

# On moduli spaces of 3d Lie algebras

Arvid SIQVELAND

Buskerud University College, PoBox 235, 3603 Kongsberg, Norway

E-mail: arvid.siqveland@hibu.no

## Abstract

We consider Lie algebras of dimension 3 up to isomorphism. We construct a noncommutative affine spectrum of the isomorphism classes as a noncommutative  $k$ -algebra  $\mathbb{M}$ , using noncommutative deformation theory. This  $k$ -algebra is an example of a noncommutative structure.

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## 1 Introduction

Throughout, let  $k$  be an algebraically closed field of characteristic 0. The classification of three-dimensional Lie algebras is well known. Up to isomorphism they are  $ab$ ,  $sl_2$ ,  $r_3$ ,  $l_a$ , and  $n_3$ , ([4]). In this paper, we are going to construct a noncommutative algebraic moduli (i.e. an algebraic geometric classifying structure) having the isomorphism classes of 3-dimensional Lie algebras as geometric points, in this case represented by 1-dimensional simple modules, see [9] or the definitions 1.10, 1.11 below. We obtain an example of noncommutative deformation theory that is special in that it shows two families of Lie-algebras meeting in only one point.

We will achieve this by using noncommutative deformation theory, see [9, 3]. In commutative deformation theory, the local formal moduli (or prorepresenting hull) of the deformation functor  $\text{Def}_M$  is a candidate for the completion of the local  $k$ -algebra  $\hat{\mathcal{O}}_M$  of the moduli space in the point corresponding to  $M$ , see [11], and in a lot of examples the moduli space is given locally around  $M$  as the spectrum of a natural finitely generated algebraization (definition 1.11)  $\mathcal{O}_M$  of  $\hat{\mathcal{O}}_M$ , see [13, 12]. Moreover, [13] contains an algorithm for computing the local formal moduli of  $\text{Def}_M$ . In noncommutative deformation theory as given in [9, 3], this is generalized to the noncommutative situation:

**Definition 1.1.** A noncommutative  $k$ -algebra  $R$  is called  $r$ -pointed if there exists exactly  $r$  isomorphism classes of simple one-dimensional quotient modules of  $R$ ,  $R \twoheadrightarrow V_i \cong k$ ,  $i = 1, \dots, r$ . A morphism of  $r$ -pointed  $k$ -algebras, is a  $k$ -algebra homomorphism inducing the identity on the  $r$  one-dimensional quotient modules.

**Definition 1.2** (the category  $\underline{a}_r$ ). The category  $\underline{a}_r$  is the category of  $r$ -pointed Artinian  $k$ -algebras  $S$  together with morphisms of  $r$ -pointed algebras.

The category  $\underline{a}_r$  is characterized by the following: An  $r$ -pointed Artinian  $k$ -algebra is a  $k$ -algebra  $S$  together with morphisms  $\iota, \rho$  commuting in the diagram

$$\begin{array}{ccc} k^r & \xrightarrow{\iota} & S \\ & \searrow & \downarrow \rho \\ & \text{Id} & k^r \end{array}$$

and such that  $(\ker(\rho))^n = 0$  for some  $n > 0$ .  $\ker(\rho)$  is called the radical of  $S$  and denoted  $\ker(\rho) = \text{rad}(S)$ . The procategory  $\hat{\underline{a}}_r$  of  $\underline{a}_r$  is the full subcategory of ( $r$ -pointed)  $k$ -algebras such that  $R/(\text{rad}(R))^n$  is an object of  $\underline{a}_r$  for all  $n \geq 1$ , and such that  $R$  is (separated) complete in the  $I = \text{rad}(R)$ -adic topology.

**Remark 1.3.** The commutative  $k$ -algebras in  $\underline{a}_1$  makes up the category  $\underline{\ell}$  of pointed Artinian  $k$ -algebras.

**Remark 1.4.** Consider the  $r \times r$ -matrices  $e_{ii}$ , the matrix with 1 at place  $i, i$  and 0 elsewhere, and  $t_{ij}(l)$ , the matrices with the indeterminates  $t_{ij}(l)$  at place  $i, j$  and 0 elsewhere,  $1 \leq l \leq l_{ij}$ . Then we use the notation  $k^r \{t_{ij}(l)\}$  for the  $k$ -algebra generated by these matrices under ordinary matrix multiplication. This is the noncommutative counterpart of the free  $k$ -algebra  $k[t_1, \dots, t_l]$  in the commutative situation.

