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Proper holomorphic embeddings of complements of large Cantor sets in $\mathbb{C}^2$

G.D. Di Salvo and E.F. Wold

Abstract. We prove that there exist Cantor sets of arbitrarily large 2-dimensional Lebesgue measure whose complements admit proper holomorphic embeddings in $\mathbb{C}^2$.

1. Introduction

1.1. The main result

A major unresolved issue, known as Forster's Conjecture, is whether or not every open Riemann surface X admits a holomorphic embedding into $\mathbb{C}^2$, and, if it does, whether it admits a *proper* holomorphic embedding. For instance, if Y is a compact Riemann surface, and if $X=Y\setminus C$ where C is a closed set whose connected components are all points, it is unknown whether X embeds (properly or not) into $\mathbb{C}^2$. We may consider two extremal cases: (i) the case where C is a finite set, and (ii) the case where C is a Cantor set, and we may further consider the simplest compact Riemann surface in this context, namely $Y=\mathbb{P}^1$. Then in case (i), it is clear that X admits a proper holomorphic embedding into $\mathbb{C}^2$, so we will consider the case (ii).

Let Q denote the square $Q=[-1,1]\times[-1,1]\subset\mathbb{C}$, and let μ denote the 2–dimensional Lebesgue measure on $\mathbb{C}$. A procedure for constructing a (large) Cantor set $C\subset Q$ is as follows (see Section 3 for a more detailed description). Let l_1 denote the vertical line dividing Q into two equal pieces, choose $\delta_1>0$ small, and remove an open δ_1–neighbourhood of l_1 to obtain a union Q_2 of two disjoint rectangles. Next, let $l_2^j, j=1,2$, be horizontal lines dividing each rectangle in Q_2 into equal pieces, choose δ_2 small, and remove an open δ_2–neighbourhood of $l_2^1\cup l_2^2$ to obtain

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a disjoint union Q_3 of four rectangles. Next, switch back to vertical lines, chose δ_3 small to obtain Q_4 and so forth, to obtain a sequence $\delta_j \to 0$ and nested sequence Q_j of rectangles such that $C = \cap_j Q_j$ is a Cantor set contained in Q. Our main result is the following.

Theorem 1.1. *There are sequences $\{\delta_j\}_j$ converging to zero arbitrarily fast such that the complement $\mathbb{P}^1 \setminus C$ of the resulting Cantor set admits a proper holomorphic embedding into $\mathbb{C}^2$. In particular, for any $\varepsilon > 0$ we may achieve that $\mu(C) > 4 - \varepsilon$.*

The motivation for proving this result is that there have been speculations that considering complements of "fat" Cantor sets could lead to counterexamples to Forster's Conjecture.

We note that Orevkov [18] showed the existence of a Cantor set $C \subset \mathbb{P}^1$ such that $\mathbb{P}^1 \setminus C$ admits a proper holomorphic embedding into $\mathbb{C}^2$. His construction is quite cryptical and it is explained in detail in [4], where it is also proved that C can be obtained to have zero Hausdorff dimension. From such a construction it seems difficult, or perhaps impossible, to achieve that C is large.

1.2. History

Dealing with Stein manifolds, one of the most important goals to achieve is to embed them properly holomorphically into $\mathbb{C}^N$ for some N. A first important result comes from Remmert [19], who proved in 1956 that every n–dimensional Stein manifold admits a proper holomorphic embedding into $\mathbb{C}^N$ for N big enough. Such a result was made precise by Bishop and Narasimhan, who independently proved in 1960–61 that N can be taken to be $2n+1$ (see [3] and [17]). In 1970, Forster [7] improved Bishop–Narasimhan's result, decreasing N to $\lfloor \frac{5n}{3} \rfloor + 2$ and proving that it is not possible for N to go below $\lfloor \frac{3n}{2} \rfloor + 1$ and conjecturing that the euclidean dimension could have been improved exactly to $\lfloor \frac{3n}{2} \rfloor + 1$. Eliashberg, Gromov (1992) and Shürmann (1997) proved the following

Theorem 1.2. (Eliashberg–Gromov [5] (1992) and Shürmann [21] (1997)) *Every n–dimensional Stein manifold X, with $n \geq 2$ embeds properly holomorphically into $\mathbb{C}^N$ with $N = \lfloor \frac{3n}{2} \rfloor + 1$.*

