^\tau$. By the assumption on $f = |\phi|^2$ and the fact that $\tau > 0$, $u \le 0$, we conclude that

$$\int_{B_o(2r)} w^2 = \int_{B_o(2r)} (e^{\tau u} f^\tau)^2 = o(r^2) \tag{3.9}$$

as $r \to \infty$. Combine this with (3.8) and let $r \to \infty$, $|\nabla w| \equiv 0$ on M_a. Since a is arbitrary, w must be a constant. Since M is nonparabolic,

$$\limsup_{r\to\infty} \frac{V_o(r)}{r^2} > 0.$$

and hence (3.9) implies that w must be identically zero. q.e.d.

Theorem 3.2 generalizes Theorem 2.3 in [22] which deals with holomorphic section of line bundles. By taking L to be the trivial bundle with flat metric, it is easy to see that the theorem also generalizes part of Theorem 2 in [10] for the case of holomorphic p-forms.

4. Liouville property of plurisubharmonic functions

In this section, we will apply the results in Section 1 and Section 2 to study plurisubharmonic functions on a complete noncompact Kähler manifold M^m with nonnegative Ricci curvature, where m is the complex dimension of M. In [22, Proposition 4.1], it was proved that if u is a plurisubharmonic function such that

$$\limsup_{r(x)\to\infty} \frac{u(x)}{\log r(x)} = 0$$

then

$$(\partial\bar{\partial}u)^m = 0 \tag{4.1}$$

on M. Let us first prove a more general result as an application of Theorem 3.2.

Proposition 4.1. *Let M^m be a complete noncompact nonparabolic Kähler manifold such that the scalar curvature $\mathcal{R}$ satisfies*

$$\int_M \mathcal{R}_- < \infty.$$

where $\mathcal{R}_-$ is the negative part of $\mathcal{R}$. Suppose u is plurisubharmonic function on M such that

$$\limsup_{x\to\infty} \frac{u(x)}{\log r(x)} = 0.$$

Then $(\partial\bar{\partial}u)^m = 0$.

Proof. Suppose $\sqrt{-1}\partial\bar{\partial}u(x_0) > 0$ at some point x_0. We can find a coordinate neighborhood U with holomorphic coordinates $z(x)$ where $z = (z_1, \ldots, z_m)$ so that x_0 corresponds to the origin and U corresponds to $|z| < 4$, and that $\sqrt{-1}\partial\bar{\partial}u > 0$ in U. Let $\lambda \geq 0$ be a smooth function on U such that $\lambda(z(x)) = 1$ in $|z(x)| < 1$ and $\lambda = 0$ outside $|z(x)| = 2$. Let ϕ be the function on M such that $\phi(x) = 2(m+1)\lambda(z(x))\log|z(x)|$ on U and zero outside U. Then ϕ is smooth on $M \setminus \{x_0\}$ with compact support. Since $\partial\bar{\partial}\phi \geq 0$, in the weak sense, within $\{|z(x)| \leq 1\}$ and $\sqrt{-1}\partial\bar{\partial}u > 0$ in U. Hence there is a positive constant A such that if $\psi = Au + \phi$ then $\sqrt{-1}\partial\bar{\partial}\psi \geq \epsilon\omega$ for some nonnegative continuous function ϵ which is positive on $|z(x)| \leq 1$. Here ω is the Kähler form of M. Let $\rho \geq 0$ be a smooth cutoff function such that $\rho(z(x)) = 1$ if

$|z(x)| \leq 1/2$ and $\rho = 0$ outside $|z(x)| = 1$. Let $\eta = \rho dz_1 \wedge \cdots \wedge dz_m$. It is easy to see that

$$\int_M \frac{||\bar{\partial}\eta||^2}{\epsilon} e^{-\psi} < \infty.$$

By Theorem 5.1 in [7], there is an $(m, 0)$ form τ such that $\bar{\partial}\tau = \bar{\partial}\eta$ and

$$\int_M ||\tau||^2 e^{-\psi} \leq C \int_M \frac{||\bar{\partial}\eta||^2}{\epsilon} e^{-\psi} < \infty.$$

Note, here we do not need assumptions on the curvature of M because we are dealing with $(m, 0)$ forms. By the definition of ψ, we conclude that $\tau(x_0) = 0$. Hence $\widetilde{\eta} = \tau - \eta$ is holomorphic $(m, 0)$ form which is nontrivial. Moreover, by the above inequality and the growth assumption on u, we have

$$\int_{B_o(r)} |\widetilde{\eta}|^2 = O(r).$$

as $r \to \infty$. This contradicts Theorem 3.2 with L being the trivial line bundle with flat metric. q.e.d.

