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A recursive formula for osculating curves

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Abstract. Let X be a smooth complex projective variety. Using a construction devised by Gathmann, we present a recursive formula for some of the Gromov-Witten invariants of X. We prove that, when X is homogeneous, this formula gives the number of osculating rational curves at a general point of a general hypersurface of X. This generalizes the classical well known pairs of inflection (asymptotic) lines for surfaces in $\mathbb{P}^3$ of Salmon, as well as Darboux's 27 osculating conics.

1. Introduction

Fix a general smooth complex surface Y of degree $d\geq 3$ in $\mathbb{P}^3$, and let $p\in Y$ be a point. In the family of tangent lines to Y at p, Salmon [Sal65, §265] proved that there are exactly two lines whose contact order at p is at least 3. Note that this number is independent of Y and p, as long as they are general. An analogous result was proved by Darboux in [Dar80, p. 372]. Indeed, he proved that there are exactly 27 conics whose contact order with Y at p is at least 7, and this number depends neither on the degree of the surface, nor on the point p. In the case Y is a cubic, he pointed out that those conics must be contained in Y by Bézout, so each of them is the residual intersection of the plane spanned by p and one of the 27 lines of Y. His argument in the case that Y has degree at least 4 rests on a vague and intricate application of classical elimination theory. After that, no advances have been made.

Our goal is to extend those results using the theory of Gromov-Witten invariants. We propose the following

Definition 1.1. Let X be a smooth complex projective variety, let β be the homological class of a curve and let Y be a very ample smooth hypersurface $Y\subset X$.

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An osculating curve C of class β is an irreducible rational curve in X, not contained in Y, such that the intersection index at a general point of Y with C is at least $c_1(X)\cdot\beta-1$. We denote by $\mathrm{OC}(\beta, X)$ the number of osculating curves in X of class β through a general point of Y.

For example, in the case $X=\mathbb{P}^3$ those curves are rational curves of degree n with contact order $4n-1$ at a general point of Y. In particular, Salmon and Darboux's curves are osculating.

In this paper, we will find a formula to compute the number of osculating curves for certain X and β. Using Gathmann's construction, we will find a recursive formula for a GW invariant of X relative to Y (Equation (5.7)) under the hypothesis that Y has no rational curves. Moreover, we will prove that this invariant coincides with $\mathrm{OC}(\beta, X)$ (i.e., it is enumerative) when X is homogeneous (Proposition 4.1). Finally we will see in Remark 5.3 that the number $\mathrm{OC}(\beta, X)$ does not depend on Y, so it is an invariant of β. This was already noted in the case of lines and conics in $\mathbb{P}^3$. Our main result is the following

Theorem 1.2. *Let X be a homogeneous variety, let β be the homological class of a curve. There exists a recursive formula for the number of curves of class β osculating a very ample hypersurface.*

For related results, see for example [FW20] and reference therein.

There exists a nice application of Salmon's pair of inflectional lines. Taking the directions of the two lines define a 2-web on Y. This web is used to show a bound on the number of lines contained in Y. In the same way, Darboux's 27 conics define a 27-web. This web could give an upper bound to the number of conics on Y, see [LP18]. We hope to address these questions elsewhere.

The paper is organized as follows. Sections 2 and 3 recall standard notations of the moduli space of stable curves, Gromov-Witten invariants and Gathmann's construction of the moduli space of curves with tangency conditions. In Section 4, we will study the connection between osculating curves and Gromov-Witten invariants. Section 5 contains the proof of the recursive formula cited before. Finally, Section 6 contains some application. In particular, we present an implementation of $\mathrm{OC}(\beta, X)$ in case X is a product of projective spaces.

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2. Kontsevich moduli space of stable maps

We begin by giving an informal discussion of the main properties of the Kontsevich moduli space of stable maps, following [FP97] and [HTK+03]. Let X be a smooth complex projective variety, let $\beta \in H_2(X,\mathbb{Z})$ be a non torsion homology class, and let $Y \subset X$ be a smooth very ample hypersurface. We denote by Y or $[Y]$ the cohomology class of the subvariety Y in $H^2(X,\mathbb{Z})$, given by Poincaré duality. The cohomology class of a point is denoted by pt.

For any non-negative integer n, we denote by $\overline{M}_{0,n}$ and $\overline{M}_{0,n}(X,\beta)$ the moduli spaces of n-pointed genus zero stable curves and stable maps to X of class β, respectively. The markings provide evaluation morphisms $\mathrm{ev}_i : \overline{M}_{0,n}(X,\beta) \to X$. We have tautological classes $\psi_i := c_1(\mathbb{L}_i)$ where $\mathbb{L}_i$ is the line bundle whose fiber at a stable map $(C, p_1, ..., p_n, f)$ is the cotangent line to C at point p_i. When $n=1$, we omit the index.

Definition 2.1. The virtual dimension of $\overline{M}_{0,n}(X,\beta)$ is the number

$$\mathrm{vdim}\overline{M}_{0,n}(X,\beta) = \dim X + c_1(X)\cdot\beta + n - 3.$$

Since X is projective, there exists a homology class, the virtual fundamental class $[\overline{M}_{0,n}(X,\beta)]^{virt}$, of dimension $\mathrm{vdim}\overline{M}_{0,n}(X,\beta)$ (see [HTK+03, Chapter 26] for further discussions). If X is also a homogeneous variety (i.e., a quotient G/P, where G is a Lie group and P is a parabolic subgroup), and $\overline{M}_{0,n}(X,\beta) \neq \varnothing$, then $\overline{M}_{0,n}(X,\beta)$ exists as a projective non singular stack or orbifold coarse moduli space of pure dimension $\mathrm{vdim}\overline{M}_{0,n}(X,\beta)$ [FP97, Theorem 1, 2, 3].

Definition 2.2. For every choice of classes $\gamma_1, ..., \gamma_n \in H^*(X,\mathbb{Z})$, and non negative integers $a_1, ..., a_n \in \mathbb{Z}$ such that $\sum_{i=1}^n \mathrm{codim}\gamma_i + a_i = \mathrm{vdim}\overline{M}_{0,n}(X,\beta)$, we have the numbers

$$I^X_{n,\beta}(\gamma_1\psi_1^{a_1} \otimes ... \otimes \gamma_n\psi_n^{a_n}) := \mathrm{ev}_1^*(\gamma_1)\cdot\psi_1^{a_1}\cdot ... \cdot \mathrm{ev}_n^*(\gamma_n)\cdot\psi_n^{a_n}\cdot[\overline{M}_{0,n}(X,\beta)]^{virt},$$

called descendant invariants.

