Ark. Mat., **60** (2022), 365–385
DOI: 10.4310/ARKIV.2022.v60.n2.a8

On the existence of Auslander-Reiten n-exangles in n-exangulated categories

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Abstract. Let $\mathscr{C}$ be an n-exangulated category. In this note, we show that if $\mathscr{C}$ is locally finite, then $\mathscr{C}$ has Auslander-Reiten n-exangles. This unifies and extends results of Xiao–Zhu, Zhu–Zhuang, Zhou and Xie–Lu–Wang for triangulated, extriangulated, $(n+2)$-angulated and n-abelian categories, respectively.

1. Introduction

The notion of extriangulated categories was introduced by Nakaoka–Palu in [19], which can be viewed as a simultaneous generalization of exact categories and triangulated categories. The data of such a category is a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, where $\mathscr{C}$ is an additive category, $\mathbb{E}: \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab}$ is an additive bifunctor and $\mathfrak{s}$ assigns to each $\delta \in \mathbb{E}(C, A)$ a class of 3-term sequences with end terms A and C such that certain axioms hold. However, there are some other examples of extriangulated categories which are neither exact nor triangulated. In particular, Nakaoka and Palu [19, Remark 2.18] proved extension closed subcategories of extriangulated categories are extriangulated categories. For example, let A be an artin algebra and $K^{[-1,0]}(\mathrm{proj} A)$ the category of complexes of finitely generated projective A-modules concentrated in degrees -1 and 0, with morphisms considered up to homotopy. Then $K^{[-1,0]}(\mathrm{proj} A)$ is an extension closed subcategory of the bounded homotopy

Panyue Zhou is corresponding author. Jian He was supported by the National Natural Science Foundation of China (Grant No. 12171230). Jiangsheng Hu was supported by the NSF of China (Grant Nos. 12171206 and 12126424), the Natural Science Foundation of Jiangsu Province (Grant No. BK20211358) and Jiangsu 333 Project.

Key words and phrases: n-exangulated categories, Auslander-Reiten n-exangles, locally finite, extriangulated categories, triangulated categories, n-abelian categories.
2010 Mathematics Subject Classification: 18G80, 18E10, 18G50.

category $K^b(\mathrm{proj}A)$ which is not exact and triangulated, see [12, Example 6.2]. This construction gives extriangulated categories which are not exact and triangulated. Recently, Herschend–Liu–Nakaoka [7] introduced the notion of n-exangulated categories for any positive integer n. It is not only a higher dimensional analogue of extriangulated categories, but also gives a common generalization of n-exact categories in the sense of Jasso [15] and $(n+2)$-angulated in the sense of Geiss–Keller–Oppermann [5]. However, there are some other examples of n-exangulated categories which are neither n-exact nor $(n+2)$-angulated, see [7]–[9], [18].

Auslander-Reiten theory was introduced by Auslander and Reiten in [1], [2]. Since its introduction, Auslander-Reiten theory has become a fundamental tool for studying the representation theory of Artin algebras. Later it has been generalized to the situation of exact categories [14], triangulated categories [6], [20] and their subcategories [3], [16] and some certain additive categories [16], [17], [21] by many authors. Iyama, Nakaoka and Palu [12] developed Auslander–Reiten theory for extriangulated categories. This unifies Auslander–Reiten theories in exact categories and triangulated categories independently. Xiao and Zhu [23], [24] showed that if a triangulated category $\mathscr{C}$ is locally finite, then $\mathscr{C}$ has Auslander-Reiten triangles. Recently, Zhu and Zhuang [27] proved that if an extriangulated category $\mathscr{C}$ is locally finite, then $\mathscr{C}$ has Auslander-Reiten $\mathbb{E}$-triangles. Later, Zhou [26] extended Xiao-Zhu's result into $(n+2)$-angulated categories. Namely, Zhou proved that if an $(n+2)$-angulated category $\mathscr{C}$ is locally finite, then $\mathscr{C}$ has Auslander-Reiten $(n+2)$-angles. Subsequently, Xie-Lu-Wang [22] proved a similar result to Zhou. More precisely, they showed that if an n-abelian category $\mathscr{C}$ is locally finite, then $\mathscr{C}$ has n-Auslander-Reiten sequences. Based on this idea, we have a natural question of whether the results of Zhou [26] and Xie-Lu-Wang [22] can be unified under the framework of n-exangulated categories or whether the result of Zhu-Zhuang [27] has a higher counterpart. In this article, we give an affirmative answer.

Our main result is the following.

Theorem 1.1. (See Theorem 3.12 for details)*Let $\mathscr{C}$ be a locally finite n-exangulated category. If $X\in\mathsf{ind}(\mathscr{C})$ is a non-projective object, then there exists an Auslander-Reiten n-exangle ending at X, and if $Y\in\mathsf{ind}(\mathscr{C})$ is a non-injective object, then there exists an Auslander-Reiten n-exangle starting at Y. In this case, we say that $\mathscr{C}$ has Auslander-Reiten n-exangles.*

This article is organized as follows: In Section 2, we recall the definition of n-exangulated category and review some results. In Section 3, we show our main result.

2. Preliminaries

In this section, we briefly review basic concepts and results concerning n-exangulated categories.

Let $\mathscr{C}$ be an additive category and $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to\mathsf{Ab}$ (Ab is the category of abelian groups) an additive bifunctor. For any pair of objects $A, C\in\mathscr{C}$, an element $\delta\in\mathbb{E}(C,A)$ is called an $\mathbb{E}$-*extension* or simply an *extension*. We also write such δ as ${}_A\delta_C$ when we indicate A and C. The zero element ${}_A0_C=0\in\mathbb{E}(C,A)$ is called the *split* $\mathbb{E}$-*extension*. For any pair of $\mathbb{E}$-extensions ${}_A\delta_C$ and ${}_{A'}\delta'{}_{C'}$, let $\delta\oplus\delta'\in\mathbb{E}(C\oplus C', A\oplus A')$ be the element corresponding to $(\delta,0,0,\delta')$ through the natural isomorphism $\mathbb{E}(C\oplus C', A\oplus A')\simeq\mathbb{E}(C,A)\oplus\mathbb{E}(C,A')\oplus\mathbb{E}(C',A)\oplus\mathbb{E}(C',A')$.

For any $a\in\mathscr{C}(A,A')$ and $c\in\mathscr{C}(C',C)$, $\mathbb{E}(C,a)(\delta)\in\mathbb{E}(C,A')$ and $\mathbb{E}(c,A)(\delta)\in\mathbb{E}(C',A)$ are simply denoted by $a_*\delta$ and $c^*\delta$, respectively.

Let ${}_A\delta_C$ and ${}_{A'}\delta'{}_{C'}$ be any pair of $\mathbb{E}$-extensions. A *morphism* $(a,c)\colon \delta\to\delta'$ of extensions is a pair of morphisms $a\in\mathscr{C}(A,A')$ and $c\in\mathscr{C}(C,C')$ in $\mathscr{C}$, satisfying the equality $a_*\delta=c^*\delta'$.

Let $\mathscr{C}$ be an additive category as before, and let n be any positive integer.

Definition 2.1. ([7, Definition 2.7]) Let $\mathbf{C}_{\mathscr{C}}$ be the category of complexes in $\mathscr{C}$. As its full subcategory, define $\mathbf{C}^{n+2}_{\mathscr{C}}$ to be the category of complexes in $\mathscr{C}$ whose components are zero in the degrees outside of $\{0,1,\dots,n+1\}$. Namely, an object in $\mathbf{C}^{n+2}_{\mathscr{C}}$ is a complex $X_\bullet=\{X_i, d_i^X\}$ of the form

$$X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \dots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1}.$$

We write a morphism $f_\bullet\colon X_\bullet\to Y_\bullet$ simply $f_\bullet=(f_0,f_1,\dots,f_{n+1})$, only indicating the terms of degrees $0,\dots,n+1$.