Because of Ni's result [22], it is interesting to see whether u is actually constant if u satisfies (4.1).

Let u be a plurisubharmonic function and let $f = \Delta u$. As before, let $k(x, t) = 1/V_x(t) \int_{B_x(t)} f$ and $k(t) = k(o, t)$. First, we assume that M supports a strictly plurisubharmonic function.

Theorem 4.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative Ricci curvature. Let u be a plurisubharmnonic function satisfying* (4.1) *such that $f(x) = \Delta u(x) \leq Cr^{-2}(x)$ for some constant $C > 0$ for all x. Suppose M supports a strictly plurisubharmonic function. Then u must be constant if one of the following is true:*

(a) *u is bounded.*

(b) *$\sup_{B_o(r)} u = o(r)$ as $r \to \infty$ and there exist $r_i \to \infty$ and a constant C such that $r_i \int_{r_i}^{\infty} k(t)dt \leq C$.*

(c) *$u(x) \leq a \log r(x)$ for some constant a for all x with $r(x) \geq 2$ and M is nonparabolic with Green's function satisfying* (1.1)*.*

Proof. (a) Since $f(x) \leq Cr^{-2}(x)$, by Theorem 2.3, we have

$$\lim_{x \to \infty} u(x) = \sup_M u.$$

By the minimum principle of [2], [1], u must be a constant.

Suppose (b) is true. Then $\int_0^\infty k(t)dt < \infty$. Since $f(x) \le Cr^{-2}(x)$, we can find a solution v of $\Delta v = f$ such that $|v(x)| = o(r(x))$ as $x \to \infty$ by Proposition 1.1 and Theorem 1.2. Hence $u - v$ is a harmonic function and $\sup_{B_o(r)}(u - v) = o(r)$ because $\sup_{B_o(r)} u = o(r)$. By the gradient estimate [5, Theorem 6], $u - v$ must be a constant. Without loss of generality, we may assume that $u = v$ which is the solution obtained in Theorem 1.2. In particular,

$$u(x) \ge -\alpha_2 r \int_{2r}^\infty k(t)dt - \beta_2 \int_0^{\frac{1}{2}r} tk(x,t)dt + \beta_3 \int_0^{2r} tk(t)dt$$

where α_2 and β_2 are positive constants depending only on m and $r = r(x)$. Since $f(x) \le Cr^{-2}(x)$,

$$u(x) \ge -C_1 - \alpha_2 r \int_{2r}^\infty k(t)dt + \beta_3 \int_0^{2r} tk(t)dt \tag{4.2}$$

for some constant C_1 independent of x. By the assumption, there exist $r_i \to \infty$ and a constant C_2 such that

$$r_i \int_{r_i}^\infty k(t)dt \le C_2. \tag{4.3}$$

Hence

$$\inf_{\partial B_o(\frac{1}{2}r_i)} u \ge -C_3 + \beta_3 \int_0^{r_i} tk(t)dt$$

for some constant C_3 for all i. Suppose $\int_0^\infty tk(t)dt = \infty$, then $u(0) = \infty$ by the minimum principle of [2], [1]. This is impossible. Hence $\int_0^\infty tk(t)dt < \infty$. By the minimum principle again, we conclude that u is bounded from below. By Theorem 1.2, u also has an upper bound

$$u(x) \le \alpha_1 r \int_{2r}^\infty k(t)dt + \beta_1 \int_0^{2r} tk(t)dt$$

for some constants α_1 and β_1 depending only on m, where $r = r(x)$. By (4.3), the fact that $\int_0^\infty tk(t)dt < \infty$ and the maximum principle for subharmonic function, we conclude that u must be also bounded from above. Hence u is constant by (a).

Suppose (c) is true. By the assumption on the upper bound of $u(x)$ and Theorem 2.1, there exists a constant C_4 such that

$$\int_0^R tk(t)dt \le C_4 \log R \tag{4.4}$$

for all R large enough. It is sufficient to show that there exist $r_i \to \infty$ such that (4.3) is true. First note that

$$
\begin{aligned}
(4.5) \qquad I &= \int_r^\infty t^{-1} \left(\int_t^\infty \frac{s}{V_o(s)}\, ds \right) \left(\int_{\partial B_o(t)} f \right) dt \\
&\leq C_5 \int_r^\infty \frac{A(t)}{tV(t)} dt \\
&\leq C_6 r^{-1}
\end{aligned}
$$