We can extend this definition to every integer $a_1, ..., a_n \in \mathbb{Z}$, by imposing

$$I^X_{n,\beta}(\gamma_1\psi_1^{a_1} \otimes ... \otimes \gamma_n\psi_n^{a_n}) = 0$$

if $a_i < 0$ for some i.

We adopt the well established notation that encodes all 1-point invariants of class β in a single cohomology class:

$$I^X_{1,\beta} := \mathrm{ev}_*\left(\frac{1}{1-\psi}\left[\overline{M}_{0,1}(X,\beta)\right]^{virt}\right)$$

$$:= \sum_{i,j} I^X_{1,\beta}(T^i\psi^j)\cdot T_i,$$

where $\{T^i\}$ and $\{T_i\}$ are bases of $H^*(X,\mathbb{Z})\otimes\mathbb{Q}$ dual to each other. Note that

$$I^X_{1,\beta}(T^i\psi^j)$$

is zero when $j\neq \mathrm{vdim}\overline{M}_{0,1}(X,\beta)-i$. We define $I^X_{1,0}:=1_X$, i.e., the unity of the ring $H^*(X,\mathbb{Z})\otimes\mathbb{Q}$.

Example 2.3. A very useful descendant invariant is the following. Let $X=\mathbb{P}^s$, so that every class β will be of the form $\beta=n[\text{line}]$ for some positive integer n. It is known by [Pan98, Section 1.4] that

$$I^{\mathbb{P}^s}_{1,\beta}(\mathrm{pt}\psi^{(s+1)\cdot n-2}) = \frac{1}{(n!)^{s+1}}.$$

3. Gathmann construction

We recall briefly the construction given in [Gat02] and [Gat03]. Let m be a non negative integer. There exists a closed subspace $\overline{M}^Y_{(m)}(X,\beta)\subseteq\overline{M}_{0,1}(X,\beta)$ which parameterizes curves such that the contact order with Y is at least m at the marked point. As a set, it has the following simple description.

Definition 3.1. ([Gat02, Definition 1.1]) The space $\overline{M}^Y_{(m)}(X,\beta)$ is the locus in $\overline{M}_{0,1}(X,\beta)$ of all stable maps (C,p,f) such that

1. $f(p)\in Y$ if $m>0$.
2. f^*Y-mp in the Chow group $A_0(f^{-1}(Y))$ is effective.

Curves with multiplicity 0 are just unrestricted curves in X, whereas a multiplicity of $Y\cdot\beta+1$ forces at least the irreducible curves to lie inside Y. This space comes equipped with a virtual fundamental class $[\overline{M}^Y_{(m)}(X,\beta)]^{virt}$ of dimension $\mathrm{vdim}(\overline{M}_{0,n}(X,\beta))-m$.

By $\overline{M}_{0,n}(Y,\beta)$ we mean the space of n-pointed stable maps to Y of all homology classes whose push-forward to X is β. For every integer $i=1,\ldots,n$, we denote by $\widetilde{\mathrm{ev}}_i:\overline{M}_{0,n}(Y,\beta)\to Y$ the evaluation maps to Y instead of X.

The explicit form of $[\overline{M}^Y_{(m)}(X,\beta)]^{virt}$ is given by the following

Theorem 3.2. ([Gat03, Theorem 0.1]) *For all $m\geq 0$ we have*

$$(m\psi+\mathrm{ev}^*Y)\cdot\left[\overline{M}^Y_{(m)}(X,\beta)\right]^{virt} = \left[\overline{M}^Y_{(m+1)}(X,\beta)\right]^{virt} + \left[D^Y_{(m)}(X,\beta)\right]^{virt}. \tag{3.1}$$

Here, the correction term $D^Y_{(m)}(X,\beta) = \coprod_r \coprod_{B,M} D^Y(X,B,M)$ *is a disjoint union of individual terms*

$$D^Y(X,B,M) := \overline{M}_{0,1+r}(Y,\beta^{(0)}) \times_{Y^r} \prod_{i=1}^{r} \overline{M}^Y_{(m^{(i)})}(X,\beta^{(i)})$$

where $r \geq 0$, $B = (\beta^{(0)},...,\beta^{(r)})$ *with* $\beta^{(i)} \in H_2(X)/torsion$ *and* $\beta^{(i)} \neq 0$ *for* $i > 0$, *and* $M = (m^{(1)},...,m^{(r)})$ *with* $m^{(i)} > 0$. *The maps to* Y^r *are the evaluation maps for the last* r *marked points of* $\overline{M}_{1+r}(Y,\beta^{(0)})$ *and each of the marked points of* $\overline{M}^Y_{(m^{(i)})}(X,\beta^{(i)})$, *respectively. The union in* $D^Y_{(m)}(X,\beta)$ *is taken over all* r, B, *and* M *subject to the following three conditions*:

$$\sum_{i=0}^{r} \beta^{(i)} = \beta \quad \text{(degree condition)}$$

$$Y \cdot \beta^{(0)} + \sum_{i=1}^{r} m^{(i)} = m \quad \text{(multiplicity condition)}$$

$$\textit{if } \beta^{(0)} = 0 \textit{ then } r \geq 2. \quad \text{(stability condition)}$$

In (3.1), *the virtual fundamental class of the summands* $D^Y(X,B,M)$ *is defined to be* $\frac{m^{(1)}...m^{(r)}}{r!}$ *times the class induced by the virtual fundamental classes of the factors* $\overline{M}_{0,1+r}(Y,\beta^{(0)})$ *and* $\overline{M}^Y_{(m^{(i)})}(X,\beta^{(i)})$. *The spaces* $D^Y(X,B,M)$ *can be considered to be subspaces of* $\overline{M}_{0,1}(X,\beta)$, *so the equation of the theorem makes sense in the Chow group of* $\overline{M}_{0,1}(X,\beta)$.