Let  $e_i \in k^r$ ,  $1 \leq i \leq r$  be the idempotents. Put  $S_{ij} = e_i S e_j$ . Then it follows that every  $r$ -pointed  $k$ -algebra can be written as the matrix algebra  $S \cong (S_{ij})$ .

Now, let  $V = \{V_1, \dots, V_r\}$  be right  $A$ -modules. Let  $S = (S_{ij}) \in \underline{a}_r$  be an  $r$ -pointed Artinian  $k$ -algebra.

**Definition 1.5.** Let  $k_i = k \cdot e_{ii}$ , i.e. the matrix algebra with  $k$  at the  $i$ 'th place on the diagonal and 0 elsewhere. The deformation functor  $\text{Def}_V : \underline{a}_r \rightarrow \underline{\text{Sets}}$  is defined by

$$\text{Def}_V(S) = \{S \otimes_k A\text{-modules } M_S \mid k_i \otimes_S M_S \cong V_i \text{ and } M_S \cong_k (S_{ij} \otimes_k V_j) = S \otimes_k V\} / \cong$$

Notice that the condition  $S$ -flat in the commutative case is replaced by  $M_S \cong_k (S_{ij} \otimes_k V_j)$  in the noncommutative case. (Here  $\cong_k$  means isomorphic as  $k$ -vector spaces, or equivalently, as left  $S$ -modules).

Any covariant functor  $F : \underline{a}_r \rightarrow \underline{\text{Sets}}$  extends in a natural way to a functor  $\hat{F}$  on the procategory  $\hat{\underline{a}}_r$  by

$$\hat{F}(\hat{R}) = \varprojlim_n F(\hat{R}/\text{rad}(R)^n)$$

We use the notation  $\underline{a}_r(2)$  for the  $r$ -pointed Artinian  $k$ -algebras  $S$  for which  $(\text{rad}(S))^2 = 0$ .

**Definition 1.6.** A prorepresenting hull ( or a local formal moduli ) for a pointed covariant functor  $F : \underline{a}_r \rightarrow \underline{\text{Sets}}$  is an object  $\hat{H}$  in  $\hat{\underline{a}}_r$  such that there exists a proversal family  $\hat{V} \in \hat{F}(\hat{H})$  with the property that the corresponding morphism  $\text{Mor}(\hat{H}, -) \rightarrow F$  of functors on  $\underline{a}_r$  is smooth and an isomorphism when restricted to a morphism of functors on  $\underline{a}_r(2)$ .

**Definition 1.7.** A surjective morphism  $\pi : R \rightarrow S$  between two  $r$ -pointed Artinian  $k$ -algebras is called small if  $\ker \pi \cdot \text{rad}(R) = \text{rad}(R) \cdot \ker \pi = 0$ . If there exists a deformation  $M_R \in \text{Def}_V(R)$  such that  $\text{Def}_V(\pi)(V_R) = V_S$ ,  $V_R$  is called a lifting of  $V_S$  to  $R$ .

The obstruction theory for the noncommutative deformation functor is the obstruction theory in small lifting situations, and is given in the references [8, 9, 3, 15]. The main results from these articles relevant for this work is the following (notice the fact that  $\text{HH}^i(A, \text{Hom}_k(V_i, V_j)) \cong \text{Ext}_A^i(M, N)$  for  $k$ -algebras  $A$  and (right)  $A$ -modules  $M$  and  $N$ ).

**Theorem 1.8.** *Given a small morphism  $\pi : R \rightarrow S$  with kernel  $I = (I_{ij})$  in  $\underline{a}_r$  and  $M_S \in \text{Def}_V(S)$ . There exists an obstruction  $o(\pi, M_S) = (o_{ij}(\pi, M_S)) \in I_{ij} \otimes_k \text{HH}^2(A, \text{Hom}_k(V_i, V_j))$  such that  $o(\pi, M_S) = 0$  if and only if there exists a lifting  $M_R \in \text{Def}_V(R)$  of  $M_S$ . The set of equivalence classes of such liftings is a torsor under*

$$(I_{ij} \otimes_k \text{Ext}_A^1(V_i, V_j))$$

**Proof.** The proof can be found in [9]. The part essential for the computations in this paper is:

Assume  $0 = o_{ij} = \bar{\psi}_{ij} \in \text{HH}^2(A, \text{Hom}_k(V_i, V_j))$ . Then  $\psi = d\phi$ ,  $\phi \in \text{Hom}_k(A, I_{ij} \otimes_k \text{Hom}_k(V_i, V_j))$ . Put  $\sigma' = \sigma + \phi$ . Then  $\sigma'(ab) - \sigma'(a)\sigma'(b) = 0$  because  $I^2 = 0$ .  $\square$