for some constants C_5 and C_6 independent of r. Here we have used the fact that $f(x) \leq Cr^{-2}(x)$ and the fact that $tA(t) \leq 2mV(t)$. On the other hand,

$$
\begin{aligned}
(4.6) \qquad I &= t^{-1} \left(\int_t^\infty \frac{s}{V_o(s)}\, ds \right) \left(\int_{B_o(t)} f \right) \Bigg|_r^\infty \\
&\quad + \int_r^\infty t^{-2} \left(\int_t^\infty \frac{s}{V_o(s)}\, ds \right) \left(\int_{B_o(t)} f \right) dt \\
&\quad + \int_r^\infty \frac{1}{V_o(t)} \left(\int_{B_o(t)} f \right) dt \\
&\geq -\frac{C_7 r}{V_o(r)} \left(\int_{B_o(r)} f \right) + \int_r^\infty k(t) dt
\end{aligned}
$$

for some positive constant. Here we have used the fact that the positive Green's function satisfies (1.1). By (4.4), (4.5), (4.6) it is easy to see that (4.3) is true for some $r_i \to \infty$. q.e.d.

By the method of [20, pp.195-199], we can obtain the following Liouville result for bounded plurisubharmonic functions on Kähler manifold with nonnegative sectional curvature with maximal volume growth.

Proposition 4.2. *Let M^m be a complete noncompact Kähler manifold with nonnegative sectional curvature and with maximal volume growth. Let u be a bounded plurisubharmnonic function satisfying* (4.1) *such that $f = \Delta u$ satisfies $f(x) \leq Cr^{-2}(x)$ for some constant $C > 0$ for all x. Then u must be constant.*

Proof. Since u is bounded and $f(x) \leq Cr^{-2}(x)$, u is asymptotically constant by Theorem 2.4. Using the method in [20], we conclude that u is constant. q.e.d.

Suppose M has positive holomorphic bisectional curvature, then M supports a strictly plurisubharmonic function by [9]. Hence we have the following.

Corollary 4.1. *Let M^m be a complete noncompact Kähler manifold with positive biholomorphic sectional curvature. Let u be a plurisubharmonic function satisfying* (4.1) *such that $f(x) \le Cr^{-2}(x)$. Suppose one of the conditions* (a), (b) *or* (c) *in Theorem* 4.1 *is true, then u must be constant.*

5. The Poincaré-Lelong equation

Let M^m be a complete Kähler manifold with nonnegative bisectional curvature. In [20, Theorem 1.1], it was proved that if M has maximal volume growth and if ρ is a closed $(1,1)$ form on M such that the norm $||\rho||$ of ρ satisfies $||\rho(x)|| \le Cr^{-2}(x)$ for some constant for all x, then one can solve the Poincaré-Lelong equation by solving $1/2\Delta u = trace(\rho)$. In this section, we will apply the results in Section 1 and Section 2 to show that given a closed $(1,1)$ form ρ with trace f, one can solve the following Poincaré-Lelong equation under rather general assumptions on ρ:

$$\sqrt{-1}\partial\bar{\partial}u = \rho. \tag{5.1}$$

We will also give some applications of the result.

In this section, m always denotes the complex dimension of M^m.

Theorem 5.1. *Let M^m be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let ρ be a real closed* $(1,1)$ *form with trace f. Suppose $f \ge 0$ and ρ satisfies the following conditions:*

$$\int_0^\infty \frac{1}{V_o(t)} \int_{B_o(t)} ||\rho|| dt < \infty, \tag{5.2}$$

and

$$\liminf_{r\to\infty} \frac{1}{V_o(r)} \int_{B_o(r)} ||\rho||^2 = 0. \tag{5.3}$$

Then there is a solution u of the Poincaré-Lelong Equation (5.1). *More-*

over, for any $0 < \epsilon < 1$*, u satisfies*

$$\begin{aligned}\alpha_1 r \int_{2r}^{\infty} k(t)dt + \beta_1 \int_0^{2r} tk(t)dt \geq u(x)& \\ \geq -\alpha_2 r \int_{2r}^{\infty} k(t)dt& \\ - \beta_2 \int_0^{\epsilon r} tk(x,t)dt& \\ + \beta_3 \int_0^{2r} tk(t)dt&\end{aligned}$$

for some positive constants $\alpha_1(m)$*,* $\alpha_2(m,\epsilon)$ *and* $\beta_i(m)$*,* $1 \leq i \leq 3$*, where* $r = r(x)$*. Here as before,* $k(x,t) = 1/V_x(t)\int_{B_x(t)} f$ *and* $k(t) = k(o,t)$*, where* $o \in M$ *is a fixed point. Moreover, the gradient of* u *satisfies the estimates in Theorem* 1.3.