The proof of the theorem breaks down when $n = 1$; since 1–dimensional Stein manifolds are precisely open connected Riemann surfaces, Forster's conjecture reduces to the following

Conjecture 1.1. *Every open connected Riemann surface embeds properly holomorphically into $\mathbb{C}^2$.*

So far, only a few open Riemann surfaces are known to admit a proper holomorphic embedding into $\mathbb{C}^2$. The first known examples are the open unit disk in $\mathbb{C}$ (Kasahara–Nishino, 1970, [22]), open annuli in $\mathbb{C}$ (Laufer, 1973, [14]) and the punctured disk in $\mathbb{C}$ (Alexander, 1977, [1]). Later (1995) Stensønes and Globevnik proved in [12] that every finitely connected planar domain without isolated boundary points verifies the conjecture. In 2009, Wold and Forstnerič proved the best result known so far: if $\overline{D}$ is a Riemann surface with smooth enough boundary which admits a smooth embedding into $\mathbb{C}^2$, holomorphic on the interior D, then D admits a proper holomorphic embedding into $\mathbb{C}^2$ (see [10] and next section). Other remarkable results include proper holomorphic embeddings of certain Riemann surfaces into $\mathbb{C}^2$ with interpolation (see [13]), deformation of continuous mappings $f\colon S\to X$ between Stein manifolds into proper holomorphic embeddings under certain hypothesis on the dimension of the spaces (see [2]), embeddings of infinitely connected planar domains into $\mathbb{C}^2$ (see [11]), the existence of a homotopy of continuous mappings $f\colon D\to\mathbb{C}\times\mathbb{C}^*$ into proper holomorphic embedding whenever D is a finitely connected planar domain without punctures (see [20]), existence of proper holomorphic embeddings of the unit disc $\mathbb{B}$ into connected pseudoconvex Runge domains $\Omega\subset\mathbb{C}^n$ (when $n\geq 2$) whose image contains arbitrarily fixed discrete subsets of Ω (see [9]), approximation of proper embeddings on smooth curves contained in a finitely connected planar domain D into $\mathbb{C}^n$ (with $n\geq 2$) by proper holomorphic embeddings $f\colon D\hookrightarrow\mathbb{C}^n$ (see [15]), and the existence of proper holomorphic embeddings into $\mathbb{C}^2$ of certain infinitely connected domains Ω lying inside a bordered Riemann surface $\overline{D}$ knowing to admit a proper holomorphic embedding into $\mathbb{C}^2$ [16].

2. Preliminaries

2.1. Notation

We will use the following notation.

Given $K\subset\mathbb{C}$ and a positive real number δ, we define the open subset

$$K(\delta):=\{z\in\mathbb{C} : \operatorname{dist}(z,K)<\delta\}.$$

• For a closed subset $K\subset\mathbb{P}^1$ we denote by $\mathcal{O}(K)$ the algebra of continuous functions $f\in\mathcal{C}(K)$ such that there exists an open set $U\subset\mathbb{P}^1$ containing K, and $F\in\mathcal{O}(U)$ with $F|_K=f$.

• We let $\pi_j\colon\mathbb{C}^2\to\mathbb{C}$ denote the projection onto the j–th coordinate line, and given a point $p\in\mathbb{C}^2$ we denote the vertical complex line through p by

$$\Lambda_p:=\pi_1^{-1}(\pi_1(p))=\{(\pi_1(p),\zeta) : \zeta\in\mathbb{C}\}.$$

• If X is a domain with piecewise smooth boundary in a Riemann surface Y, $f\colon \overline{X}\to\mathbb{C}^2$ is a holomorphic map, and if $a\in\partial X$ is a smooth boundary point, we say that $f(a)$ is π_1–exposed for $f(\overline{X})$ if $f(\overline{X})\cap\Lambda_{f(a)}=\{f(a)\}$, and $\pi_1\circ f$ is an embedding sufficiently close to a. Similarly, for a smooth map $\gamma\colon [0,1]\to\mathbb{C}^2$, we say that $\gamma([0,1])$ is exposed at $\gamma(1)$ if $\gamma([0,1])\cap\Lambda_{\gamma(1)}=\{\gamma(1)\}$, and $\pi_1\circ\gamma$ is an embedding sufficiently close to 1.