In [9] it is proved that  $\text{Def}_V$  has a prorepresenting hull  $H(V) = (H_{i,j})$ . Also, in [9] (Sections 5 and 6), the construction of a noncommutative scheme theory and moduli of isomorphism classes of modules is given. Let  $\pi : A - \underline{\text{mod}} \rightarrow k - \underline{\text{mod}}$  be the obvious forgetful functor. Then a subalgebra

$$O^A(V, \pi) \subseteq (H_{i,j} \otimes_k \text{Hom}_k(V_i, V_j))$$

is constructed, and the restriction of the canonical homomorphism  $\eta$ ,

$$\eta(V) : A \rightarrow O^A(V, \pi)$$

gives an action of  $O^A(V, \pi)$  on  $V$ , extending the action of  $A$ . In this situation, this construction is a closure operation, i.e.  $O^A(V, \pi) \cong O^O(V, \pi)$ . This  $O$ -construction is then extended to infinite families of isomorphism classes of modules by "sheaffying", obtaining for every finite family  $V$  a smaller  $k$ -algebra,  $\mathcal{O}(V, \pi)$  containing the image of  $\eta(V)$ . The final noncommutative structure sheaf  $\mathcal{O}_\pi$  is then a certain quotient of this  $\mathcal{O}(-, \pi)$ .

**Definition 1.9.** A family  $V$  of  $A$ -modules will be called a prescheme for  $A$ , if

$$\eta(V) : A \rightarrow \mathcal{O}(V, \pi)$$

is an isomorphism. Then  $(V, A)$  is called an affine prescheme. The family  $V$  will be called scheme for  $A$ , if

$$\eta(V) : A \rightarrow \mathcal{O}_\pi(V)$$

is an isomorphism.

In the situation of the present paper, the construction of noncommutative orbit spaces translates to the following simplification:

**Definition 1.10.** A finitely generated  $k$ -algebra  $R$  is called an affine moduli (or spectrum) for a family  $V$  of  $A$ -modules if the maximal ideals in  $R$  is in one to one correspondence with the family  $V$ , and if

$$\hat{H}_{\{R/\underline{m}_i\}_{i=1}^s}^R \cong \hat{H}_{\{V_i\}_{i=1}^s}^A$$

for every subfamily  $\{V_i\}_{i=1}^s$  with corresponding subfamily of maximal ideals  $\{\underline{m}_i\}_{i=1}^s$ . As such,  $R$  is a moduli (affine spectrum) for its simple modules.

In our situation, the results are achieved by the following:

**Definition 1.11.** Let  $\hat{R} \in \hat{a}_r$ . A finitely generated  $k$ -algebra  $R$  is called an algebraization of  $\hat{R}$  if  $R$  has  $r$  maximal (left) ideals  $\underline{m}_i$ ,  $i = 1, \dots, r$  such that the formal moduli of the simple left  $R$ -modules  $V_i = R/\underline{m}_i$ ,  $i = 1, \dots, r$  is isomorphic to  $\hat{R}$ , that is

$$\hat{H}_{\{V_i\}_{i=1}^r}^R \cong \hat{R}$$

The paper is organized as follows: Section 2 and 3 contains the classification of 3-dimensional Lie-algebras as given for example in [4]. Section 4 considers the affine space of 3-dimensional Lie algebras and its components. In section 5 we compute the closures of the orbits under the  $\mathrm{GL}_3(k)$ -action giving the the isomorphism classes. These closures are the geometric points in our noncommutative moduli, the objects of our study. In section 6 we compute the tangent space dimensions of the moduli, and in section 7 we state the main result and explain it geometrically.

## 2 Moduli of Lie algebras

An  $n$ -dimensional Lie algebra  $\mathfrak{g}$  over  $k$  is determined by its structure coefficients  $c_{ij}^l$ ,  $1 \leq l \leq n$ , given by its bracket product  $b = [\cdot, \cdot] : \mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}$ , where a  $k$ -basis  $\{e_1, \dots, e_n\}$  for  $\mathfrak{g}$  is chosen, and

$$[e_i, e_j] = \sum_{l=1}^n c_{ij}^l e_l, \quad 1 \leq i < j \leq n$$

Writing up the Jacobi identity

$$[e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 0$$

we can rewrite for every  $i < j < k$  and  $m$ ,  $1 \leq i, j, k, m \leq n$ , the Jacobi identity as follows:

$$\sum_{l=1}^n (c_{jk}^l c_{il}^m + c_{ki}^l c_{jl}^m + c_{ij}^l c_{kl}^m) = 0$$

An isomorphism of Lie algebras is an isomorphism of  $k$ -vector spaces  $g$  commuting with the bracket:

$$\begin{array}{ccc} \mathfrak{g} \wedge \mathfrak{g} & \xrightarrow{\tilde{b}} & \mathfrak{g} \\ g \wedge g \downarrow & & \downarrow g \\ \mathfrak{g} \wedge \mathfrak{g} & \xrightarrow{b} & \mathfrak{g} \end{array}$$