Note that this theorem implies immediately the following

Fact 3.3. *If* $D^Y_{(m)}(X,\beta) = 0$ *for all* $0 \leq m \leq n$, *then*

$$\left[\overline{M}^Y_{(n+1)}(X,\beta)\right]^{virt} = c_{n+1}(\mathcal{P}^n(Y)),$$

where $\mathcal{P}^n(Y)$ *is the bundle of* n*-jets of* $\mathrm{ev}^*(\mathcal{O}_X(Y))$. *This follows from the initial condition*

$$\left[\overline{M}^Y_{(1)}(X,\beta)\right]^{virt} = (0\psi + \mathrm{ev}^*Y) \cdot \left[\overline{M}^Y_{(0)}(X,\beta)\right]^{virt} = \mathrm{ev}^*Y = c_1(\mathcal{P}^0(Y)),$$

and from the exact sequence

$$0 \longrightarrow \mathbb{L}^{\otimes m} \otimes \mathrm{ev}^*(\mathcal{O}_X(Y)) \longrightarrow \mathcal{P}^m(Y) \longrightarrow \mathcal{P}^{m-1}(Y) \longrightarrow 0. \tag{3.2}$$

Finally, we define descendant invariants of X relative to Y.

Definition 3.4. ([Gat03, Section 1]) For every $m\geq 0$ and $\gamma\in H^*(X,\mathbb{Z})$ we define

$$I_{\beta,(m)}(\gamma\psi^j):=\mathrm{ev}^*(\gamma)\cdot\psi^j\cdot[\overline{M}^Y_{(m)}(X,\beta)]^{virt},$$

where $j=\mathrm{vdim}\overline{M}_{0,1}(X,\beta)-m-\mathrm{codim}\gamma$. We assemble all those invariants in a unique cohomology class of X,

$$\begin{aligned}I_{\beta,(m)} &:= \mathrm{ev}_*\left(\frac{1}{1-\psi}\left[\overline{M}^Y_{(m)}(X,\beta)\right]^{virt}\right)\\ &:= \sum_{i,j} I_{\beta,(m)}(T^i\psi^j)\cdot T_i.\end{aligned}$$

Moreover, we have another cohomology class, $J_{\beta,(m)}$, defined as follow

$$J_{\beta,(m)} := \mathrm{ev}_*\left(\frac{1}{1-\psi}\left[D^Y_{(m)}(X,\beta)\right]^{virt}\right)+m\cdot\mathrm{ev}_*\left[\overline{M}^Y_{(m)}(X,\beta)\right]^{virt}.$$

Definition 3.5. ([Gat02, Definition 5.1]) For cohomology classes $\gamma_i\in H^*(X,\mathbb{Z})$ we define $I^Y_{n,\beta}(\gamma_1\psi_1^{a_1}\otimes\ldots\otimes\gamma_n\psi_n^{a_n})$ in the same way of Definition 2.2, replacing $\overline{M}_{0,n}(X,\beta)$ by $\overline{M}_{0,n}(Y,\beta)$, but keeping the ev_i to denote the evaluation maps to X. More generally, we can take some of the cohomology classes γ_i to be classes of Y instead of X. In that case we simply use $\widetilde{\mathrm{ev}}_i(\gamma_i)$ instead of $\mathrm{ev}_i(\gamma_i)$.

By construction, $I_{\beta,(0)}=I^X_{1,\beta}$. From (3.1) it follows

Lemma 3.6. ([Gat03, Lemma 1.2]) *For all torsion free effective class* $\beta\neq 0$, *and* $m\geq 0$ *we have*

$$(Y+m)\cdot I_{\beta,(m)}=I_{\beta,(m+1)}+J_{\beta,(m)}\in H^*(X,\mathbb{Z}). \tag{3.3}$$

The number m in $Y+m$ should be taken as $m1_X$.

A construction very similar to Gathmann's was used *ante litteram* by Kock for counting bitangents of a plane curve. See [AC06] and [Koc99] for more details.

4. Osculating curves

We denote by C_β the number

$$C_\beta:=c_1(X)\cdot\beta-2.$$

This constant has the property that the virtual fundamental class of $\overline{M}^Y_{(C_\beta+1)}(X,\beta)$ has the same dimension of Y.

When X is homogeneous, there is a smooth dense open subspace $M^*_{0,1}(X,\beta)$ in $\overline{M}_{0,1}(X,\beta)$ whose points are stable maps with no non-trivial automorphisms and with smooth domain [FP97, Lemma 13 & Theorem 2]. We denote by $M_{0,1}(X,\beta)^b$ the (possibly empty) open subspace of $M^*_{0,1}(X,\beta)$ whose points are birational maps. The following proposition clarifies the enumerative meaning of $[\overline{M}^Y_{(C_\beta+1)}(X,\beta)]^{virt}$.

Proposition 4.1. *Let X be a homogeneous variety, and let $Y\subset X$ be a general hypersurface which does not contain rational curves. Let T be any $\mathbb{Q}$-cohomological class, of codimension* $\dim X-1$, *such that* $T\cdot Y=\mathrm{pt}$. *Then the number of osculating curves at a general point of Y is $I_{\beta,(C_\beta+1)}(T)$.*

Proof. Since we have just one marked point, there are no different labelings of the marked points that give the same osculating curve.

Let $s\in\Gamma(X,\mathcal{O}_X(Y))$ be the global section defining Y. It defines a global section $\partial(s)$ of the jet bundle $\mathcal{P}^{C_\beta}(Y)$. We know that $\overline{M}_{0,1}(X,\beta)$ is irreducible of the expected dimension (Section 2). The osculating curves are parameterized by those stable maps in $M_{0,1}(X,\beta)^b$ at which the section $\partial(s)$ vanishes. The rank of $\mathcal{P}^{C_\beta}(Y)$ is $C_\beta+1$. By generality of Y, and by the hypothesis that Y has no rational curves, the locus of osculating curves in $M_{0,1}(X,\beta)^b$ has codimension $C_\beta+1$, which means that it has dimension $\dim Y$. This locus is contained in $\overline{M}^Y_{(C_\beta+1)}(X,\beta)$ by Definition 3.1. Let $i:Y\to X$ be the inclusion. By construction of $\overline{M}^Y_{(C_\beta+1)}(X,\beta)$, there is a map $\widetilde{\mathrm{ev}}:\overline{M}^Y_{(C_\beta+1)}(X,\beta)\to Y$ which makes the following diagram commutative.

$$\begin{array}{ccc}
\overline{M}^Y_{(C_\beta+1)}(X,\beta) & \xrightarrow{\widetilde{\mathrm{ev}}} & Y \\
 & \searrow^{\mathrm{ev}} & \downarrow{\scriptstyle i} \\
 & & X
\end{array}$$