Proof. Let us first consider the case that M is nonparabolic and its Green's function satisfies (1.1). By (5.2), since f is the trace of ρ we have

$$\int_0^{\infty} k(t)dt < \infty.$$

By Theorem 1.1 we can find a solution u of $\frac{1}{2}\Delta u = f$. Moreover, u satisfies the estimates in Theorems 1.1 and 1.3. We claim that u satisfies (5.1). By (5.3), we can find $R_j \to \infty$ such that

$$\lim_{j\to\infty} \frac{1}{V_o(R_j)} \int_{B_o(R_j)} ||\rho||^2 = 0.$$

It is known that $||\sqrt{-1}\partial\bar{\partial}u - \rho||^2$ is subharmonic, see [20, p.187] for example. For any $x \in M$, if j is large enough so that $R_j \geq 8r(x)$, then by the mean value inequality for subharmonic function in [13, Theorem 2.1], using Theorem 1.3(iii) we have

$$\begin{aligned}||\sqrt{-1}\partial\bar{\partial}u - \rho||^2(x) &\leq \frac{C_1}{V_x(\frac{R_j}{8})} \int_{B_x(\frac{R_j}{8})} ||\sqrt{-1}\partial\bar{\partial}u - \rho||^2 \\ &\leq \frac{C_2}{V_o(\frac{R_j}{4})} \int_{B_o(\frac{R_j}{4})} \left(|\nabla^2 u|^2 + ||\rho||^2\right) \\ &\leq C_3 \left[R^{-2}\left(\int_{R_j}^{\infty} k(t)dt\right)^2 + \frac{1}{V_o(R_j)}\int_{B_o(R_j)} ||\rho||^2\right] \\ &\to 0\end{aligned}$$

as $j \to \infty$, where $C_1 - C_3$ are constants independent of j. Hence $\sqrt{-1}\partial\bar{\partial}u \equiv \rho$ and the proof is completed in this case.

In general, let $\widetilde{M} = M \times \mathbb{C}^2$. Then $\widetilde{M}$ is nonparabolic and its Green's function satisfies (1.1). We may consider ρ as a closed $(1,1)$ form on $\widetilde{M}$. Moreover, the trace of ρ is still f which is independent of the variable in $\mathbb{C}^2$. It is easy to see that ρ still satisfies (5.2) and (5.3) by Lemma 1.1. Hence we can find $\widetilde{u}$ such that $\sqrt{-1}\partial\bar{\partial}\widetilde{u} = \rho$. It is easy to see that for any fixed $x_0 \in M$, $\widetilde{u}(x_0, \cdot)$ is pluriharmonic on $\mathbb{C}^2$. Moreover, since $\widetilde{u}$ satisfies the estimates in the theorem, we have

$$\limsup_{y \in \mathbb{C}^2, y \to \infty} \frac{\widetilde{u}(x_0, y)}{|y|} = 0.$$

Hence $\widetilde{u}(x_0, \cdot)$ is constant on $\mathbb{C}^2$ by Harnack inequality, and $\widetilde{u}(x, y) = u(x)$ which satisfies (5.1) on M. Moreover, u satisfies the estimates in the theorem. The estimates of the gradient of u follows from the construction and Theorem 1.3. q.e.d.

Remark 5.1. (i) If ρ satisfies (5.2), and if

$$\liminf_{r \to \infty} r^{-1} \sup_{\partial B_o(r)} ||\rho|| < \infty,$$

then (5.3) will also be satisfied. In particular, $||\rho||$ may be unbounded. (ii) By Remark 1.1, it is easy to see that the theorem is still true without the assumptions that ρ is real and f is nonnegative. What we need is (5.2) and (5.3).

In the following we give some applications of the theorem.

(I) Steinness of Kähler manifolds.

Theorem 5.2. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. M is Stein if one of the following is true:*

(i) *There exists a closed real $(1,1)$ form ρ which is positive everywhere such that $||\rho||(x) \le Cr^{-2}(x)$ and $k(t) \le Ct^{-2}$ for some constant C for all x and t. Here f is the trace of ρ and $k(t)$ is as in Theorem 5.1.*

(ii) *M has nonnegative sectional curvature and there exists a real closed $(1,1)$ form ρ which is positive everywhere and satisfies (5.2) and (5.3).*

Proof. (i) By Theorem 5.1, we can solve the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = \rho$. Since ρ is positive everywhere, u is a strictly plurisubharmonic function. Since u satisfies the estimate in the theorem, by Corollary 1.1 and Corollary 3.1, we see that u is an exhaustion function. Hence M is Stein.