Thus  $g \in \mathrm{GL}(n)$ , and  $\tilde{b} = g^{-1} \circ b \circ (g \wedge g)$ . We choose the basis  $\{e_i \wedge e_j\}$  for  $\mathfrak{g} \wedge \mathfrak{g}$  in lexicographic order. Then the matrix of  $b$  with respect to these bases is  $b = (c_{ij}^l)$ . We get

$$\tilde{b} = g^{-1} \cdot b \cdot \mathrm{Coef}(g)$$

where  $\mathrm{Coef}(g) = (C(l, m, i, j))$ , that is

$$\mathrm{Coef}(g)(e_i \wedge e_j) = \sum_{1 \leq l < m \leq n} C(l, m, i, j) e_l \wedge e_m$$

for  $i < j$ , and  $C(l, m, i, j)$  is the determinant of  $g$  after removing all rows except the  $l$ 'th and the  $m$ 'th, and removing all columns except the  $i$ 'th and the  $j$ 'th.

We use the notation

$$A_n = k[x_{ij}^l]_{\substack{1 \leq i < j \leq n \\ 1 \leq l \leq n}}, \quad J_{ijkm}^n = \sum_{l=1}^n (x_{jk}^l x_{il}^m + x_{ki}^l x_{jl}^m + x_{ij}^l x_{kl}^m)$$

Put  $\text{Lie}(n) = A_n / (J_{ijkm}^n)$ ,  $\underline{\text{Lie}}(n) = \text{Spec}(\text{Lie}(n))$  and let  $\underline{\text{GL}}_n(k)$  denote the affine variety of  $\text{GL}_n(k)$ , that is  $\underline{\text{GL}}_n(k) = \text{Spec}(k[x_{ij}]_{\det})$ . Then  $\underline{\text{GL}}_n(k)$  acts on  $\underline{\text{Lie}}(n)$  as above, and the set of isomorphism classes of  $n$ -dimensional Lie algebras is in bijective correspondence with the orbits of  $\underline{\text{GL}}(n)$  in  $\underline{\text{Lie}}(n)$ .

The case  $n=1$  contains only one Lie algebra, the abelian one, and in the case  $n = 2$ , the situation is well known. It is known that there exists no orbit space  $L_n = \underline{\text{Lie}}_n / \underline{\text{GL}}(n)$  for  $n \geq 2$

### 3 Classification of 3-dimensional Lie algebras

As before, we choose a basis  $\{e_1, e_2, e_3\}$  for  $\mathfrak{g}$ . Then the lexicographic ordering of  $e_i \wedge e_j$ ,  $1 \leq i < j \leq 3$  gives the basis  $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$  on  $\mathfrak{g} \wedge \mathfrak{g}$ . The matrix of the bracket  $b = [\cdot, \cdot]$  with respect to this basis is

$$\begin{pmatrix} c_{12}^1 & c_{13}^1 & c_{23}^1 \\ c_{12}^2 & c_{13}^2 & c_{23}^2 \\ c_{12}^3 & c_{13}^3 & c_{23}^3 \end{pmatrix}$$

Fulton and Harris gives the following classification in [4]:

**Lemma 3.1.** *There exists a  $k$ -vector space basis  $\{e_1, e_2, e_3\}$  for a non abelian Lie algebra  $\mathfrak{g}$  such that the matrix of structure coefficients of  $\mathfrak{g}$  is in one of the following forms:*

$$\mathfrak{sl}_2 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad l_a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & a & 0 \end{pmatrix}, \quad n_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the only pairs of isomorphic Lie-algebras are  $\{l_a, l_{a^{-1}}\}$ ,  $a \neq 0$ . Moreover, the Heisenberg Lie-algebra  $n_3$  is the only nilpotent one.