That is, $\widetilde{\mathrm{ev}}$ sends each curve (C,p,f) to the point of tangency $f(p)\in Y$. If $M_{0,1}(X,\beta)^b\neq\varnothing$, then the moduli space $\overline{M}^Y_{(C_\beta+1)}(X,\beta)$ has a component $\overline{M}^b$ of the expected dimension $\dim Y$, where each general point represents a stable map in $M_{0,1}(X,\beta)^b$ whose image is an osculating curve. We may have another component, $\overline{M}^c$, whose points parameterize maps $(\mathbb{P}^1,p,f)$ where f is a not generically injective map. In this case, we can find a decomposition $f:\mathbb{P}^1\xrightarrow{g}\mathbb{P}^1\xrightarrow{h}X$ with $g:\mathbb{P}^1\to\mathbb{P}^1$ a finite cover, and $h:\mathbb{P}^1\to X$ generically injective (h is the normalization of the curve $f(\mathbb{P}^1)$).

We want to prove that $\widetilde{\mathrm{ev}}(\overline{M}^c)$ has dimension strictly less that $\dim Y$. Let us fix a map $(\mathbb{P}^1,p,f)\in\overline{M}^c$ where g is a cover of degree $k\geq 2$, and let m be the multiplicity of intersection of h with Y at $g(p)$. If we denote by β' the class $h_*[\mathbb{P}^1]$, we clearly have $k\beta'=\beta$. The contact order of f and Y at p is at most km, depending on the degree of ramification of p. If we want that km be at least $C_\beta+1=c_1(X)\cdot\beta-1$, then clearly $m\geq c_1(X)\cdot\beta'=C_{\beta'}+2$. This implies that $(\mathbb{P}^1, g(p), h)$ is in $M_{0,1}(X,\beta')^b$ and kills a general section of $\mathcal{P}^{C_{\beta'}+1}(Y)$. The dimension of the zero set of that general section is

$$\dim X-2+c_1(X)\cdot\beta'-(C_{\beta'}+2)=\dim X-2.$$

This dimension is strictly less than $\dim Y$. Hence there is no rational irreducible curve of class β' through a general point of Y with multiplicity $C_{\beta'}+2$ at that point. Since $\widetilde{\mathrm{ev}}(\mathbb{P}^1,p,f)=h(g(p))$, we deduce that $\widetilde{\mathrm{ev}}(\overline{M}^c)$ has dimension strictly less than $\dim Y$, as claimed. So, for a general point $y\in Y$, the inverse image $\widetilde{\mathrm{ev}}^{-1}(y)$ is supported on $\overline{M}^b$, and it is possibly empty.

This implies by projection formula that every cycle $\tau\in H_{\dim Y}(\overline{M}^c,\mathbb{Z})$ is contracted by $\widetilde{\mathrm{ev}}$. So the contribution to $[\overline{M}^Y_{(C_\beta+1)}(X,\beta)]^{virt}\in H_{\dim Y}(\overline{M}^Y_{(C_\beta+1)}(X,\beta),\mathbb{Z})$ from $\overline{M}^c$ does not intersect $\widetilde{\mathrm{ev}}^*(\mathrm{pt})$. Therefore

$$\widetilde{\mathrm{ev}}^*(\mathrm{pt})\cdot\left[\overline{M}^Y_{(C_\beta+1)}(X,\beta)\right]^{virt}=\widetilde{\mathrm{ev}}^*(\mathrm{pt})\cdot\left[\overline{M}^*\cup\overline{M}^c\right]^{virt}=\widetilde{\mathrm{ev}}^*(\mathrm{pt})\cdot\left[\overline{M}^b\right]^{virt}. \tag{4.1}$$

But $\overline{M}^b$ has the expected dimension, so its virtual fundamental class coincides with the usual fundamental class. Let T be the $\mathbb{Q}$-cohomology class of the statement. Using (4.1) we get

$$\begin{aligned}\mathrm{ev}^*(T)\cdot\left[\overline{M}^Y_{(C_\beta+1)}(X,\beta)\right]^{virt} &= \widetilde{\mathrm{ev}}^*(i^*(T))\cdot\left[\overline{M}^Y_{(C_\beta+1)}(X,\beta)\right]^{virt}\\ &= \widetilde{\mathrm{ev}}^*(\mathrm{pt})\cdot\left[\overline{M}^Y_{(C_\beta+1)}(X,\beta)\right]^{virt}\\ &= \widetilde{\mathrm{ev}}^*(\mathrm{pt})\cdot\left[\overline{M}^b\right].\end{aligned}$$

We deduce that if $\overline{M}^b=\varnothing$, then $I_{\beta,(C_\beta+1)}(T)=0$. If $\overline{M}^b\neq\varnothing$, then $I_{\beta,(C_\beta+1)}(T)$ is equal to the degree of the map $\widetilde{\mathrm{ev}}_{|\overline{M}^b}:\overline{M}^b\to Y$, i.e., to the number of osculating curves through a general point of Y. □

Example 4.2. Take $X=\mathbb{P}^2$, Y a general curve of degree $d>2$ and β the class of a conic, so that $C_\beta+1=5$. It is clear that we have just one osculating conic at every point of Y. Because if we had two, then every curve in the linear system that they span would be an osculating conic. Let l be the tangent line at a general point $p\in Y$. A double cover of l branched at y will have multiplicity 4, hence it is not osculating. If we take p to be a flex point, then a double cover of l will have multiplicity 6, so it is osculating at p. But the flex points are not dense in Y. This implies that

$\overline{M}^{Y}_{(C_\beta+1)}(X,\beta)$ has the following components: $\overline{M}^{b}$ which is mapped isomorphically to Y by $\widetilde{\mathrm{ev}}$, and a 1-dimensional irreducible component for each flex point p. Such a component parameterizes double covers of the tangent l at p, branched at p.