2.2. Results

In this section, we collect the technical tools needed to prove Theorem 1.1.

The following result is essentially Theorem 4.2 in [10]. Although (1) and (2) were not stated explicitly in [10] they are evident from the proof therein and were added to the corresponding Theorem 2.8 in [11].

Theorem 2.1. *Let X be a smoothly bounded domain in a Riemann surface Y, $f\colon \overline{X}\hookrightarrow\mathbb{C}^2$ a holomorphic embedding, and $a_1,\dots,a_m\in\partial X$. Let $\gamma_j\colon [0,1]\to\mathbb{C}^2$ $(j=1,\dots,m)$ be smooth embedded arcs with pairwise disjoint images satisfying the following properties*:

• *$\gamma_j([0,1])\cap f(\overline{X})=\gamma_j(0)=f(a_j)$ for $j=1,\dots,m$, and*

• *the image $E:=f(\overline{X})\cup\bigcup_{j=1}^m\gamma_j([0,1])$ is π_1–exposed at $\gamma_j(1)$ for $j=1,\dots,m$.*

Then given an open set $V\subset\mathbb{C}^2$ containing $\bigcup_{j=1}^m\gamma_j([0,1])$, an open set $U\subset Y$ containing the points a_j that satisfies $f(\overline{U\cap X})\subset V$, and any $\varepsilon>0$, there exists a holomorphic embedding $F\colon \overline{X}\hookrightarrow\mathbb{C}^2$ with the following properties:

(1) $\|F-f\|_{\overline{X}\setminus U}<\varepsilon$,

(2) $F(\overline{U\cap X})\subset V$, *and*

(3) $F(a_j)=\gamma_j(1)$, *and $F(\overline{X})$ is π_1–exposed at $F(a_j)$ for $j=1,\dots,m$.*

The following is essentially Lemma 1 in [23]. The difference is that Lemma 1 was stated for π_1 instead of π_2, and for curves $\lambda\colon [0,+\infty)\to\mathbb{C}^2$ instead of $\lambda\colon (-\infty,+\infty)\to\mathbb{C}^2$ – neither make a difference for the proof.

Lemma 2.1. *Let $K\subset\mathbb{C}^2$ be a polynomially convex compact set, and let*

$$\Lambda=\{\lambda_j(t): j=1,\dots,m,\quad t\in(-\infty,+\infty)\}$$

be a collection of pairwise disjoint smooth curves in $\mathbb{C}^2\setminus K$ without self-intersection, enjoying the immediate projection property (*with respect to π_2*):

• $\lim_{|t|\to\infty}|\pi_2(\lambda_j(t))|=\infty$ *for all j, and*

• *there exists an $M>0$ such that $\mathbb{C}\setminus(R\overline{\mathbb{B}}\cup\pi_2(\Lambda))$ does not contain any relatively compact components for $R\geq M$.*

Then for any $r>0$ *and* $\varepsilon>0$ *there exists* $\phi\in\operatorname{Aut}\mathbb{C}^2$ *such that the following are satisfied*:

(i) $\|\phi-\operatorname{Id}\|_K<\varepsilon$, *and*

(ii) $\phi(\Lambda)\subset\mathbb{C}^2\setminus r\overline{\mathbb{B}^2}$.

3. The induction step

We will now describe an inductive procedure to construct a nested sequence of closed rectangles $Q_n\subset Q$, along with holomorphic embeddings $f_n\colon\overline{\mathbb{P}^1\setminus Q_n}\to\mathbb{C}^2$ that will be used to construct a proper holomorphic embedding

$$f\colon \mathbb{P}^1\setminus\bigcap_n Q_n\hookrightarrow\mathbb{C}^2,$$

where $C=\bigcap_n Q_n$ will be a Cantor set, where the construction will enable us to ensure that its Lebesgue measure $\mu(C)$ is arbitrarily close to 4.