Definition 2.2. ([7, Definition 2.11]) By Yoneda lemma, any extension $\delta\in\mathbb{E}(C,A)$ induces natural transformations

$$\delta_\sharp\colon \mathscr{C}(-,C)\Longrightarrow\mathbb{E}(-,A) \quad\text{and}\quad \delta^\sharp\colon \mathscr{C}(A,-)\Longrightarrow\mathbb{E}(C,-).$$

For any $X\in\mathscr{C}$, these $(\delta_\sharp)_X$ and $\delta^\sharp_X$ are given as follows.

(1) $(\delta_\sharp)_X\colon \mathscr{C}(X,C)\to\mathbb{E}(X,A)\ :\ f\mapsto f^*\delta$.

(2) $\delta^\sharp_X\colon \mathscr{C}(A,X)\to\mathbb{E}(C,X)\ :\ g\mapsto g_*\delta$.

We simply denote $(\delta_\sharp)_X(f)$ and $\delta^\sharp_X(g)$ by $\delta_\sharp(f)$ and $\delta^\sharp(g)$, respectively.

Definition 2.3. ([7, Definition 2.9]) Let $\mathscr{C},\mathbb{E},n$ be as before. Define a category $\text{Æ}:=\text{Æ}^{n+2}_{(\mathscr{C},\mathbb{E})}$ as follows.

(1) An object in $\text{Æ}^{n+2}_{(\mathscr{C},\mathbb{E})}$ is a pair $\langle X_\bullet,\delta\rangle$ of $X_\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}$ and $\delta\in\mathbb{E}(X_{n+1},X_0)$ satisfying

$$(d_0^X)_*\delta=0 \quad\text{and}\quad (d_n^X)^*\delta=0.$$

We call such a pair an $\mathbb{E}$-attached complex of length $n+2$. We also denote it by

$$X_0\xrightarrow{d_0^X}X_1\xrightarrow{d_1^X}\dots\xrightarrow{d_{n-2}^X}X_{n-1}\xrightarrow{d_{n-1}^X}X_n\xrightarrow{d_n^X}X_{n+1}\overset{\delta}{\dashrightarrow}.$$

(2) For such pairs $\langle X_\bullet,\delta\rangle$ and $\langle Y_\bullet,\rho\rangle$, a morphism $f_\bullet\colon\langle X_\bullet,\delta\rangle\to\langle Y_\bullet,\rho\rangle$ is defined to be a morphism $f_\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X_\bullet,Y_\bullet)$ satisfying $(f_0)_*\delta=(f_{n+1})^*\rho$.

We use the same composition and the identities as in $\mathbf{C}^{n+2}_{\mathscr{C}}$.

Definition 2.4. ([7, Definition 2.13]) An *n-exangle* is a pair $\langle X_\bullet,\delta\rangle$ of $X_\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}$ and $\delta\in\mathbb{E}(X_{n+1},X_0)$ which satisfies the following conditions.

(1) The following sequence of functors $\mathscr{C}^{\rm op}\to\mathsf{Ab}$ is exact.

$$\mathscr{C}(-,X_0)\xrightarrow{\mathscr{C}(-,\ d_0^X)}\dots\xrightarrow{\mathscr{C}(-,\ d_n^X)}\mathscr{C}(-,X_{n+1})\xrightarrow{\delta_\sharp}\mathbb{E}(-,X_0)$$

(2) The following sequence of functors $\mathscr{C}\to\mathsf{Ab}$ is exact.

$$\mathscr{C}(X_{n+1},-)\xrightarrow{\mathscr{C}(d_n^X,\ -)}\dots\xrightarrow{\mathscr{C}(d_0^X,\ -)}\mathscr{C}(X_0,-)\xrightarrow{\delta^\sharp}\mathbb{E}(X_{n+1},-)$$

In particular any n-exangle is an object in Æ. A *morphism of n-exangles* simply means a morphism in Æ. Thus n-exangles form a full subcategory of Æ.

Definition 2.5. ([7, Definition 2.22]) Let $\mathfrak{s}$ be a correspondence which associates a homotopic equivalence class $\mathfrak{s}(\delta)=[{}_AX_{\bullet C}]$ to each extension $\delta={}_A\delta_C$. Such $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it satisfies the following condition for any $\mathfrak{s}(\delta)=[X_\bullet]$ and any $\mathfrak{s}(\rho)=[Y_\bullet]$.

(R0) For any morphism of extensions $(a,c)\colon\delta\to\rho$, there exists a morphism $f_\bullet\in\mathbf{C}^{n+2}_{\mathscr{C}}(X_\bullet,Y_\bullet)$ of the form $f_\bullet=(a,f_1,\dots,f_n,c)$. Such $f_\bullet$ is called a *lift* of (a,c).
In such a case, we simply say that "$X_\bullet$ realizes δ" whenever they satisfy $\mathfrak{s}(\delta)=[X_\bullet]$.

Moreover, a realization $\mathfrak{s}$ of $\mathbb{E}$ is said to be *exact* if it satisfies the following conditions.

(R1) For any $\mathfrak{s}(\delta)=[X_\bullet]$, the pair $\langle X_\bullet,\delta\rangle$ is an n-exangle.

(R2) For any $A\in\mathscr{C}$, the zero element ${}_A0_0=0\in\mathbb{E}(0,A)$ satisfies

$$\mathfrak{s}({}_A0_0)=[A\xrightarrow{\mathrm{id}_A}A\longrightarrow0\longrightarrow\dots\longrightarrow0\longrightarrow0].$$

Dually, $\mathfrak{s}({}_00_A)=[0\to0\to\dots\to0\to A\xrightarrow{\mathrm{id}_A}A]$ holds for any $A\in\mathscr{C}$.
Note that the above condition (R1) does not depend on representatives of the class $[X_\bullet]$.

Definition 2.6. ([7, Definition 2.23]) Let $\mathfrak{s}$ be an exact realization of $\mathbb{E}$.

(1) An n-exangle $\langle X_\bullet, \delta\rangle$ is called an $\mathfrak{s}$-*distinguished* n-exangle if it satisfies $\mathfrak{s}(\delta)=[X_\bullet]$. We often simply say *distinguished n-exangle* when $\mathfrak{s}$ is clear from the context.

(2) An object $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ is called an $\mathfrak{s}$-*conflation* or simply a *conflation* if it realizes some extension $\delta\in\mathbb{E}(X_{n+1}, X_0)$.

(3) A morphism f in $\mathscr{C}$ is called an $\mathfrak{s}$-*inflation* or simply an *inflation* if it admits some conflation $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $d_0^X = f$.

(4) A morphism g in $\mathscr{C}$ is called an $\mathfrak{s}$-*deflation* or simply a *deflation* if it admits some conflation $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $d_n^X = g$.

Definition 2.7. ([7, Definition 2.27]) For a morphism $f_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X_\bullet, Y_\bullet)$ satisfying $f_0=\mathrm{id}_A$ for some $A=X_0=Y_0$, its *mapping cone* $M^f_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ is defined to be the complex

$$X_1 \xrightarrow{d_0^{M_f}} X_2\oplus Y_1 \xrightarrow{d_1^{M_f}} X_3\oplus Y_2 \xrightarrow{d_2^{M_f}} \cdots \xrightarrow{d_{n-1}^{M_f}} X_{n+1}\oplus Y_n \xrightarrow{d_n^{M_f}} Y_{n+1}$$

where $d_0^{M_f} = \begin{bmatrix} -d_1^X \\ f_1 \end{bmatrix}$, $d_i^{M_f} = \begin{bmatrix} -d_{i+1}^X & 0 \\ f_{i+1} & d_i^Y \end{bmatrix}$ $(1\le i\le n-1)$, $d_n^{M_f} = \begin{bmatrix} f_{n+1} & d_n^Y \end{bmatrix}$.