Let d be a positive integer, and Y' a general element in the linear system $|\mathcal{O}_X(dY)|$. For every non zero effective 1-cycle γ of Y', by adjunction

$$-K_{Y'}\cdot\gamma = (-K_X - dY)_{|Y'}\cdot\gamma$$

will be negative for some large d. Indeed, as Y is very ample, $Y_{|Y'}\cdot\gamma>0$ by Kleiman's Positivity Theorem [Kle66, Chapter 3, §1]. This implies that

$$\mathrm{vdim}\overline{M}_{0,1}(Y',\gamma)<0$$

for $d\gg 0$, i.e., $\overline{M}_{0,1}(Y',\gamma)$ is virtually empty. It can happen that the requested intersection multiplicity between Y and the osculating curve is so high that the curve must be contained in Y (take lines tangent to linear subspaces). In order to avoid that, we could substitute Y with Y' for $d\gg 0$. We will see that $I_{\beta,(C_\beta+1)}(T)$ does not depend on d, as well as on Y, as long as Y has no rational curves. So, it makes sense to omit Y in $\mathrm{OC}(\beta,X)$.

5. Recursive formula

In this section we will give a recursive formula for $I_{\beta,(C_\beta+1)}(T)$. We use the same notation as before. The variety $Y\subset X$ is a smooth very ample hypersurface with no rational curves. We suppose that $1_X, Y\in\{T^i\}$. We denote by T the dual of Y in $\{T_i\}$. When Y generates $H^2(X,\mathbb{Z})\otimes\mathbb{Q}$, the class T is uniquely determined. The number $I_{\beta,(C_\beta+1)}(T)$ can be described as the coefficient of Y in $I_{\beta,(C_\beta+1)}$, as seen in Definition 3.4.

We apply Theorem 3.2 to compute $I_{\beta,(C_\beta+1)}(T)$. Using Equation 3.3 we have

$$\begin{aligned} I_{\beta,(C_\beta+1)} &= (Y+C_\beta)I_{\beta,(C_\beta)} - J_{\beta,(C_\beta)} \\ &= (Y+C_\beta)(Y+C_\beta-1)I_{\beta,(C_\beta-1)} - (Y+C_\beta)J_{\beta,(C_\beta-1)} - J_{\beta,(C_\beta)} \\ &\;\vdots\;\vdots \\ &= \left(\prod_{i=0}^{C_\beta}(Y+i)\right) I_{\beta,(0)} - \sum_{i=0}^{C_\beta-1}\left(\prod_{j=i+1}^{C_\beta}(Y+j)\right) J_{\beta,(i)} - J_{\beta,(C_\beta)}. \end{aligned} \tag{5.1}$$

Let us compute the first term of this sum in Equation (5.2). In the next display, following [Gat03], *mod* H^3 means that we omit cohomology classes of codimension greater than 1.

$$\left(\prod_{i=0}^{C_\beta}(Y+i)\right) I_{\beta,(0)} = \left(\prod_{i=1}^{C_\beta}(Y+i)\right) Y I_{\beta,(0)}$$
$$= \left(\prod_{i=1}^{C_\beta}(Y+i)\right) Y I^X_{1,\beta}(\mathrm{pt}\psi^{C_\beta}) \quad (mod\, H^3)$$
$$= C_\beta! I^X_{1,\beta}(\mathrm{pt}\psi^{C_\beta})\, Y \quad (mod\, H^3). \tag{5.2}$$

Back in (5.1), we need to find the contribution of each $J_{\beta,(i)}$. Gathmann computed explicitly all the $J_{\beta,(i)}$ in [Gat03, Lemma 1.8] using that $-K_Y$ is nef. Since we want that Y has no rational curves, we need that $-K_Y\cdot\gamma$ is negative for every rational curve γ in Y, as explained at the end of Section 4. So $-K_Y$ is not nef.

Remark 5.1. For each $0\le m\le C_\beta-1$, dimensional reasons ensure that the class $\mathrm{ev}_*[\overline{M}_{(m)}(X,\beta)]^{virt}$ has trivial H^0 and H^2 part. Moreover $\mathrm{ev}_*[\overline{M}_{(C_\beta)}(X,\beta)]^{virt}$ has trivial H^2 part.

Let us compute the contribution of $\mathrm{ev}_*(\frac{1}{1-\psi}[D^Y_{(m)}(X,\beta)]^{virt})$.

Let $D:=D^Y(X,B,M)$ be one of the individual term as in Theorem 3.2, with $B=(\beta^{(0)},...,\beta^{(r)})$ and $M=(m^{(1)},...,m^{(r)})$, $m^{(i)}>0$. If $\beta^{(0)}\neq 0$, then $\overline{M}_{0,1}(Y,\beta^{(0)})$ is empty by our hypothesis that Y has no rational curves. So that D has no contribution. Let $\beta^{(0)}=0$, in particular we have the following conditions on D:

$$\sum_{i=1}^r \beta^{(i)} = \beta \quad \text{(degree condition)}$$
$$\sum_{i=1}^r m^{(i)} = m \quad \text{(multiplicity condition)}$$
$$r\geq 2. \quad \text{(stability condition).}$$

The value of $\mathrm{ev}_*(\frac{1}{1-\psi}[D]^{virt})$ is given by the formula [Gat03, Remark 1.4, Equation (2)]

$$\sum I_0^Y(T^i\psi^j\otimes\gamma_1\otimes...\otimes\gamma_r)\cdot\frac{1}{r!}\prod_{k=1}^r\left(m^{(k)}\cdot I_{\beta^{(k)},(m^{(k)})}(\gamma_k^\vee)\right)\cdot T_i, \tag{5.3}$$

where the γ_k run in a basis of the part of $H^*(Y)\otimes\mathbb{Q}$ induced by X [Gat02, Remark 5.4], and $\gamma_k^\vee$ is the dual as a $\mathbb{Q}$-class in X. By Lefschetz Hyperplane Theorem, we can take such a basis as $\{T^i_{|Y}\}$. We will look for the conditions on D such that this contribution is non zero.