Set $Q_1:=Q$ and set $C_1:=\overline{\mathbb{P}^1\setminus Q_1}$. To construct Q_2 from Q_1 we let l_1 be the vertical line segment dividing Q_1 into two equal pieces. Then, for $0<\delta_2<<1$, we set

$$Q_2:=Q_1\setminus l_1(\delta_2),$$

and we set $C_2:=\overline{\mathbb{P}^1\setminus Q_2}$. Then C_2 is the complement of the disjoint union of 2 open rectangles $(Q_2^j)^\circ$, $j=1,2$, contained in Q_1.

Assume now that we have constructed a nested sequence $\{Q_j\}_{j=1}^n$, $n\geq2$, where

$$Q_n=\bigsqcup_{j=1}^{2^{n-1}}Q_n^j$$

is the disjoint union of 2^{n-1} closed rectangles contained in Q_{n-1}, along with an increasing sequence of closed subsets $C_n:=\overline{\mathbb{P}^1\setminus Q_n}$ in $\mathbb{P}^1$. We let l_n^j be the line segment – vertical for n odd, horizontal for n even – dividing Q_n^j into two equal pieces, we set $l_n:=\bigsqcup_{j=1}^{2^{n-1}}l_n^j$, and for $\delta_{n+1}>0$ small enough we define

$$Q_{n+1}:=Q_n\setminus l_n(\delta_{n+1})=:\bigsqcup_{j=1}^{2^n}Q_{n+1}^j,$$

and

$$C_{n+1}:=\overline{\mathbb{P}^1\setminus Q_{n+1}}\,.$$

Proposition 3.1. *With the procedure above assume that we have constructed Q_n and C_n for $n\geq 1$. Let $K_n\subset C_n^\circ$ be a compact set, let $r_n>0$, and assume that $f_n\colon C_n\hookrightarrow\mathbb{C}^2$ is a holomorphic embedding such that*

$$f_n(\overline{C_n\setminus K_n})\subset\mathbb{C}^2\setminus r_n\overline{\mathbb{B}^2}. \tag{3.1}$$

Then for any $\varepsilon_n>0$ and any $r_{n+1}>r_n$, there exist $\delta_{n+1}>0$ arbitrarily close to zero and a holomorphic embedding $f_{n+1}\colon C_{n+1}\hookrightarrow\mathbb{C}^2$ such that

(a) $\|f_{n+1}-f_n\|_{K_n}<\varepsilon_n$,
(b) $f_{n+1}(\overline{C_{n+1}\setminus K_n})\subset\mathbb{C}^2\setminus r_n\overline{\mathbb{B}^2}$,
(c) $f_{n+1}(\partial C_{n+1})\subset\mathbb{C}^2\setminus r_{n+1}\overline{\mathbb{B}^2}$.

Proof. We extend f_n to a smooth embedding $\tilde{f}_n\colon C_n\cup l_n\hookrightarrow\mathbb{C}^2$ with $\tilde{f}_n(l_n)$ lying close enough to $f_n(\partial C_n)$ so that by (3.1) we get

$$\tilde{f}_n(l_n)\subset\mathbb{C}^2\setminus r_n\overline{\mathbb{B}^2}. \tag{3.2}$$

Now Mergelyan's theorem (see e.g., [6]) ensures the existence of a holomorphic embedding $\hat{f}_{n+1}\colon C_n\cup l_n\hookrightarrow\mathbb{C}^2$ such that

$$\|\hat{f}_{n+1}-\tilde{f}_n\|_{C_n\cup l_n}<\frac{\varepsilon_n}{4},$$

and

$$\hat{f}_{n+1}(\overline{(C_n\cup l_n)\setminus K_n})\subset\mathbb{C}^2\setminus r_n\overline{\mathbb{B}^2}. \tag{3.3}$$

Then by choosing a preliminary $\tilde{\delta}_{n+1}>0$ sufficiently small (to be shrunk further later), and letting the set corresponding to C_{n+1} be denoted by $\widetilde{C}_{n+1}$ (and similarly for Q_{n+1}), we have that $\hat{f}_{n+1}\in\mathcal{O}(\widetilde{C}_{n+1})$, and

$$\hat{f}_{n+1}(\overline{\widetilde{C}_{n+1}\setminus K_n})\subset\mathbb{C}^2\setminus r_n\overline{\mathbb{B}^2}. \tag{3.4}$$

Next, recall that $\widetilde{Q}_{n+1}$ is constructed from Q_n by splitting each Q_n^j into two smaller rectangles $\widetilde{Q}_n^{j,1}$ and $\widetilde{Q}_n^{j,2}$, by removing the strip $l_n^j(\tilde{\delta}_{n+1})$. Choose smooth boundary points

$$\tilde{a}_{ji}\in\partial\widetilde{Q}_n^{j,i}\cap\overline{l_n^j(\tilde{\delta}_{n+1})}, \tag{3.5}$$

for $i=1,2$, and $j=1,\dots,2^{n-1}$, and relabel these to get 2^n boundary points a_j, one in each $\partial\widetilde{Q}_{n+1}^j$.