The mapping cocone is defined dually, for morphisms $h_\bullet$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $h_{n+1}=\mathrm{id}$.

Definition 2.8. ([7, Definition 2.32]) An *n-exangulated category* is a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ of additive category $\mathscr{C}$, additive bifunctor $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to\mathsf{Ab}$, and its exact realization $\mathfrak{s}$, satisfying the following conditions.

(EA1) Let $A\xrightarrow{f}B\xrightarrow{g}C$ be any sequence of morphisms in $\mathscr{C}$. If both f and g are inflations, then so is $g\circ f$. Dually, if f and g are deflations, then so is $g\circ f$.

(EA2) For $\rho\in\mathbb{E}(D,A)$ and $c\in\mathscr{C}(C,D)$, let ${}_A\langle X_\bullet, c^*\rho\rangle_C$ and ${}_A\langle Y_\bullet, \rho\rangle_D$ be distinguished n-exangles. Then (id_A, c) has a *good lift* $f_\bullet$, in the sense that its mapping cone gives a distinguished n-exangle $\langle M^f_\bullet, (d_0^X)_*\rho\rangle$.

(EA2$^{\mathrm{op}}$) Dual of (EA2).

Note that the case $n=1$, a triplet $(\mathscr{C},\mathbb{E},\mathfrak{s})$ is a 1-exangulated category if and only if it is an extriangulated category, see [7, Proposition 4.3].

Example 2.9. From [7, Proposition 4.34] and [7, Proposition 4.5], we know that n-exact categories and $(n+2)$-angulated categories are n-exangulated categories. There are some other examples of n-exangulated categories which are neither n-exact nor $(n+2)$-angulated, see [7]–[9], [18].

The following are some very useful lemmas and they will be needed later on.

Lemma 2.10. *Let $\langle X_\bullet, \delta\rangle$ and $\langle Y_\bullet, \rho\rangle$ be distinguished n-exangles. Suppose that we are given a commutative square*

$$\begin{array}{ccc} X_n & \xrightarrow{d_n^X} & X_{n+1} \\ {\scriptstyle c}\downarrow & \circlearrowright & \downarrow{\scriptstyle d} \\ Y_n & \xrightarrow[d_n^Y]{} & Y_{n+1} \end{array}$$

in $\mathscr{C}$. Then there is a morphism $f_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle Y_\bullet, \rho\rangle$ which satisfies $f_n = c$ and $f_{n+1} = d$.

Proof. This proof is the dual of [7, Proposition 3.6], and we omit it. □

Lemma 2.11. ([7, *Claim* 2.15]) *Let $\mathscr{C}$ be an n-exangulated category, and*

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\theta}{\dashrightarrow} \tag{1}$$

be a distinguished n-exangle in $\mathscr{C}$. Then the following statements are equivalent:

(1) *α_0 is a section (also known as a split monomorphism);*

(2) *α_n is a retraction (also known as a split epimorphism);*

(3) $\theta = 0$.

If a distinguished n-exangle (1) satisfies one of the above equivalent conditions, it is called split.

Definition 2.12. ([25, Definition 3.14] and [18, Definition 3.2]) Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. An object $P \in \mathscr{C}$ is called *projective* if, for any distinguished n-exangle

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\delta}{\dashrightarrow}$$

and any morphism c in $\mathscr{C}(P, A_{n+1})$, there exists a morphism $b \in \mathscr{C}(P, A_n)$ satisfying $\alpha_n \circ b = c$. We denote the full subcategory of projective objects in $\mathscr{C}$ by $\mathcal{P}$. The concept of injective objects is defined dually. The full subcategory of injective objects in $\mathscr{C}$ is denoted by $\mathcal{I}$.

Lemma 2.13. ([18, *Lemma* 3.4]) *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. Then the following statements are equivalent for an object $P \in \mathscr{C}$.*

(1) *$\mathbb{E}(P, A) = 0$ for any $A \in \mathscr{C}$.*

(2) *P is projective.*

(3) *Any distinguished n-exangle $A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} P \overset{\delta}{\dashrightarrow}$ splits.*

We denote by $\mathrm{rad}_{\mathscr{C}}$ the Jacobson radical of $\mathscr{C}$. Namely, $\mathrm{rad}_{\mathscr{C}}$ is an ideal of $\mathscr{C}$ such that $\mathrm{rad}_{\mathscr{C}}(A,A)$ coincides with the Jacobson radical of the endomorphism ring $\mathrm{End}(A)$ for any $A\in\mathscr{C}$.

Definition 2.14. ([10, Definition 3.3]) When $n\geq 2$, a distinguished n-exangle in $\mathscr{C}$ of the form

$$A_\bullet: \quad A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \dashrightarrow$$

is minimal if $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are in $\mathrm{rad}_{\mathscr{C}}$.

The following lemma shows that $\mathbb{E}$-extension in an equivalence class can be chosen in a minimal way in a Krull-Schmidt n-exangulated category.

Lemma 2.15. ([10, *Lemma* 3.4]) *Let $\mathscr{C}$ be a Krull-Schmidt n-exangulated category, $A_0, A_{n+1}\in\mathscr{C}$. Then for every equivalence class associated with $\mathbb{E}$-extension $\delta={}_{A_0}\delta_{A_{n+1}}$, there exists a representation*

$$A_\bullet: \quad A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \dashrightarrow$$

such that $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are in $\mathrm{rad}_{\mathscr{C}}$. Moreover, $A_\bullet$ is a direct summand of every other equivalent $\mathbb{E}$-extension.

Remark 2.16. Let $\mathscr{C}$ be a Krull-Schmidt n-exangulated category. By the Krull-Schmidt property of $\mathscr{C}$, every minimal distinguished n-exangle in each equivalence class is unique up to isomorphism.

3. Locally finite n-exangulated categories

The result of this section generalizes the work of Sections 3 in [22] and [26], but the proof is not too far from their case.

In this section, let k be a field. We always assume that $\mathscr{C}$ is a k-linear Hom-finite Krull-Schmidt n-exangulated category. We denote by $\mathsf{ind}(\mathscr{C})$ the set of isomorphism classes of indecomposable objects in $\mathscr{C}$.

Assume that $\mathscr{C}$ is an additive category. Recall that a morphism $\alpha_n\colon A_n\to A_{n+1}$ in $\mathscr{C}$ is *right almost split* if it is not a split epimorphism and each $f\colon Y\to A_{n+1}$ in $\mathscr{C}$ which is not a split epimorphism factors through α_n. Dually, a morphism $\alpha_0\colon A_0\to A_1$ in $\mathscr{C}$ is *left almost split* if it is not a split monomorphism and each $g\colon A_0\to Z$ in $\mathscr{C}$ which is not a split monomorphism factors through α_0. Next, we introduce the notion of *Auslander-Reiten n-exangle* in an n-exangulated category.

Definition 3.1. Let $\mathscr{C}$ be an n-exangulated category. A distinguished n-exangle

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\delta}{\dashrightarrow}$$

in $\mathscr{C}$ is called an *Auslander-Reiten n-exangle* if α_0 is left almost split, α_n is right almost split and when $n \geq 2$, $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are in $\mathrm{rad}_{\mathscr{C}}$.

Remark 3.2. (1) If $\mathscr{C}$ is an n-abelian category, then Definition 3.1 coincides with the definition of n-Auslander-Reiten sequence of n-abelian category (cf. [11], [22]), which is first introduced by Iyama in [11, Definition 3.1].

(2) If $\mathscr{C}$ is an $(n+2)$-angulated category, then Definition 3.1 coincides with the definition of Auslander-Reiten $(n+2)$-angle of $(n+2)$-angulated category (cf. [4], [26]). It is worth noting that the original definition is introduced by Iyama and Yoshino in [13, Definition 3.8], but this allowed the endterms to be non-indecomposable objects, while the modified definition by Fedele restricts to indecomposable endterms in [4, Definition 5.1].