Lemma 5.2. *The coefficient of* Y *in* $\mathrm{ev}_*(\frac{1}{1-\psi}[D]^{virt})$ *is*

$$\frac{1}{r!}\prod_{k=1}^{r}\Big((C_{\beta^{(k)}}+1)I_{\beta^{(k)},(C_{\beta^{(k)}}+1)}(T)\Big)$$

if $r=C_\beta+2-m$, *and zero otherwise.*

Proof. The coefficient of Y is given by the sum in Equation (5.3), when $T^i=T$, $\gamma_k\in\{T^i_{|Y}\}$ for $1\le k\le r$, and

$$\begin{aligned} j &= \dim[D]^{virt}-\mathrm{codim}(T)\\ &= \mathrm{vdim}\overline{M}_{(m+1)}(X,\beta)-(\dim X-1)\\ &= \dim X+c_1(X)\cdot\beta-2-(m+1)-\dim X+1\\ &= C_\beta-m. \end{aligned}$$

By Definition 3.5, we know that

$$I_0^Y(T\psi^j\otimes\gamma_1\otimes\ldots\otimes\gamma_r)=\mathrm{ev}_1^*(T)\cdot\psi^j\cdot\widetilde{\mathrm{ev}}_2^*(\gamma_1)\cdot\ldots\cdot\widetilde{\mathrm{ev}}_{r+1}^*(\gamma_r)\cdot[\overline{M}_{0,1+r}(Y,0)]^{virt}. \tag{5.4}$$

It is well known that $\overline{M}_{0,1+r}(Y,0)\cong\overline{M}_{0,1+r}\times Y$, and each map $\widetilde{\mathrm{ev}}_i$ is the second projection. Moreover, $\mathrm{ev}_1^*(T)=\widetilde{\mathrm{ev}}_i^*(\mathrm{pt})$. So for every i,

$$\mathrm{ev}_1^*(T)\cdot\widetilde{\mathrm{ev}}_i^*(1_{X|Y})=\widetilde{\mathrm{ev}}_i(\mathrm{pt}\cdot 1_Y)=\widetilde{\mathrm{ev}}_i(\mathrm{pt}).$$

If one of the γ_i is not $1_{X|Y}=1_Y$, then $\mathrm{ev}_1^*(T)\cdot\widetilde{\mathrm{ev}}_{i+1}^*(\gamma_i)=0$ for dimensional reasons. This implies that the expression (5.4) is zero if $\gamma_k\neq 1_Y$ for some $1\le k\le r$.

Now we want to compute $I_0^Y(T\psi^j\otimes 1_Y\otimes\ldots\otimes 1_Y)$, where 1_Y appears r times. Using $(r-2)$-times the string equation [Pan98, 1.2.I], we get

$$\begin{aligned} I_0^Y(T\psi^j\otimes 1_Y^{\otimes r}) &= I_0^Y(T\psi^{j-1}\otimes 1_Y^{\otimes r-1})\\ &\ \ \vdots\ \vdots\\ &= I_0^Y(T\psi^{j-(r-2)}\otimes 1_Y\otimes 1_Y). \end{aligned}$$

Hence,

$$I_0^Y(T\psi^{j-(r-2)}\otimes 1_Y\otimes 1_Y)=T_{|Y}\cdot 1_Y\cdot 1_Y\cdot\psi^{j-(r-2)}\cdot\left[\overline{M}_{0,3}\times Y\right].$$

This expression is 1 if $j-(r-2)=0$, and 0 otherwise. We proved that the coefficient of Y is zero if $C_\beta-m-(r-2)\neq 0$. So, necessarily $r=C_\beta+2-m$, as asserted.

We need to determine the value of $m^{(k)}$ in $m^{(k)}I_{\beta^{(k)},(m^{(k)})}(\gamma_k^\vee)$. First of all, since $\gamma_k=1_Y$, then $\gamma_k^\vee=T$, hence

$$m^{(k)}I_{\beta^{(k)},(m^{(k)})}(\gamma_k^\vee)=m^{(k)}I_{\beta^{(k)},(m^{(k)})}(T).$$

Consider

$$I_{\beta^{(k)},(m^{(k)})}(T) = \mathrm{ev}^*(T)\cdot\psi^s\cdot[\overline{M}_{(m^{(k)})}(X,\beta^{(k)})]^{virt}, \tag{5.5}$$

where

$$\begin{aligned} s &= \dim[\overline{M}_{(m^{(k)})}(X,\beta^{(k)})]^{virt} - \mathrm{codim}(T) \\ &= C_{\beta^{(k)}} - m^{(k)} + 1. \end{aligned}$$

If we want (5.5) to be non zero, $s \geq 0$ so that each term $m^{(k)}$ must be at most $C_{\beta^{(k)}}+1$. This forces each $m^{(k)}$ to be exactly $C_{\beta^{(k)}}+1$, indeed

$$\begin{aligned} \sum_{i=1}^{r}(C_{\beta^{(i)}}+1) &= \sum_{i=1}^{r}\left(c_1(X)\cdot\beta^{(i)}-1\right) \\ &= \left(\sum_{i=1}^{r} c_1(X)\cdot\beta^{(i)}\right) - r \\ &= C_\beta + 2 - r. \\ &= \sum_{i=1}^{r} m^{(i)}. \end{aligned}$$

Finally, the coefficient of Y in $\mathrm{ev}_*(\frac{1}{1-\psi}[D]^{virt})$ is

$$\frac{1}{r!}\prod_{k=1}^{r}\left((C_{\beta^{(k)}}+1)I_{\beta^{(k)},(C_{\beta^{(k)}}+1)}(T)\right). \qquad \square$$

Let us go back to Equation (5.1). We are interested in the coefficient of Y in $J_{\beta,(C_\beta)}$ and also in each term

$$\left(\prod_{j=i+1}^{C_\beta}(Y+j)\right) J_{\beta,(i)}, \quad i = 0, \ldots, C_\beta - 1. \tag{5.6}$$

For $J_{\beta,(C_\beta)}$, by Lemma 5.2 and Remark 5.1 the required coefficient is

$$\sum \frac{1}{2}\prod_{k=1}^{2}\left((C_{\beta^{(k)}}+1)I_{\beta^{(k)},(C_{\beta^{(k)}}+1)}(T)\right),$$

where the sum runs over all the ordered partitions $(\beta^{(1)},\beta^{(2)})$ of β, with two non zero summands. The reason why we take ordered partitions is the following. Since the marked points in $\overline{M}_{0,1+r}(Y,0)$ are ordered, the two spaces

$$\overline{M}_{0,3}(Y,0)\times_Y \overline{M}^Y_{(m^{(k)})}(X,\beta^{(k)})\times_Y \overline{M}^Y_{(m^{(3-k)})}(X,\beta^{(3-k)}), \quad k\in\{1,2\},$$

are isomorphic, but not the same if $\beta^{(1)}\neq\beta^{(2)}$. So, we have to compute the contribution of each of them. For $(Y+C_\beta)J_{\beta,(C_\beta-1)}$, since $J_{\beta,(C_\beta-1)}$ has trivial H^0 coefficient as noted in Remark 5.1, we can ignore the class Y in $(Y+C_\beta)$. So the coefficient of Y is