Now choose 2^n pairwise disjoint smoothly embedded arcs $\gamma_j\colon[0,1]\hookrightarrow\mathbb{C}^2$ disjoint from $r_n\overline{\mathbb{B}^2}$, such that

$$\gamma_j([0,1])\cap\hat{f}_{n+1}(\widetilde{C}_{n+1})=\hat{f}_{n+1}(a_j)=\gamma_j(0),$$

and such that each point $\gamma_j(1)$ is π_1–exposed for the surface

$$\hat{f}_{n+1}(\widetilde{C}_{n+1}) \cup \bigcup_{j=1}^{2^n} \gamma_j([0,1]).$$

Choose an open set $V \subset \mathbb{C}^2$ containing the arcs $\gamma_j([0,1])$ with $\overline{V} \cap r_n \overline{\mathbb{B}^2} = \varnothing$ and take $U \subset \mathbb{P}^1$ to be the union of sufficiently small open balls centered at the points a_j, so that $U \cap K_n = \varnothing$ and $\hat{f}_{n+1}(\overline{U \cap \widetilde{C}_{n+1}}) \subset V$. Then Theorem 2.1 furnishes a holomorphic embedding $F_{n+1} \colon \widetilde{C}_{n+1} \hookrightarrow \mathbb{C}^2$ such that $p_j := F_{n+1}(a_j) = \gamma_j(1)$ is an exposed point for $F_{n+1}(\widetilde{C}_{n+1})$ for each j, and

$$\|F_{n+1} - \hat{f}_{n+1}\|_{K_n} < \frac{\varepsilon_n}{4}, \tag{3.6}$$

and also

$$F_{n+1}(\overline{\widetilde{C}_{n+1} \setminus K_n}) \subset \mathbb{C}^2 \setminus r_n \overline{\mathbb{B}^2}. \tag{3.7}$$

Now choose $\alpha_j \in \mathbb{C}$, $j = 1, ..., 2^n$, such that setting

$$g_{n+1}(z,w) := \left(z, w + \sum_{j=1}^{2^n} \frac{\alpha_j}{\pi_1(p_j) - z} \right),$$

we have that

$$\|g_{n+1} \circ F_{n+1} - F_{n+1}\|_{K_n} < \frac{\varepsilon_n}{4},$$

and such that the conditions in Lemma 2.1 are satisfied for the collection Λ of curves

$$\lambda_{ji} := g_{n+1} \circ F_{n+1}(\partial \widetilde{Q}_n^{j,i}), \quad i = 1, 2,\ j = 1, ..., 2^{n-1},$$

that are the boundary of the unbounded complex curve

$$X_{n+1} := g_{n+1} \circ F_{n+1}(\widetilde{C}_{n+1}).$$

Note that we still have

$$g_{n+1} \circ F_{n+1}(\overline{\widetilde{C}_{n+1} \setminus K_n}) \subset \mathbb{C}^2 \setminus r_n \overline{\mathbb{B}^2}. \tag{3.8}$$

Choose $0 < \eta << 1$ such that $(r_n + \eta)\overline{\mathbb{B}^2} \cap \Lambda = \varnothing$. We may choose a compact polynomially convex set $K' \subset X_{n+1}^{\circ}$ with $g_{n+1} \circ F_{n+1}(K_n) \subset K'$ such that $L = (r_n + \eta)\overline{\mathbb{B}^2} \cup K'$ is polynomially convex (see e.g., Theorem 4.14.6 in [8]). Then by Lemma 2.1 there exists $\phi_{n+1} \in \operatorname{Aut} \mathbb{C}^2$ such that

$$\phi_{n+1}(\Lambda) \subset \mathbb{C}^2 \setminus r_{n+1} \overline{\mathbb{B}^2}, \tag{3.9}$$

and

$$\|\phi_{n+1} - \mathrm{Id}\,\|_L < \frac{\min\{\eta, \varepsilon_n\}}{4}\,. \tag{3.10}$$