Lemma 3.3. *Let $\mathscr{C}$ be an n-exangulated category and*

$$A_\bullet: \quad A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\delta}{\dashrightarrow}$$

be a distinguished n-exangle in $\mathscr{C}$. Then the following statements are equivalent:

(1) *$A_\bullet$ is an Auslander-Reiten n-exangle*;

(2) $\mathrm{End}(A_0)$ *is local, if $n \geq 2$, $\alpha_1, \dots, \alpha_{n-1}$ are in* $\mathrm{rad}_{\mathscr{C}}$ *and α_n is right almost split*;

(3) $\mathrm{End}(A_{n+1})$ *is local, if $n \geq 2$, $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are in* $\mathrm{rad}_{\mathscr{C}}$ *and α_0 is left almost split.*

Proof. The proof given in [4, Lemma 5.3] can be adapted to the context of n-exangulated categories, we omit it. □

For any $X \in \mathsf{ind}(\mathscr{C})$, we denote by $\mathsf{Supp}\mathrm{Hom}_{\mathscr{C}}(X, -)$ the subcategory of $\mathscr{C}$ generated by objects Y in $\mathsf{ind}(\mathscr{C})$ with $\mathrm{Hom}_{\mathscr{C}}(X, Y) \neq 0$. Similarly, $\mathsf{Supp}\,\mathrm{Hom}_{\mathscr{C}}(-, X)$ denotes the subcategory generated by objects Y in $\mathsf{ind}(\mathscr{C})$ with $\mathrm{Hom}_{\mathscr{C}}(Y, X) \neq 0$. If $\mathsf{Supp}\,\mathrm{Hom}_{\mathscr{C}}(X, -)$ ($\mathsf{Supp}\,\mathrm{Hom}_{\mathscr{C}}(-, X)$, respectively) contains only finitely many indecomposables, we say that $|\mathsf{Supp}\,\mathrm{Hom}_{\mathscr{C}}(X, -)| < \infty$ ($|\mathsf{Supp}\,\mathrm{Hom}_{\mathscr{C}}(-, X)| < \infty$, respectively).

Based on the definition of locally finite $(n+2)$-angulated categories and locally finite n-abelian categories, [22], [26], we define the notion of locally finite n-exangulated categories.

Definition 3.4. An n-exangulated category $\mathscr{C}$ is called *locally finite* if $|\mathsf{Supp}\,\mathrm{Hom}_{\mathscr{C}}(X, -)| < \infty$ and $|\mathsf{Supp}\,\mathrm{Hom}_{\mathscr{C}}(-, X)| < \infty$, for any object $X \in \mathsf{ind}(\mathscr{C})$.

Definition 3.5. Let $\mathscr{C}$ be an n-exangulated category and $X_{n+1}, Y_0 \in \mathsf{ind}(\mathscr{C})$. We define a set of distinguished n-exangles as follows:

$$S(X_{n+1}) := \left\{ X_\bullet : X_0 \xrightarrow{\alpha_0} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \overset{\delta}{\dashrightarrow} \;\middle|\; \begin{array}{l} X_\bullet \text{ is a non-split distinguished } n\text{-exangle} \\ \text{with } X_0 \in \mathsf{ind}(\mathscr{C}), \text{and when} \\ n \geq 2,\ \alpha_1, \alpha_2, \dots, \alpha_{n-1} \text{ in } \mathrm{rad}_{\mathscr{C}}. \end{array} \right\}$$

Dually, we can define a set of distinguished n-exangles as follows:

$$T(Y_0) := \left\{ Y_\bullet : Y_0 \xrightarrow{\beta_0} \dots \xrightarrow{\beta_{n-1}} Y_n \xrightarrow{\beta_n} Y_{n+1} \overset{\eta}{\dashrightarrow} \;\middle|\; \begin{array}{l} Y_\bullet \text{ is a non-split distinguished } n\text{-exangle} \\ \text{with } Y_{n+1} \in \mathsf{ind}(\mathscr{C}), \text{and when} \\ n \geq 2,\ \beta_1, \beta_2, \dots, \beta_{n-1} \text{ in } \mathrm{rad}_{\mathscr{C}}. \end{array} \right\}$$

Lemma 3.6. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category.*

(1) *If $X_{n+1} \in \mathsf{ind}(\mathscr{C})$ is a non-projective object, then $S(X_{n+1})$ is non-empty.*

(2) *If $Y_0 \in \mathsf{ind}(\mathscr{C})$ is a non-injective object, then $T(Y_0)$ is non-empty.*

Proof. We only show that (1), dually one can prove (2).

Since $X_{n+1} \in \mathsf{ind}(\mathscr{C})$ is a non-projective, there is an object $X_0 \in \mathscr{C}$, such that $\mathbb{E}(X_{n+1}, X_0) \neq 0$ by Lemma 2.13. That is to say, there exists a non-split distinguished n-exangle:

$$X_\bullet : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \overset{\delta}{\dashrightarrow}.$$

Since $\mathscr{C}$ is a Krull-Schmidt category, we decompose X_0 into a direct sum of indecomposable objects $X_0 = \bigoplus_{i=1}^{d} A_i$. Without loss of generality, we can assume that $X_0 = U \oplus V$ where U and V are indecomposable. Since $\mathbb{E}(X_{n+1}, X_0) \simeq \mathbb{E}(X_{n+1}, U \oplus V) \simeq \mathbb{E}(X_{n+1}, U) \oplus \mathbb{E}(X_{n+1}, V)$. We claim that at least one of the following two distinguished n-exangles is non-split

$$U \xrightarrow{\gamma_0} C_1 \xrightarrow{\gamma_1} C_2 \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_{n-1}} C_n \xrightarrow{\gamma_n} X_{n+1} \overset{\eta}{\dashrightarrow}$$

$$V \xrightarrow{\beta_0} D_1 \xrightarrow{\beta_1} D_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} D_n \xrightarrow{\beta_n} X_{n+1} \overset{\eta'}{\dashrightarrow},$$

where $\eta := [1, 0]_* \delta$ and $\eta' := [0, 1]_* \delta$. Otherwise, $\delta = \eta \oplus \eta' = 0 \in \mathbb{E}(X_{n+1}, X_0)$. This is a contradiction since δ is non-split.

We can take a distinguished n-exangle as we want by Lemma 2.15. This completes the proof. $\square$

Remark 3.7. Let

$$X_\bullet : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \overset{\delta}{\dashrightarrow}$$

be a non-split distinguished n-exangle and $X_0{=}U\oplus V$, where U and V are indecomposable. From the proof of Lemma 3.6, we see that at least one of the following two distinguished n-exangles is non-split

$$U \xrightarrow{\gamma_0} C_1 \xrightarrow{\gamma_1} C_2 \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_{n-1}} C_n \xrightarrow{\gamma_n} X_{n+1} \overset{\eta}{\dashrightarrow}$$

$$V \xrightarrow{\beta_0} D_1 \xrightarrow{\beta_1} D_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} D_n \xrightarrow{\beta_n} X_{n+1} \overset{\eta'}{\dashrightarrow},$$

where $\eta{:=}[\,1,\,0\,]_*\,\delta$ and $\eta'{:=}[\,0,\,1\,]_*\,\delta$.

Definition 3.8. Let $\mathscr{C}$ be an n-exangulated category, and

$$X_\bullet : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \overset{\delta}{\dashrightarrow}$$

$$U_\bullet : U_0 \xrightarrow{\beta_0} U_1 \xrightarrow{\beta_1} U_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} U_n \xrightarrow{\beta_n} X_{n+1} \overset{\delta'}{\dashrightarrow}$$

be two distinguished n-exangles in $S(X_{n+1})$. We say that $X_\bullet{>}U_\bullet$ if there exists a morphism of distinguished n-exangles as follows:

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots \longrightarrow & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \overset{\delta}{\dashrightarrow} \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & \downarrow{\scriptstyle \varphi_n} & & \| & \\
U_0 & \xrightarrow{\beta_0} & U_1 & \xrightarrow{\beta_1} & U_2 & \xrightarrow{\beta_2} & \cdots \longrightarrow & U_n & \xrightarrow{\beta_n} & X_{n+1} & \overset{\delta'}{\dashrightarrow}
\end{array}$$

We say that $X_\bullet{\sim}U_\bullet$ if φ_0 is an isomorphism.