$$C_\beta\sum\frac{1}{3!}\prod_{k=1}^{3}\left((C_{\beta^{(k)}}+1)I_{\beta^{(k)},(C_{\beta^{(k)}}+1)}(T)\right),$$

where the sum runs over all the ordered partitions of β with three non zero summands. For any other term in (5.6) we proceed in the same way. We get that the contribution of those terms to (5.1) is the coefficient of Y in $J_{\beta,(i)}$ times the number $\prod_{j=i+1}^{C_\beta} j=\frac{C_\beta!}{i!}$. At the very end, we get that $I_{\beta,(C_\beta+1)}(T)$ is equal to

$$C_\beta!I^X_{1,\beta}(\mathrm{pt}\psi^{C_\beta})-\sum_K\frac{C_\beta!}{(C_\beta+2-r_K)!}\frac{1}{r_K!}\prod_{k=1}^{r_K}(C_{\beta^{(k)}}+1)I_{\beta^{(k)},(C_{\beta^{(k)}}+1)}(T),$$

or, equivalently,

$$C_\beta!I^X_{1,\beta}(\mathrm{pt}\psi^{C_\beta})-\sum_K\binom{C_\beta}{r_K-2}\frac{1}{r_K(r_K-1)}\prod_{k=1}^{r_K}(C_{\beta^{(k)}}+1)I_{\beta^{(k)},(C_{\beta^{(k)}}+1)}(T), \tag{5.7}$$

where the sum is taken among all the ordered partitions K of β:

$$K=(\beta^{(1)},\dots,\beta^{(r_K)})\quad\text{such that}\ \sum_{k=1}^{r_K}\beta^{(k)}=\beta,\quad \beta^{(k)}>0,\ r_K\geq 2.$$

Remark 5.3. Note that $I_{\beta,(C_\beta+1)}(T)$ does not depend on Y, but only on X and β. To prove that, we can use a simple induction argument on the maximal length $max(\beta)$ of all the partitions of β. If $max(\beta)=1$, i.e., β is primitive, then

$$I_{\beta,(C_\beta+1)}(T)=C_\beta!I^X_{1,\beta}(\mathrm{pt}\psi^{C_\beta}).$$

In the general case, $I_{\beta,(C_\beta+1)}(T)$ is a combination of $I_{\beta^{(k)},(C_{\beta^{(k)}}+1)}(T)$ and other terms independent of Y. But $I_{\beta^{(k)},(C_{\beta^{(k)}}+1)}(T)$ is independent of Y by induction, since clearly $max(\beta^{(k)})<max(\beta)$.

Example 5.4. Let us give an example of a calculation using (5.7). If $X=\mathbb{P}^3$, then $I^{\mathbb{P}^3}_{1,\beta}(\mathrm{pt}\psi^{4n-2})=\frac{1}{(n!)^4}$ where $\beta=n[\text{line}]$ by Example 2.3. By a simple calculation, we see that $\mathrm{OC}(1,\mathbb{P}^3)=2$. If β is the class of a conic, the unique partition is the sum of two lines, so

$$\mathrm{OC}(2,\mathbb{P}^3)=\frac{C_2!}{2^4}-\frac{C_2!}{(C_2+2-2)!}\frac{1}{2}\prod_{k=1}^{2}(C_1+1)\mathrm{OC}(1,\mathbb{P}^3)$$

$$\begin{aligned}&= 45 - \frac{1}{2} \cdot 3 \cdot 2 \cdot 3 \cdot 2\\ &= 27,\end{aligned}$$

as stated in Introduction.

6. Applications

In this section X will be a homogeneous variety, so by Proposition 4.1 $\mathrm{OC}(\beta, X)$ coincides with $I_{\beta,(C_\beta+1)}(T)$. To compute $\mathrm{OC}(\beta, X)$, we need $I^X_{1,\beta}(\mathrm{pt}\psi^{C_\beta})$. The opposite direction is also possible: once we know $\mathrm{OC}(\beta, X)$ for some β, then we can get $I^X_{1,\beta}(\mathrm{pt}\psi^{C_\beta})$. For example, no point of $\overline{M}_{0,1}(\mathbb{P}^1, n)$ represents a birational stable map if $n \geq 2$, so by the proof of Proposition 4.1 we expect $\mathrm{OC}(1, \mathbb{P}^1) = 1$ and $\mathrm{OC}(n, \mathbb{P}^1) = 0$ for $n \geq 2$. Equation (5.7) implies immediately $I^{\mathbb{P}^1}_{1,1}(\mathrm{pt}) = 1$, whilst for $n \geq 2$ the only non zero term of the sum

$$\sum_K \frac{C_\beta!}{(C_\beta+2-r_K)!} \frac{1}{r_K!} \prod_{i=1}^{r_K} (C_{\beta_i}+1)\mathrm{OC}(\beta_i, \mathbb{P}^1)$$

appears when $K = (1, ..., 1)$. So the entire sum is equal to

$$\frac{C_\beta!}{(C_\beta+2-n)!} \frac{1}{n!} \prod_{i=1}^{n} (C_1+1)\mathrm{OC}(1, \mathbb{P}^1) = \frac{C_\beta!}{n!} \frac{1}{n!} 1 = \frac{C_\beta!}{(n!)^2}.$$

Finally, the equation

$$\mathrm{OC}(n, \mathbb{P}^1) = C_n! I^{\mathbb{P}^1}_{1,n}(\mathrm{pt}\psi^{2n-2}) - \frac{C_n!}{(n!)^2}$$

implies $I^{\mathbb{P}^1}_{1,n}(\mathrm{pt}\psi^{2n-2}) = \frac{1}{(n!)^2}$.

Using the same technique, we prove $I^{\mathbb{P}^s}_{1,1}(\mathrm{pt}\psi^{(s+1)n-2}) = 1$ for every $s \geq 1$. Indeed, for $n = 1$ (5.7) reduces to $\mathrm{OC}(1, \mathbb{P}^s) = (s-1)! I^{\mathbb{P}^s}_{1,1}(\mathrm{pt}\psi^{(s+1)n-2})$. So, it is enough to prove the following

Proposition 6.1. $\mathrm{OC}(1, \mathbb{P}^s) = (s-1)!$.