To prove (ii), by the assumption, we can obtain a strictly plurisubharmonic function as before. Since the sectional curvature is nonnegative, one can apply the method in [8] to show that the manifold is Stein.

q.e.d.

Corollary 5.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that M has positive Ricci curvature. Then M is Stein if M satisfies one of the following:*

(i) *The Ricci form ρ satisfies $||\rho||(x) \le Cr^{-2}(x)$ and $1/V_o(t)\int_{B_o(t)}||\rho|| \le Ct^{-2}$ for some constant C for all $x \in M$ and $t > 0$.*

(ii) *M has nonnegative sectional curvature and ρ satisfies* (5.2) *and* (5.3).

Corollary 5.1(i) was basically proved in [19] (see also [20]) under the assumption is that $||\rho||(x) \le Cr^{-2}(x)$ and M has maximal volume growth, which will imply that $1/V_o(t)\int_{B_o(t)}||\rho|| \le Ct^{-2}$ provided $m \ge 2$. Corollary 5.1(i) seems to be more general at first sight, but we will see later that if $m \ge 2$, the assumptions in Corollary 5.1(i) will imply that M has maximal volume growth. Also, it was proved in [8] that if M has nonnegative sectional curvature and has positive holomorphic bisectional curvature, then M is Stein. In (ii) of the corollary, we still have the same assumption on sectional curvature, but we replace the assumption on the positivity of holomorphic bisectional curvature by the assumption that the Ricci curvature is positive and whose norm decays faster than linearly in a certain sense. We would like to mention that in [4], it is proved that if M has nonnegative holomorphic bisectional curvature, has maximal volume growth such that its scalar curvature $\mathcal{R}$ satisfies $\mathcal{R}(x) \le Cr^{-1-\epsilon}(x)$ for some constants C and $\epsilon > 0$ for all x, then M is Stein.

(II) Plurisubharmonic functions revisited

Proposition 5.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and positive*

Ricci curvature with Ricci form ρ satisfies (5.2) *and* (5.3). *Let u be a plurisubharmonic function satisfying* (4.1) *such that $f(x) \le Cr^{-2}(x)$. Suppose one of* (a), (b) *or* (c) *in Theorem* 4.1 *is true, then u must be constant.*

Proof. By Theorem 5.1, (5.1) has a solution with ρ being the Ricci form of M. Since $\rho > 0$ everywhere, M supports a strictly plurisubharmonic function. The result follows from Theorem 4.1. q.e.d.

Proposition 5.2. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and let L be a holomorphic line bundle over M with Hermitian metric h with nonnegative curvature. Let Ω be the trace of the curvature of L with respect to h. Suppose*

(i) *M supports a strictly plurisubharmonic function;*

(ii)
$$\limsup_{r\to\infty} \frac{r^2}{V_o(r)} \int_{B_o(r)} \Omega = 0;$$

and

(iii) $\Omega(x) \le Cr^{-2}(x)$.

Then L is flat.

Proof. By Theorem 5.1, we can find a solution u of (5.1) satisfying the estimate in the theorem with $f = \Omega$ and ρ be the curvature form of L. Since $\rho^m = 0$ by Theorem 3.1, u must be constant by Theorem 4.1. Hence $\rho \equiv 0$. q.e.d.

(III) Volume and curvature estimates

Lemma 5.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let ρ be a real closed* $(1,1)$ *form with trace f. Suppose $\rho \ge 0$ and $\rho > 0$ at some point o and satisfies* (5.2) *and* (5.3). *Then for any $\alpha > 2$ and $p \ge 1$, there exists constants $C_1 > 0$, $C_2 > 0$ independent of R and C^* independent of R and α such that*

$$\frac{C_1 R}{V_o(R)} \le \left[C^* \int_{\alpha R}^{\infty} k(t)dt + C_2 R \left(\frac{1}{V_o(R)} \int_{B_o(R)} f^p \right)^{\frac{1}{p}} \right] \cdot \left[\frac{1}{V_o(R)} \int_{B_o(2R)\setminus B_o(\frac{R}{2})} f^{q(m-1)} \right]^{\frac{1}{q}}$$

where $q = p/(p-1)$ *if* $p > 1$*, and if* $p = 1$*, then the last integral is interpreted as* $\sup_{B_o(2R)\setminus B_o(\frac{R}{2})} f^{m-1}$.