Dually, let

$$Y_\bullet : Y_0 \xrightarrow{\alpha_0} Y_1 \xrightarrow{\alpha_1} Y_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} Y_n \xrightarrow{\alpha_n} Y_{n+1} \overset{\eta}{\dashrightarrow}$$

$$V_\bullet : Y_0 \xrightarrow{\beta_0} V_1 \xrightarrow{\beta_1} V_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} V_n \xrightarrow{\beta_n} V_{n+1} \overset{\eta'}{\dashrightarrow}$$

be two distinguished n-exangles in $T(Y_0)$. We say that $Y_\bullet{>}V_\bullet$ if there exists a morphism of distinguished n-exangles as follows:

$$\begin{array}{ccccccccccc}
Y_0 & \xrightarrow{\alpha_0} & Y_1 & \xrightarrow{\alpha_1} & Y_2 & \xrightarrow{\alpha_2} & \cdots \longrightarrow & Y_n & \xrightarrow{\alpha_n} & Y_{n+1} & \overset{\eta}{\dashrightarrow} \\
\|{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} & \\
Y_0 & \xrightarrow{\beta_0} & V_1 & \xrightarrow{\beta_1} & V_2 & \xrightarrow{\beta_2} & \cdots \longrightarrow & V_n & \xrightarrow{\beta_n} & V_{n+1} & \overset{\eta'}{\dashrightarrow}
\end{array}$$

We say that $Y_\bullet{\sim}V_\bullet$ if φ_{n+1} is an isomorphism.

Lemma 3.9. Let $Y,Z\in\mathscr{C}$, $X\in\mathsf{ind}(\mathscr{C})$. *If $f:Y\to X$ and $g:Z\to X$ are not split epimorphisms, then $[f,g]:Y\oplus Z\to X$ is also not split epimorphism.*

Proof. If not, there exists a morphism $\left[\begin{smallmatrix}s\\t\end{smallmatrix}\right]: X\to Y\oplus Z$ such that $[f,g]\left[\begin{smallmatrix}s\\t\end{smallmatrix}\right]=1_X$ and then $fs+gt=1_X$. Since X is an indecomposable object, we have that $\operatorname{End}(X)$ is local which implies that either fs or gt is an isomorphism. Thus either f or g is a split epimorphism, a contradiction. □

In the following, we will consider a direct ordered set, namely, a partially ordered set with every pair of elements has a lower bound.

Lemma 3.10. *$S(X_{n+1})$ is a direct ordered set with the relation defined in Definition 3.8, and $T(Y_0)$ is a direct ordered set with the relation defined in Definition 3.8.*

Proof. We just prove the first statement, the second statement proves similarly.

Assume that

$$X_\bullet: X_0\xrightarrow{\alpha_0}X_1\xrightarrow{\alpha_1}X_2\xrightarrow{\alpha_2}\dots\xrightarrow{\alpha_{n-1}}X_n\xrightarrow{\alpha_n}X_{n+1}\overset{\delta}{\dashrightarrow}$$

and

$$U_\bullet: U_0\xrightarrow{\beta_0}U_1\xrightarrow{\beta_1}U_2\xrightarrow{\beta_2}\dots\xrightarrow{\beta_{n-1}}U_n\xrightarrow{\beta_n}X_{n+1}\overset{\delta'}{\dashrightarrow}$$

belong to $S(X_{n+1})$.

Firstly, the axioms of reflexivity and transitivity are clear. Secondly, we show that if $X_\bullet>U_\bullet$ and $U_\bullet>X_\bullet$, then $X_\bullet\sim U_\bullet$.

Since $X_\bullet>U_\bullet$ and $U_\bullet>X_\bullet$, we have the following two commutative diagrams

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \longrightarrow & X_n & \xrightarrow{\alpha_n} & X_{n+1} \overset{\delta}{\dashrightarrow} \\
{\scriptstyle\varphi_0}\downarrow & & {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_2}\downarrow & & & & {\scriptstyle\varphi_n}\downarrow & & \Vert \\
U_0 & \xrightarrow{\beta_0} & U_1 & \xrightarrow{\beta_1} & U_2 & \xrightarrow{\beta_2} & \cdots & \longrightarrow & U_n & \xrightarrow{\beta_n} & X_{n+1} \overset{\delta'}{\dashrightarrow}
\end{array}$$

$$\begin{array}{ccccccccccc}
U_0 & \xrightarrow{\beta_0} & U_1 & \xrightarrow{\beta_1} & U_2 & \xrightarrow{\beta_2} & \cdots & \longrightarrow & U_n & \xrightarrow{\beta_n} & X_{n+1} \overset{\delta'}{\dashrightarrow} \\
{\scriptstyle\psi_0}\downarrow & & {\scriptstyle\psi_1}\downarrow & & {\scriptstyle\psi_2}\downarrow & & & & {\scriptstyle\psi_n}\downarrow & & \Vert \\
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \longrightarrow & X_n & \xrightarrow{\alpha_n} & X_{n+1} \overset{\delta}{\dashrightarrow}
\end{array}$$

We claim that $\psi_0\varphi_0$ is an isomorphism. Since X_0 is an indecomposable, we have that $\operatorname{End}(X_0)$ is local implies that $\psi_0\varphi_0$ is nilpotent or is an isomorphism. If $\psi_0\varphi_0$

is nilpotent, there exists a positive integer m such that $(\psi_0\varphi_0)^m=0$. We write $\omega_i=\psi_i\varphi_i, i=1,2,\cdots,n$. Thus we have the following commutative diagram

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots \longrightarrow & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \overset{\delta}{\dashrightarrow} \\
\downarrow{\scriptstyle (\psi_0\varphi_0)^m} & & \downarrow{\scriptstyle \omega_1^m} & & \downarrow{\scriptstyle \omega_2^m} & & & \downarrow{\scriptstyle \omega_n^m} & & \| & \\
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots \longrightarrow & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \overset{\delta}{\dashrightarrow}
\end{array}$$

Then $\delta=(\psi_0\varphi_0)_*^m\delta=0$. This is a contradiction by Lemma 2.11 since $X_\bullet$ is non-split. Hence $\psi_0\varphi_0$ is an isomorphism. By a similar argument we obtain that $\varphi_0\psi_0$ is an isomorphism. This shows that φ_0 is isomorphism. So $X_\bullet\sim U_\bullet$.

Finally, we show that if $X_\bullet, U_\bullet\in S(X_{n+1})$, then there exists $C_\bullet\in S(X_{n+1})$ such that $X_\bullet>C_\bullet$ and $U_\bullet>C_\bullet$.

For the morphism $\beta_n\colon U_n\to X_{n+1}$, by (EA2), we can observe that $(\mathrm{id}_{X_0},\beta_n)$ has a *good lift* $f_\bullet=(\mathrm{id}_{X_0},\psi_1,...,\psi_n,\beta_n)$, that is, there exists the following commutative diagram of distinguished n-exangles

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{\gamma_0} & Z_1 & \xrightarrow{\gamma_1} & Z_2 & \xrightarrow{\gamma_2} & \cdots \longrightarrow & Z_n & \xrightarrow{\gamma_n} & U_n & \overset{\beta_n^*\delta}{\dashrightarrow} \\
\| & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \beta_n} & \\
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots \longrightarrow & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \overset{\delta}{\dashrightarrow}
\end{array}$$

such that $M_\bullet\colon Z_1\longrightarrow M_1\longrightarrow M_2\longrightarrow\ldots\longrightarrow M_{n-1}\longrightarrow U_n\oplus X_n\xrightarrow{[\beta_n,\ \alpha_n]}X_{n+1}\overset{(\gamma_0)_*\delta}{\dashrightarrow}$ is a distinguished n-exangle in $\mathscr{C}$, where $M_i=Z_{i+1}\oplus X_i$ $(i=1,2,...,n-1)$. Since $X_{n+1}\in \mathsf{ind}(\mathscr{C})$, β_n and α_n are not split epimorphisms, we have that $[\beta_n,\alpha_n]$ is also not split epimorphism by Lemma 3.9. That is, $M_\bullet$ is non-split.