Proof. Let Y be a hypersurface of degree $d \gg 0$. All spaces $D^Y_{(m)}(\mathbb{P}^s, 1)$ are empty. So by Fact 3.3, the number of osculating lines is $c_{C_\beta+1}(\mathcal{P}^{C_\beta}(Y))\mathrm{ev}^*([Y]^\vee)$, where $C_\beta = s-1$. It a general fact that $\overline{M}_{0,0}(\mathbb{P}^s, 1)$ and $\overline{M}_{0,1}(\mathbb{P}^s, 1)$ are canonically isomorphic to, respectively, the Grassmannian $G = G(2, s+1)$ of lines in $\mathbb{P}^s$ and its universal family. There exists a rank 2 tautological vector bundle $\mathcal{E}$ on G such that $\overline{M}_{0,1}(\mathbb{P}^s, 1) \cong \mathbb{P}(\mathcal{E})$. Moreover, ψ coincides with the first Chern class of the relative cotangent bundle of the natural map $\pi:\overline{M}_{0,1}(\mathbb{P}^s,1)\to G$. From the exact sequence (3.2) we get, by a simple recursion,

$$\begin{aligned} c_s(\mathcal{P}^{s-1}(Y)) &= c_1(\mathbb{L}^{\otimes s-1}\otimes \mathrm{ev}^*(\mathcal{O}_{\mathbb{P}^s}(d)))\cdot c_{s-1}(\mathcal{P}^{s-2}(Y)) \\ &= \prod_{i=0}^{s-1} c_1(\mathbb{L}^{\otimes i}\otimes \mathrm{ev}^*(\mathcal{O}_{\mathbb{P}^s}(d))) \\ &= d\xi \prod_{i=1}^{s-1}(i\psi+d\xi), \end{aligned}$$

where $\xi:=c_1(\mathrm{ev}^*(\mathcal{O}_{\mathbb{P}^s}(1)))$. By definition $\mathrm{ev}^*([Y]^\vee)=\frac{1}{d}\xi^{s-1}$, so

$$\begin{aligned} c_s(\mathcal{P}^{s-1}(Y))\mathrm{ev}^*([Y]^\vee) &= \left(d\xi\prod_{i=1}^{s-1}(i\psi+d\xi)\right)\frac{1}{d}\xi^{s-1} \\ &= \xi^s\prod_{i=1}^{s-1}(i\psi+d\xi). \end{aligned}$$

Since $\xi^i=0$ if $i>s$, we have $c_s(\mathcal{P}^{s-1}(Y))[Y]^\vee=\xi^s(s-1)!\psi^{s-1}$. Moreover, it is known that $\psi=\pi^*c_1(\mathcal{E}^\vee)-2\xi$ (see, e.g., [EH16, Theorem 11.4]), so

$$c_s(\mathcal{P}^{s-1}(Y))[Y]^\vee=(s-1)!\xi^s\pi^*c_1(\mathcal{E}^\vee)^{s-1}.$$

The degree of the zero cycle $\xi^s\pi^*c_1(\mathcal{E}^\vee)^{s-1}$ is equal to the number of lines through a point and $s-1$ general linear subspaces of codimension 2. To prove that such number is 1, we can use Schubert calculus as explained in [EH16, Chapter 4]. The Schubert cycle of lines through a codimension 2 linear subspace is $\sigma_{(1,0)}$. The Schubert cycles of lines through a point is $\sigma_{(s-1,0)}$. Using Pieri's formula, for each integer $k\leq s-1$ we have $(\sigma_{(1,0)})^k\cdot\sigma_{(s-1,0)}=\sigma_{(s-1,k)}$. Finally

$$\xi^s\pi^*c_1(\mathcal{E}^\vee)^{s-1}=(\sigma_{(1,0)})^{s-1}\cdot\sigma_{(s-1,0)}=\sigma_{(s-1,s-1)}=1. \quad \square$$

On the other hand in the case $n=2$, (5.7) reduces to

$$\begin{aligned} \mathrm{OC}(2,\mathbb{P}^s) &= C_2!I_{1,2}^{\mathbb{P}^s}(\mathrm{pt}\psi^{(s+1)2-2})-\frac{1}{2}(C_1+1)^2\mathrm{OC}(1,\mathbb{P}^s)^2 \\ &= (2s)!I_{1,2}^{\mathbb{P}^s}(\mathrm{pt}\psi^{(s+1)2-2})-\frac{1}{2}(s(s-1)!)^2. \end{aligned} \tag{6.1}$$

We have seen in Example 4.2 that $\mathrm{OC}(2,\mathbb{P}^2)=1$. The case $\mathrm{OC}(2,\mathbb{P}^3)=27$ was proved by Darboux. By (6.1) we have $I_{1,2}^{\mathbb{P}^s}(\mathrm{pt}\psi^{(s+1)2-2})=\frac{1}{2^{s+1}}$ for $s=2,3$.

6.1. Computational aspects

If $H^2(X,\mathbb{Z})=\mathbb{Z}$, then by Poincaré duality all the effective homology classes of a curve are multiples of a unique homology class. In those cases, in Equation (5.7) K is equivalent to an ordered partition of an integer n, where n is a multiple of the cohomology class generating $H^2(X,\mathbb{Z})$. The following is a *Wolfram Mathematica* code for the case $X=\mathbb{P}^s$, ($\mathrm{OC}(n,s)$ is $\mathrm{OC}(n[\text{line}],\mathbb{P}^s)$):

```
OC[n_, s_]:=((s+1)*n-2)!/n!^(s+1)-Sum[(((s+1)*n-2)!/((s+1)*n
-Length[K])!)*(Length[Permutations[K]]/Length[K]!)*Product
[((s+1)*K[[i]]-1)*OC[K[[i]],s],{i,1,Length[K]}],
{K,IntegerPartitions[n,{2,n}]}];
```

We have made several computer checks using GROWI [Gat]. All results were as expected.

The code we give for $\mathrm{OC}(n[\text{line}],\mathbb{P}^s)$ can be generalized for other varieties. What really changes is that we have to find a way to write all the partitions of an effective homology class β. We wrote a code when X is a product of projective spaces. In all computations we made, the result was always an integer number, as expected by Proposition 4.1. For example, the number of osculating curves of class $(3,4)$ in $\mathbb{P}^5\times\mathbb{P}^6$ is precisely $123765177219015389315749781205 4065\times 10^7$. A *Wolfram Mathematica notebook* of this code can be provided upon request.

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