We consider the map $f_{n+1}\colon \widetilde{C}_{n+1} \to \mathbb{C}^2$ defined by

$$f_{n+1} := \phi_{n+1} \circ g_{n+1} \circ F_{n+1}\,.$$

We have that (a) and (b) (with $\widetilde{C}_{n+1}$ instead of C_{n+1}) clearly hold, but now f_{n+1} has singularites on $\partial\widetilde{C}_{n+1}$. However, we now consider $0<\delta_{n+1}<\tilde{\delta}_{n+1}$ to see what happens on C_{n+1}. As the points to expose are taken on the boundary components (see (3.5)), the singularities of f_{n+1} are not contained in C_{n+1} for any such δ_{n+1}, and so $f_n\colon C_{n+1} \to \mathbb{C}^2$ is holomorphic. Finally, since ∂C_{n+1} will converge to $\partial\widetilde{C}_{n+1}$ as $\delta_{n+1} \to \tilde{\delta}_{n+1}$ we have (c) for δ_{n+1} sufficiently close to $\tilde{\delta}_{n+1}$. □

4. Proof of Theorem 1.1

We will prove Theorem 1.1 via an inductive construction, where Proposition 3.1 provides us with the inductive step. Without loss of generality, we may assume that $\varepsilon<1$.

4.1. The induction scheme

To start the induction, with the notation as in Section 3, we define $f_1\colon C_1 \hookrightarrow \mathbb{C}^2$ by $f_1(\zeta):=(2/\zeta, 0)$ for $\zeta\in\mathbb{C}$, and $f(\infty):=(0,0)$. Setting $r_1=1$ we note that $f_1(\partial C_1)\subset \mathbb{C}^2\setminus r_1\overline{\mathbb{B}^2}$, so if we choose $0<\delta_1'<<1$ sufficiently close to zero, and set

$$K_1 := \mathbb{P}^1 \setminus Q_1(\delta_1'),$$

we have that $K_1 \subset C_1^\circ$ and $f_1(\overline{C_1\setminus K_1})\subset\mathbb{C}^2\setminus r_1\overline{\mathbb{B}^2}$. Then the conditions in Proposition 3.1 are satisfied with $n=1$, and setting $\delta_2\leq\varepsilon\cdot 2^{-4}$, $r_2=2$, we let f_2 be the map furnished by the proposition, with ε_1 explained in the induction scheme below. We then choose $\delta_2'<\delta_2/2$ sufficiently close to zero such that if we set

$$K_2 := \mathbb{P}^1 \setminus Q_2(\delta_2'),$$

we have $K_2\subset C_2^\circ$ and $f_2(\overline{C_2\setminus K_2})\subset\mathbb{C}^2\setminus r_2\overline{\mathbb{B}^2}$.

Let us now state our induction hypothesis I_n for some $n\geq 2$. We assume that we have found and constructed the following.

(i)$_n$ A decreasing sequence $\delta_2>\delta_3>\ldots>\delta_n$ of numbers with $\delta_k\leq\varepsilon\cdot 2^{-2k}$ such that $\{Q_k\}_{k=1}^n$ is a nested sequence of rectangles.

(ii)$_n$ A decreasing sequence $\delta'_1 > \delta'_2 > ... > \delta'_n$ of numbers with δ'_1, δ'_2 as above, and $\delta'_k < \delta_k/2$ for $k=1, ..., n$, and holomorphic embeddings $f_k\colon C_k \hookrightarrow \mathbb{C}^2$ such that, setting $K_k := \mathbb{P}^1 \setminus Q_k(\delta'_k)$, we have that $f_m(\overline{C_m \setminus K_k}) \subset \mathbb{C}^2 \setminus r_k \overline{\mathbb{B}^2}$ for $1 \leq k \leq m \leq n$, where $r_k \geq k$.