Without loss of generality, we can assume that $Z_1=U\oplus V$ where U and V are indecomposable. For the morphism $p_1=[1,0]\colon U\oplus V\to U$, by (EA2$^{\rm op}$), we can observe that $(p_1,\mathrm{id}_{X_{n+1}})$ has a *good lift* $g_\bullet=(p_1,\varphi_1,...,\varphi_n,\mathrm{id}_{X_{n+1}})$, that is, there exists the following commutative diagram of distinguished n-exangles

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{[u,\ v]} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots \longrightarrow & M_{n-1} & \longrightarrow & U_n\oplus X_n & \longrightarrow & X_{n+1} & \dashrightarrow \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & & \downarrow & & \downarrow & & \| & \\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots \longrightarrow & L_{n-1} & \longrightarrow & L_n & \longrightarrow & X_{n+1} & \dashrightarrow .
\end{array}$$

Similarly, for the morphism $p_2=[0,1]\colon U\oplus V\to V$, there exists the following commutative diagram of distinguished n-exangles

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{[u,\ v]} & M_1 & \longrightarrow & M_2 & \longrightarrow \cdots \longrightarrow & M_{n-1} & \longrightarrow & U_n\oplus X_n & \longrightarrow & X_{n+1} & \dashrightarrow & \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle m_1} & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
V & \xrightarrow{\eta_0} & N_1 & \longrightarrow & N_2 & \longrightarrow \cdots \longrightarrow & N_{n-1} & \longrightarrow & N_n & \longrightarrow & X_{n+1} & \dashrightarrow & .
\end{array}$$

By Remark 3.7, we conclude that at least one of the following two distinguished n-exangles is non-split

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \ldots \longrightarrow L_{n-1} \longrightarrow L_n \longrightarrow X_{n+1} \dashrightarrow$$

$$V \xrightarrow{\eta_0} N_1 \longrightarrow N_2 \longrightarrow \cdots \longrightarrow N_{n-1} \longrightarrow N_n \longrightarrow X_{n+1} \dashrightarrow .$$

Without loss of generality, we assume that

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \ldots \longrightarrow L_{n-1} \longrightarrow L_n \longrightarrow X_{n+1} \dashrightarrow$$

is non-split. By Lemma 2.15, there is a non-split distinguished n-exangle

$$C_\bullet\colon U \xrightarrow{\lambda_0} C_1 \xrightarrow{\lambda_1} C_2 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_{n-2}} C_{n-1} \xrightarrow{\lambda_{n-1}} C_n \xrightarrow{\lambda_n} X_{n+1} \dashrightarrow$$

with $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}$ in $\operatorname{rad}_{\mathscr{C}}$. By (R0) and Lemma 2.10, we have the following commutative diagram

$$\begin{array}{ccccccccccccc}
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \dashrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & \| & \\
U\oplus V & \xrightarrow{[u,\ v]} & M_1 & \longrightarrow & M_2 & \longrightarrow \cdots \longrightarrow & M_{n-1} & \longrightarrow & U_n\oplus X_n & \xrightarrow{[\beta_n,\alpha_n]} & X_{n+1} & \dashrightarrow \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & \downarrow & & \downarrow & & \| & \\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow \cdots \longrightarrow & L_{n-1} & \longrightarrow & L_n & \longrightarrow & X_{n+1} & \dashrightarrow \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & \\
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_{n-2}} & C_{n-1} & \xrightarrow{\lambda_{n-1}} & C_n & \xrightarrow{\lambda_n} & X_{n+1} & \dashrightarrow
\end{array}$$

of distinguished n-exangles. This shows that $X_\bullet > C_\bullet$.

By (R0) and Lemma 2.10, we have the following commutative diagram

$$
\begin{array}{ccccccccccccc}
U_0 & \xrightarrow{\beta_0} & U_1 & \xrightarrow{\beta_1} & U_2 & \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{n-2}} & U_{n-1} & \xrightarrow{\beta_{n-1}} & U_n & \xrightarrow{\beta_n} & X_{n+1} & \dashrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \| & \\
U\oplus V & \xrightarrow{[u,\ v]} & M_1 & \longrightarrow & M_2 & \longrightarrow \cdots \longrightarrow & M_{n-1} & \longrightarrow & U_n\oplus X_n & \xrightarrow{[\beta_n,\alpha_n]} & X_{n+1} & \dashrightarrow \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & \downarrow & & \downarrow & & \| & \\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow \cdots \longrightarrow & L_{n-1} & \longrightarrow & L_n & \longrightarrow & X_{n+1} & \dashrightarrow \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & \\
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{n-2}} & C_{n-1} & \xrightarrow{\lambda_{n-1}} & C_n & \xrightarrow{\lambda_n} & X_{n+1} & \dashrightarrow
\end{array}
$$

of distinguished n-exangles. This shows that $U_\bullet > C_\bullet$. □

Lemma 3.11. *Let $\mathscr{C}$ be a locally finite n-exangulated category.*

(1) *If $X_{n+1} \in \mathsf{ind}(\mathscr{C})$ is a non-projective object, then $S(X_{n+1})$ has a minimal element.*

(2) *If $Y_0 \in \mathsf{ind}(\mathscr{C})$ is a non-injective object, then $T(Y_0)$ has a minimal element.*

Proof. We just prove the first statement, the second statement proves similarly.

Since $X_{n+1} \in \mathsf{ind}(\mathscr{C})$ is a non-projective, there exists an object $X_0 \in \mathscr{C}$, such that $\mathbb{E}(X_{n+1}, X_0) \neq 0$ by Lemma 2.13. That is to say, there is a non-split distinguished n-exangle:

$$X_\bullet : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \overset{\delta}{\dashrightarrow} .$$

Since $\mathscr{C}$ is a Krull-Schmidt category, we decompose X_n into a direct sum of indecomposable objects $X_n = \bigoplus_{k=1}^{r} B_k$. Thus $X_\bullet$ can be written as

$$X_\bullet : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} \bigoplus_{k=1}^{r} B_k \xrightarrow{[b_1, b_2, \dots, b_r]} X_{n+1} \dashrightarrow$$

where $b_k \in \mathrm{rad}_{\mathscr{C}}(B_k, X_{n+1})$, $k = 1, 2, \dots, r$.