Proof. By Theorem 5.1 we can solve the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = \rho$. Let us first assume that $p > 1$. Since $\rho \geq 0$ and is strictly positive at o, there is a constant $C_1 > 0$ such that for all $R \geq 1$

$$\begin{aligned} C_1 &\leq \int_{B_o(R)} \rho^m \\ &= \int_{B_o(R)} \left(\sqrt{-1}\partial\bar{\partial}u \wedge \rho^{m-1}\right) \\ &= \int_{\partial B_o(R)} \sqrt{-1}\bar{\partial}u \wedge \rho^{m-1} \\ &\leq \left(\int_{\partial B_o(R)} |\nabla u|^p\right)^{\frac{1}{p}} \left(\int_{\partial B_o(R)} f^{q(m-1)}\right)^{\frac{1}{q}} \end{aligned}$$

for some constant C_4 independent of R and $q = p/(p-1)$. For $R \geq 2$, integrating from $\frac{1}{2}R$ to R, there is a constant $C_5 > 0$ independent of R and α such that

$$\frac{C_1}{2}R \leq \left(\int_{B_o(R)\setminus B_o(\frac{R}{2})} |\nabla u|^p\right)^{\frac{1}{p}} \left(\int_{B_o(R)\setminus B_o(\frac{R}{2})} f^{q(m-1)}\right)^{\frac{1}{q}}. \tag{5.4}$$

By Theorem 5.1, the gradient of u satisfies

$$\frac{1}{V_o(R)}\int_{B_o(R)} |\nabla u|^p \leq C^*\left(\int_{\alpha R}^{\infty} k(t)dt\right)^p + \frac{C_2 R^p}{V_o(R)}\int_{B_o(\alpha R)} f^p$$

where $\alpha > 2$ is a constant, C^* is a constant independent of R and α and C_2 is a constant independent of R. Combine this with (5.4), the theorem is true if $p > 1$. The case that $p = 1$ is similar. q.e.d.

Theorem 5.3. *Let* M^m *be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let* ρ *be a closed real* $(1,1)$ *form with trace* f*. Suppose* $\rho \geq 0$ *and* $\rho > 0$ *at some point* o *and suppose that*

$$\left(\frac{1}{V_o(r)}\int_{B_o(r)} f^p\right)^{\frac{1}{p}} \leq Cr^{-1-\epsilon} \tag{5.5}$$

and

$$\left(\frac{1}{V_o(r)}\int_{B_o(r)\setminus B_o(\frac{r}{2})} f^{q(m-1)}\right)^{\frac{1}{q(m-1)}} \le Cr^{-1-\epsilon} \tag{5.6}$$

for some $1 \le p \le m$ *and for some constant* C *for all* r, *where* $0 < \epsilon \le 1$, $q = p/(p-1)$. *If* $p = 1$, (5.6) *means that*

$$\sup_{B_o(r)\setminus B_o(\frac{r}{2})} f \le Cr^{-1-\epsilon}.$$

Then

$$V_o(R) \ge CR^{m(1+\epsilon)}$$

for some constant $C > 0$ *for all* $R \ge 2$. *If, in addition,*

$$V_o(R) \le C'R^{m(1+\epsilon)}$$

for some constant C' *for all* R, *then*

$$\left(\frac{1}{V_o(r)}\int_{B_o(r)} f^p\right)^{\frac{1}{p}} \ge C''r^{-1-\epsilon}$$

for some constant $C'' > 0$ *for all* r *large enough. In particular, if* $\epsilon = 1$, *then* M *has maximal volume growth and*

$$\left(\frac{1}{V_o(r)}\int_{B_o(r)} f^p\right)^{\frac{1}{p}} \ge C''r^{-2}.$$

Proof. Let us consider the case that $p = 1$. Then ρ satisfies the conditions in Lemma 5.1. Hence we have for $R \ge 2$,

$$\begin{aligned}\frac{C_1 R}{V_o(R)} &\le \left(\frac{C^*}{\alpha R} + \frac{C_2 R}{V_o(R)}\int_{B_o(R)} f\right)\cdot R^{-(m-1)(1+\epsilon)}\\ &\le C_3 R^{-m(1+\epsilon)+1}\end{aligned}$$

where C_1 and C^* are positive constants independent of α and R, and C_2 and C_3 are constants independent of R. From this it is easy to see that