(iii)$_n$ A sequence of positive numbers $\{\eta_k\}_{k=2}^n$ such that if $f\colon K_k \to \mathbb{C}^2$ is a holomorphic map with $\|f - f_k\|_{K_k} < \eta_k$, then $f\colon K_{k-1} \hookrightarrow \mathbb{C}^2$ is an embedding.

(iv)$_n$ A sequence of positive numbers $\{\varepsilon_k\}_{k=1}^{n-1}$ such that $\varepsilon_{k+j} < \eta_k \cdot 2^{-j-1}$, $j \leq n-k-1$, with $\|f_k - f_{k-1}\|_{K_{k-1}} < \varepsilon_{k-1}$ for $k=2, ..., n$.

Our constructions above gives (i)$_n$, (ii)$_n$ and (iv)$_n$ in the case $n=2$ (possibly shrinking ε_1). Then, choosing η_2 small enough, gives f and f' close to f_2 and f'_2 respectively (the latter by Cauchy estimates) on K_2 such that f is injective and f' never vanishes on K_1. Being K_1 compact, this is enough to achieve (iii)$_n$ when $n=2$.

4.2. Passing from I_n to I_{n+1}

Let us assume that I_n is true and prove I_{n+1}. First of all we have that (i)$_{n+1}$, (iii)$_{n+1}$ and the first part of (iv)$_{n+1}$ are just a matter of choosing respectively δ_{n+1}, η_{n+1} and ε_n sufficiently small. By (ii)$_n$ with $k=n$, and with ε_n above fixed, we may apply Proposition 3.1 to get a holomorphic embedding $f_{n+1}\colon C_{n+1} \hookrightarrow \mathbb{C}^2$ to obtain the second part of (iv)$_{n+1}$ and (ii)$_{n+1}$ with $m=n+1$ and $k=n$. Next, by choosing δ'_{n+1} sufficiently small we get (ii)$_{n+1}$ for $k=m=n+1$. It remains to explain how to achieve (ii)$_{n+1}$ for $m=n+1$ and $k=1, ..., n-1$. Since

$$\overline{C_{n+1} \setminus K_k} = \overline{C_{n+1} \setminus K_n} \cup \overline{K_n \setminus K_k}$$

what is needed is $f_{n+1}(\overline{K_n \setminus K_k}) \subset \mathbb{C}^2 \setminus r_k \overline{\mathbb{B}^2}$. This follows from (ii)$_n$, possibly after having decreased ε_n.

4.3. Proof of Theorem 1.1

Consider the objects constructed in the inductive scheme above. Then by (iv)$_n$ we have that $\lim_{j\to\infty} f_j = f$ exists on K_k for any k. We have that

$$\left(\bigcup_k C_k\right)^{\circ} = \mathbb{P}^1 \setminus \bigcap_k Q_k =: \mathbb{P}^1 \setminus C$$

and so $\lim_{j\to\infty} f_j = f$ exists on $\mathbb{P}^1 \setminus C$. Now for any fixed k we get by (ii)$_n$ that $f_n^{-1}(r_k \overline{\mathbb{B}^2}) \subset K_k$ for all $n > k$ and therefore $f^{-1}(r_k \overline{\mathbb{B}^2}) \subset K_k$, so f is proper. By (iv)$_n$

we get that $\|f-f_k\|_{K_k}<\eta_k$, hence by (iii)$_n$ we have that $f\colon K_{k-1}\hookrightarrow\mathbb{C}^n$ is an embedding for all k, so f is an embedding. Finally, note that when constructing a rectangle Q_{n+1} from Q_n, a crude estimate gives that one obtains Q_{n+1} by removing strips of total area bounded by $2^n\cdot\delta_n$. It follows that

$$\mu(C)=\mu\left(\bigcap_n Q_n\right)\geq 4-\sum_{n=1}^{\infty}2^n\cdot\delta_n\geq 4-\varepsilon\cdot\sum_{n=1}^{\infty}2^{-n}>4-\varepsilon.$$

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G.D. Di Salvo
Department of Mathematics
University of Oslo
Postboks 1053 Blindern
NO-0316 Oslo
Norway
giovandd@math.uio.no

E.F. Wold
Department of Mathematics
University of Oslo
Postboks 1053 Blindern
NO-0316 Oslo
Norway
erlendfw@math.uio.no

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