Since $\mathscr{C}$ is locally finite, there are only finitely many objects $X_i \in \mathsf{ind}(\mathscr{C})$, $i = 1, 2, \dots, m$ such that $\mathrm{Hom}_{\mathscr{C}}(X_i, X_{n+1}) \neq 0$. We assume that λ_{ij}, $1 \leq j \leq q_i$ form a basis of the k-vector space $\mathrm{rad}_{\mathscr{C}}(B_k, X_{n+1})$. Put $M := (\bigoplus_{k=1}^{r} B_k) \oplus (\bigoplus_{i=1}^{m} (X_i)^{\oplus q_i})$, we consider the morphism

$$\gamma := [b_1, b_2, \dots, b_r, \lambda_{11}, \dots, \lambda_{ij}, \dots, \lambda_{mq_m}] \in \mathrm{rad}_{\mathscr{C}}(M, X_{n+1})$$

which is not split epimorphism. By (EA2), we deduce that there is a distinguished n-exangle in $\mathscr{C}$ as follows:

$$M_\bullet:\ B \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \dots \longrightarrow M_{n-1} \longrightarrow M \xrightarrow{\gamma} X_{n+1} \dashrightarrow .$$

Thus $M_\bullet$ is non-split since γ is not split epimorphism. Without loss of generality, we can assume that $B{=}U{\oplus}V$ where U and V are indecomposable. For the morphism $p_1{=}[1,0]\colon U{\oplus}V{\to}U$, by (EA2$^{\mathrm{op}}$), we can observe that $(p_1, \mathrm{id}_{X_{n+1}})$ has a *good lift* $g_\bullet{=}(p_1, \varphi_1, ..., \varphi_n, \mathrm{id}_{X_{n+1}})$, that is, there exists the following commutative diagram of distinguished n-exangles

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{[u,\ v]} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & M_{n-1} & \longrightarrow & M & \xrightarrow{\gamma} & X_{n+1} \dashrightarrow \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & & & \downarrow & & \downarrow & & \parallel \\
U & \xrightarrow{\theta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_{n-1} & \longrightarrow & L_n & \longrightarrow & X_{n+1} \dashrightarrow .
\end{array}$$

Similarly, for the morphism $p_2{=}[0,1]\colon U{\oplus}V{\to}V$, there exists the following commutative diagram of distinguished n-exangles

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{[u,\ v]} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & M_{n-1} & \longrightarrow & M & \xrightarrow{\gamma} & X_{n+1} \dashrightarrow \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle m_1} & & \downarrow & & & & \downarrow & & \downarrow & & \parallel \\
V & \xrightarrow{\eta_0} & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & N_{n-1} & \longrightarrow & N_n & \longrightarrow & X_{n+1} \dashrightarrow .
\end{array}$$

By Remark 3.7, we conclude that at least one of the following two distinguished n-exangles is non-split

$$U \xrightarrow{\theta_0} L_1 \longrightarrow L_2 \longrightarrow \dots \longrightarrow L_{n-1} \longrightarrow L_n \longrightarrow X_{n+1} \dashrightarrow$$

$$V \xrightarrow{\eta_0} N_1 \longrightarrow N_2 \longrightarrow \cdots \longrightarrow N_{n-1} \longrightarrow N_n \longrightarrow X_{n+1} \dashrightarrow .$$

Without loss of generality, we assume that

$$U \xrightarrow{\theta_0} L_1 \longrightarrow L_2 \longrightarrow \dots \longrightarrow L_{n-1} \longrightarrow L_n \longrightarrow X_{n+1} \dashrightarrow$$

is non-split. By Lemma 2.15, we can find a non-split distinguished n-exangle

$$C_\bullet\colon U \xrightarrow{\omega_0} C_1 \xrightarrow{\omega_1} C_2 \xrightarrow{\omega_2} \dots \xrightarrow{\omega_{n-2}} C_{n-1} \xrightarrow{\omega_{n-1}} C_n \xrightarrow{\omega_n} X_{n+1} \dashrightarrow$$

with $\omega_1, \omega_2, ..., \omega_{n-1}$ in $\mathrm{rad}_{\mathscr{C}}$. Then $C_{\bullet} \in S(X_{n+1})$. By (R0), we have the following commutative diagram

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{[u,\ v]} & M_1 & \longrightarrow & M_2 & \longrightarrow \cdots \longrightarrow & M_{n-1} & \longrightarrow & M & \xrightarrow{\gamma} & X_{n+1} & \dashrightarrow \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & \downarrow & & \downarrow & & \| & \\
U & \xrightarrow{\theta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow \cdots \longrightarrow & L_{n-1} & \longrightarrow & L_n & \longrightarrow & X_{n+1} & \dashrightarrow \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & \\
U & \xrightarrow{\omega_0} & C_1 & \xrightarrow{\omega_1} & C_2 & \xrightarrow{\omega_2} \ldots \xrightarrow{\omega_{n-2}} & C_{n-1} & \xrightarrow{\omega_{n-1}} & C_n & \xrightarrow{\omega_n} & X_{n+1} & \dashrightarrow
\end{array}$$

of distinguished n-exangles. Any $D_{\bullet} \in S(X_{n+1})$ can be written as

$$D_{\bullet}\colon\ D \longrightarrow D_1 \longrightarrow D_2 \longrightarrow \ldots \longrightarrow D_{n-1} \longrightarrow \bigoplus_{i=1}^{p} H_i \xrightarrow{d=[d_1, d_2, \ldots, d_p]} X_{n+1} \dashrightarrow$$

with $H_i \in \mathsf{ind}(\mathscr{C}), d_i \in \mathrm{rad}_{\mathscr{C}}(H_i, X_{n+1})$, $i=1,2,...,p$. Since $D_{\bullet} \in S(X_{n+1})$ is non-split, d is not split epimorphism implies that $d \in \mathrm{rad}_{\mathscr{C}}(\bigoplus_{i=1}^{p} H_i, X_{n+1})$. By the definitions of λ_{ij} and γ, there exists a morphism $\rho\colon \bigoplus_{i=1}^{p} H_i \to M$ such that $d=\gamma\rho$. By Lemma 2.10, we have the following commutative diagram

$$\begin{array}{ccccccccccc}
D & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow \ldots \longrightarrow & D_{n-1} & \longrightarrow & \bigoplus_{i=1}^{p} H_i & \xrightarrow{d} & X_{n+1} & \dashrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \rho} & & \| & \\
B & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow \ldots \longrightarrow & M_{n-1} & \longrightarrow & M & \xrightarrow{\gamma} & X_{n+1} & \dashrightarrow
\end{array}$$

of distinguished n-exangles, where $B=U\oplus V$. Thus we get the following commutative diagram

$$\begin{array}{ccccccccccc}
D & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow \ldots \longrightarrow & D_{n-1} & \longrightarrow & \bigoplus_{i=1}^{p} H_i & \xrightarrow{d} & X_{n+1} & \dashrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & \\
U & \xrightarrow{\omega_0} & C_1 & \xrightarrow{\omega_1} & C_2 & \xrightarrow{\omega_2} \ldots \xrightarrow{\omega_{n-2}} & C_{n-1} & \xrightarrow{\omega_{n-1}} & C_n & \xrightarrow{\omega_n} & X_{n+1} & \dashrightarrow
\end{array}$$

of distinguished n-exangles. This shows that $C_{\bullet}$ is a minimal element in $S(X)$. $\square$

We are now ready to state and prove our main result.

Theorem 3.12. *Let $\mathscr{C}$ be a locally finite n-exangulated category. If $X_{n+1}\in \mathsf{ind}(\mathscr{C})$ is a non-projective object, then there exists an Auslander-Reiten n-exangle ending at X_{n+1}, and if $Y_0\in\mathsf{ind}(\mathscr{C})$ is a non-injective object, then there exists an Auslander-Reiten n-exangle starting at Y_0. In this case, we say that $\mathscr{C}$ has Auslander-Reiten n-exangles.*

Proof. Since $X_{n+1}\in\mathsf{ind}(\mathscr{C})$ is a non-projective object, by Lemma 3.6 we know that the set $S(X_{n+1})$ is non-empty. Thus by Lemma 3.11, there is a distinguished n-exangle

$$X_\bullet : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \overset{\delta}{\dashrightarrow}$$

where $\alpha_1,\alpha_2,\dots,\alpha_{n-1}\in\mathrm{rad}_{\mathscr{C}}$ and $X_0\in\mathsf{ind}(\mathscr{C})$, such that $X_\bullet$ is a minimal element in $S(X_{n+1})$. Then $\mathrm{End}(X_0)$ is local.

We want to prove that $X_\bullet$ is an Auslander-Reiten n-exangle, by Lemma 3.3, it is enough to show that α_n is right almost split.