$$V_o(R) \ge CR^{m(1+\epsilon)}$$

for some constant $C > 0$ for all $R \geq 2$. If in addition,

$$V_o(R) \leq C' R^{m(1+\epsilon)}$$

for some constant C' for all R. Then for R large enough, we have

$$\frac{C_4}{R^\epsilon} \leq \frac{C^*}{\alpha R} + \frac{C_2 R}{V_o(R)} \int_{B_o(R)} f$$

where $C_4 > 0$ is a constant independent of R and α. If we take α large enough, we can conclude that

$$\frac{1}{V_o(R)} \int_{B_o(R)} f \geq C_5 R^{-1-\epsilon}$$

for some constant $C_5 > 0$ independent of R. Suppose $p > 1$. By (5.5) we have

$$\frac{1}{V_o(r)} \int_{B_o(r)} ||\rho|| \leq C r^{-1-\epsilon}.$$

Let $q^* = q(m-1)$. Since $p \leq m$, we have $q* \geq m \geq 2$. Let $k \geq 1$ be an integer. By (5.6), for $\ell \leq k$, we have

$$\int_{B_o(2^\ell)\setminus B_o(2^{\ell-1})} ||\rho||^{q^*} \leq C 2^{-\ell q^*(1+\epsilon)} \times V_o(2^\ell)$$

where C is a constant independent of ℓ. Hence for any $k_0 \leq k$, we have

$$\frac{1}{V_o(2^k)} \int_{B_o(2^k)\setminus B_o(1)} ||\rho||^{q^*} \leq C \frac{1}{V_o(2^k)} \sum_{\ell=1}^{k_0} 2^{-\ell q^*(1+\epsilon)} \times V_o(2^\ell) + \sum_{\ell=k_0+1}^{\infty} 2^{-\ell q^*(1+\epsilon)}.$$

Fix k_0, let $k \to \infty$ and then let $k_0 \to \infty$, we have

$$\lim_{r\to\infty} \frac{1}{V_o(r)} \int_{B_o(r)} ||\rho||^{q^*} = 0.$$

Since $q^* \geq 2$, we have

$$\lim_{r\to\infty} \frac{1}{V_o(r)} \int_{B_o(r)} ||\rho||^2 = 0.$$

Hence the conditions of Theorem 5.1 are satisfied and we can proceed as in the case that $p = 1$ and complete the proof of the theorem. q.e.d.

In particular, we have:

Corollary 5.2. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose the Ricci curvature is positive at some point and scalar curvature $\mathcal{R}$ satisfies $\mathcal{R}(x) \le Cr^{-2}(x)$ and $1/V_o(t)\int_{B_o(t)} \mathcal{R} \le Ct^{-2}$. Then M has maximal volume growth, and*

$$\liminf_{r\to\infty} \frac{r^2}{V_o(r)} \int_{B_o(r)} \mathcal{R} > 0.$$

Proof. Set $\epsilon = 1$ in Theorem 5.3, the result follows easily by noting that $V_o(r) \le Cr^{2m}$, by the volume comparison. q.e.d.

This result says that under the assumptions of the corollary, even though the scalar curvature decays, but it actually cannot decay too fast. One should compare this with Corollary 3.1.

Remark 5.2. If we assume that ρ has rank $\ell \ge 1$ at o rather than ρ is positive at o and if we assume that (5.5) and (5.6) are true with $\epsilon = 1$ and with m replaced by ℓ, then we can modify the proof of Lemma 5.1 and conclude that $V_o(r) \ge Cr^{2\ell}$ for some positive constant C for all r. In particular, if M is not Ricci flat, then $V(r) \ge Cr^2$ for some constant $C > 0$ for all $r \ge 1$.

(IV) Positive (1,1) forms satisfying a pinching condition

Theorem 5.4. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature, with $m \ge 2$. Let $\rho \ge 0$ be a closed real $(1,1)$ form on M such that $\inf_{1\le j\le m} \lambda_j(x) \ge \epsilon f(x)$ for some positive constant ϵ, for all x, where λ_j are the eigenvalues of ρ and f is the trace of ρ. Define $k(x,t) = 1/V_x(t)\int_{B_x(t)} f$ and $k(t) = k(o,t)$ as before. Then $\rho \equiv 0$ if one of the following is satisfied:*

(i) *$k(t) \le Ct^{-2}$ and $f(x) \le Cr^{-2}(x)$ for some constant C for all $t > 0$ and x.*

(ii) *$V_o(r) \le Cr^m$ for some constant C for all r, ρ satisfies (5.2) and (5.3) and there exists a constant C such that $\int_0^\infty k(x,t)dt \le C$ for all x.*