Assume that $g\colon M_{n+1}\to X_{n+1}$ is not a split epimorphism, we claim that g factors through α_n. By (EA2), we can observe that (id_{X_0},g) has a *good lift* $g_\bullet=(\mathrm{id}_{X_0},\varphi_1,\dots,\varphi_n,g)$, that is, there exists the following commutative diagram of distinguished n-exangles

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{\gamma_0} & B_1 & \xrightarrow{\gamma_1} & B_2 & \xrightarrow{\gamma_2} \cdots \longrightarrow & B_n & \xrightarrow{\gamma_n} & M_{n+1} & \overset{g^*\delta}{\dashrightarrow} \\
\| & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle g} & \\
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} \cdots \longrightarrow & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \overset{\delta}{\dashrightarrow}
\end{array}$$

such that

$$N_\bullet :\; B_1 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow \dots \longrightarrow N_{n-1} \longrightarrow M_{n+1}\oplus X_n \xrightarrow{[g,\ \alpha_n]} X_{n+1} \overset{(\gamma_0)_*\delta}{\dashrightarrow}$$

is a distinguished n-exangle in $\mathscr{C}$, where $N_i=B_{i+1}\oplus X_i$, $i=1,2,\dots,n-1$. Since $X_{n+1}\in\mathsf{ind}(\mathscr{C})$, g and α_n are not split epimorphisms, we have that $[g,\alpha_n]$ is also not split epimorphism by Lemma 3.9. That is, $N_\bullet$ is non-split.

Without loss of generality, we can assume that $B_1=U\oplus V$ where U and V are indecomposable. For the morphism $p_1=[1,0]\colon U\oplus V\to U$, by (EA2$^{\mathrm{op}}$), we can observe that $(p_1,\mathrm{id}_{X_{n+1}})$ has a *good lift* $h_\bullet=(p_1,\phi_1,\dots,\phi_n,\mathrm{id}_{X_{n+1}})$, that is, there exists the following commutative diagram of distinguished n-exangles

$$\begin{array}{ccccccccccc}
U\oplus V & \xrightarrow{[u,\ v]} & N_1 & \longrightarrow & N_2 & \longrightarrow \cdots \longrightarrow & N_{n-1} & \longrightarrow & M_{n+1}\oplus X_n & \longrightarrow & X_{n+1} \dashrightarrow \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle\phi_1} & & \downarrow & & \downarrow & & \downarrow & & \| \\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow \cdots \longrightarrow & L_{n-1} & \longrightarrow & L_n & \longrightarrow & X_{n+1} \dashrightarrow .
\end{array}$$

Similarly, for the morphism $p_2=[0,1]\colon U\oplus V\to V$, there exists the following commutative diagram of distinguished n-exangles

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{[u,\ v]} & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & N_{n-1} & \longrightarrow & M_{n+1}\oplus X_n & \longrightarrow & X_{n+1} \dashrightarrow \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q_1} & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
V & \xrightarrow{\eta_0} & Q_1 & \longrightarrow & Q_2 & \longrightarrow & \cdots & \longrightarrow & Q_{n-1} & \longrightarrow & Q_n & \longrightarrow & X_{n+1} \dashrightarrow .
\end{array}$$

By Remark 3.7, we conclude that at least one of the following two distinguished n-exangles is non-split

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_{n-1} \longrightarrow L_n \longrightarrow X_{n+1} \dashrightarrow$$

$$V \xrightarrow{\eta_0} Q_1 \longrightarrow Q_2 \longrightarrow \cdots \longrightarrow Q_{n-1} \longrightarrow Q_n \longrightarrow X_{n+1} \dashrightarrow .$$

Without loss of generality, we assume that

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_{n-1} \longrightarrow L_n \longrightarrow X_{n+1} \dashrightarrow$$

is non-split. By Lemma 2.15, we can find a non-split distinguished n-exangle

$$C_{\bullet}\colon U \xrightarrow{\lambda_0} C_1 \xrightarrow{\lambda_1} C_2 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_{n-2}} C_{n-1} \xrightarrow{\lambda_{n-1}} C_n \xrightarrow{\lambda_n} X_{n+1} \dashrightarrow$$

with $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}$ in $\mathrm{rad}_{\mathscr{C}}$. By (R0) and Lemma 2.10, we have the following commutative diagram

$$\begin{array}{ccccccccccccc}
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} \dashrightarrow \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & \| \\
U\oplus V & \xrightarrow{[u,\ v]} & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & N_{n-1} & \longrightarrow & M_{n+1}\oplus X_n & \xrightarrow{[g,\alpha_n]} & X_{n+1} \dashrightarrow \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \phi_1} & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_{n-1} & \longrightarrow & L_n & \longrightarrow & X_{n+1} \dashrightarrow \\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} & \ldots & \xrightarrow{\lambda_{n-2}} & C_{n-1} & \xrightarrow{\lambda_{n-1}} & C_n & \xrightarrow{\lambda_n} & X_{n+1} \dashrightarrow
\end{array}$$

of distinguished n-exangles. We obtain that $X_{\bullet}>C_{\bullet}$ implies that $X_{\bullet}\sim C_{\bullet}$ since $X_{\bullet}$ is the minimal element in $S(X_{n+1})$. Thus there exists the following commutative diagram

$$\begin{array}{ccccccccccccccc}
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} & \cdots & \xrightarrow{\lambda_{n-2}} & C_{n-1} & \xrightarrow{\lambda_{n-1}} & C_n & \xrightarrow{\lambda_n} & X_{n+1} & \dashrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| & & \\
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \dashrightarrow &
\end{array}$$

of distinguished n-exangles. Hence we get the following commutative diagram

$$\begin{array}{ccccccccccccccc}
U\oplus V & \xrightarrow{[u,\ v]} & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & N_{n-1} & \longrightarrow & M_{n+1}\oplus X_n & \xrightarrow{[g,\alpha_n]} & X_{n+1} & \dashrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle [a,b]} & & \| & & \\
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \dashrightarrow &
\end{array}$$

of distinguished n-exangles. It follows that $g=\alpha_n a$. This shows that α_n is right almost split.

Similarly, we can show that if $Y_0\in\mathsf{ind}(\mathscr{C})$ is a non-injective object, then there exists an Auslander-Reiten n-exangle starting at Y_0. Thus $\mathscr{C}$ has Auslander-Reiten n-exangles. □

By applying Theorem 3.12 to $(n+2)$-angulated categories, we have the following.

Corollary 3.13. ([26, *Theorem* 1.1]) *Let $\mathscr{C}$ be a locally finite $(n+2)$-angulated category. Then $\mathscr{C}$ has Auslander-Reiten $(n+2)$-angles.*

By applying Theorem 3.12 to n-abelian categories, we have the following.

Corollary 3.14. ([22, *Theorem* 1.1]) *Let $\mathscr{C}$ be a locally finite n-abelian category. Then $\mathscr{C}$ has n-Auslander-Reiten sequences.*

By applying Theorem 3.12 to n-exact categories, we have the following.

Corollary 3.15. *Let $\mathscr{C}$ be a locally finite n-exact category. Then $\mathscr{C}$ has n-Auslander-Reiten sequences.*

Remark 3.16. As a special case of Theorem 3.12 when $n=1$, that is, if $\mathscr{C}$ is a locally finite extriangulated category, then $\mathscr{C}$ has Auslander-Reiten $\mathbb{E}$-triangles, see [27, Theorem 3.12].

Remark 3.17. If $\mathscr{C}$ is a locally finite triangulated category, then $\mathscr{C}$ has Auslander-Reiten triangles, see [23, Proposition 1.3] and [24, Lemma 1.4.3].

Acknowledgments

The authors would like to thank the referee for reading the paper carefully and for many suggestions on mathematics and English expressions.

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Received October 27, 2021
in revised form January 10, 2022