Proof. If (i) is true and if $f > 0$ at some point, then M has maximal volume growth by Theorem 5.3. Combining this with the assumptions in (i) and the fact that $m \geq 2$, it is not hard to prove that $k(x,t) \leq Ct^{-2}$ for some constant C for all t and x. By Theorem 5.1, (5.1) has a solution u. Moreover, by the gradient estimate in Corollary 1.2, there exists a constant C_1 such that

$$|\nabla u(x)| \leq C_1 r^{-1}(x). \tag{5.7}$$

As in the proof of Theorem 5.3, using the pinching condition that $\lambda_j \geq \epsilon f$, we have

$$\begin{aligned}
\int_{B_o(r)} f^m &\leq C_2 \int_{B_o(r)} \rho^m \\
&= C_2 \int_{B_o(r)} \left(\sqrt{-1}\partial\bar{\partial} u \wedge \rho^{m-1}\right) \\
&= C_2 \int_{\partial B_o(r)} \sqrt{-1}\bar{\partial} u \wedge \rho^{m-1} \\
&\leq C_3 \left(\int_{\partial B_o(r)} |\nabla u|^m\right)^{\frac{1}{m}} \left(\int_{\partial B_o(r)} f^m\right)^{\frac{m-1}{m}} \\
&\leq C_4 \left(r \int_{\partial B_o(r)} f^m\right)^{\frac{m-1}{m}}
\end{aligned} \tag{5.8}$$

for some constants $C_2 - -C_4$ independent of r, where we have used (5.7) and the fact that $A_o(r) \leq Cr^{2m-1}$ for some constant depending only on m. Since $f > 0$ at some point, there exists $r_0 > 0$ such that $F(r) = \int_{B_o(r)} f^m > 0$ for all $r \geq r_0$. By (5.8), we have

$$\frac{F'}{F^{\frac{m}{m-1}}} \geq C_5 r^{-1}$$

for some positive constant C_5 for all $r \geq r_0$. Integrating from r_0 to r with $r > r_0$, we have

$$F^{-\frac{1}{m-1}}(r_0) - F^{-\frac{1}{m-1}}(r) \geq \frac{C_5}{m-1} \log \frac{r}{r_0}.$$

Let $r \to \infty$, we have a contradiction. Hence $f \equiv 0$ and $\rho \equiv 0$.

Under the assumptions of (ii), we conclude that $|\nabla u(x)| \leq C_6$ for some constant C_6 for all x by Corollary 1.2. As in (5.8), using the same notations as before, we have

$$\begin{aligned} F(r) &\le C_7 A_o^{\frac{1}{m}}(r)\left(F'(r)\right)^{\frac{m-1}{m}} \\ &\le C_8\left(rF'(r)\right)^{\frac{m-1}{m}} \end{aligned}$$

for some constants C_7, C_8 independent of r, where we have used the assumption that $V_o(r) \le Cr^m$ and the fact that $rA_o(r) \le 2mV_o(r)$. We can proceed as before to show that f and hence ρ must be identically zero. q.e.d.

Remark 5.3. If we let $g(r) = \int_{\partial B_o(r)} |\nabla u|^m$, then it is easy to see that $f \equiv 0$ provided that

$$\int_{r_0}^{\infty} g^{-\frac{1}{m-1}}(t)dt = +\infty.$$

Hence the conditions of Theorem 5.4 may be relaxed a little bit further.

In [24], Shi and Yau proved the following: Suppose M^m is a complete noncompact Kähler manifold with $m \ge 3$ and with bounded nonnegative holomorphic bisectional curvature and suppose that $R_{\alpha\overline{\alpha}\beta\overline{\beta}} \ge \epsilon\mathcal{R}$ for some positive constant ϵ, where $R_{\alpha\overline{\alpha}\beta\overline{\beta}}$ is the holomorphic bisectional curvature and $\mathcal{R}$ is the scalar curvature. Then $1/V_x(t)\int_{B_x(t)}\mathcal{R} \le Ct^{-2}$ for some constant for all x. Using Theorem 5.1, we have:

Corollary 5.3. *Let M^m a complete noncompact Kähler manifold with complex dimension $m \ge 3$. Suppose that $R_{\alpha\overline{\alpha}\beta\overline{\beta}} \ge \epsilon\mathcal{R}$ for some positive constant ϵ and suppose that the scalar curvature satisfies* (i) $\mathcal{R}(x) \le Cr^{-2}(x)$ *for all x or* (ii) $V_o(r) \le Cr^m$ *for some constant C for all r. Then M is flat.*

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