### 4 The $k$ -scheme $\underline{\text{Lie}}(3)$

In the 3 dimensional case, the Jacobi identity is given by the 3 equations

$$\begin{aligned} \sum_{l=1}^3 (x_{23}^l x_{1l}^1 + x_{31}^l x_{2l}^1 + x_{12}^l x_{3l}^1) &= 0 \\ \sum_{l=1}^3 (x_{23}^l x_{1l}^2 + x_{31}^l x_{2l}^2 + x_{12}^l x_{3l}^2) &= 0 \\ \sum_{l=1}^3 (x_{23}^l x_{1l}^3 + x_{31}^l x_{2l}^3 + x_{12}^l x_{3l}^3) &= 0 \end{aligned}$$

Put

$$S = \begin{pmatrix} 2x_{23}^1 & x_{23}^2 + x_{31}^1 & x_{23}^3 + x_{12}^1 \\ x_{23}^2 + x_{31}^1 & 2x_{31}^2 & x_{31}^3 + x_{12}^2 \\ x_{23}^3 + x_{12}^1 & x_{31}^3 + x_{12}^2 & 2x_{12}^3 \end{pmatrix}, \quad A = \begin{pmatrix} x_{12}^2 - x_{31}^3 \\ x_{23}^3 - x_{12}^1 \\ x_{31}^1 - x_{23}^2 \end{pmatrix}$$

Then the Jacobi identity can be written  $S \cdot A = 0$ . This gives the well known decomposition in [4] of  $\underline{\text{Lie}}(3)$  in two 6-dimensional irreducible components,

$$\underline{\text{Lie}}(3) = \underline{\text{Lie}}(3)^{(1)} \cup \underline{\text{Lie}}(3)^{(2)}$$

where  $\underline{\text{Lie}}(3)^{(1)}$  is given by the three equations corresponding to  $A = 0$  and  $\underline{\text{Lie}}(3)^{(2)}$  is given by the four equations corresponding to  $\det(S) = 0$ ,  $S \cdot A = 0$ , see [2, 7]. Of course,  $S = (x_{ij})$  also works and gives another description of  $\underline{\text{Lie}}(3)^{(2)}$ .

We shall give understandable descriptions of the orbits of the different Lie algebras in  $\underline{\text{Lie}}(3)$  under the action of  $\underline{\text{GL}}_3(k)$ . In the following, we replace the coordinates  $\{x_{i < j}^k\}$  by ordinary matrix coordinates. That is, we make the following identifications:

$$\begin{pmatrix} x_{12}^1 & x_{13}^1 & x_{23}^1 \\ x_{12}^2 & x_{13}^2 & x_{23}^2 \\ x_{12}^3 & x_{13}^3 & x_{23}^3 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

Thus the Jacobi identity is

$$\begin{pmatrix} 2x_{13} & x_{23} - x_{12} & x_{33} + x_{11} \\ x_{23} - x_{12} & -2x_{22} & x_{21} - x_{32} \\ x_{33} + x_{11} & x_{21} - x_{32} & 2x_{31} \end{pmatrix} \begin{pmatrix} x_{21} + x_{32} \\ x_{33} - x_{11} \\ -x_{12} - x_{23} \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \underline{0}$$

and the components are given accordingly by defining ideals in  $k[x_{11}, \dots, x_{33}]/(J_1, J_2, J_3)$ .

## 5 The closure of the orbits

To construct a classifying (not necessarily commutative) algebraic space for 3-dimensional Lie-algebras, we can construct the space for the closures of the orbits under the given  $\underline{G} = \underline{\text{GL}}_3(k)$ -action. This is because the different orbits have different closures. Thus we start by finding defining ideals of the orbit-closures. We use the notation  $o(x)$  for the  $\underline{G}$ -orbit of  $x \in \underline{\text{Lie}}(3)$  and  $\bar{o}(x)$  for the closure of this orbit.

First of all, because the group action  $\underline{G} \times X \rightarrow X$  is continuous, it follows that the orbits are irreducible, i.e. the surjection  $\underline{G} \twoheadrightarrow o(x) \subseteq X$  is continuous and  $\underline{G}$  is a linear irreducible group, see e.g. [10]. Secondly, the dimensions of the orbits are given by their isotropy groups. The isotropy group of  $x \in \underline{\text{Lie}}(3)$  is the vector space  $I_x = \{g \in \underline{G} | g \cdot x = x\} = \{g \in \underline{\text{GL}}_3(k) | g^{-1} \cdot x \cdot \text{Coef}(g) = x\} = \{g \in \underline{\text{GL}}_3(k) | x \cdot \text{Coef}(g) = g \cdot x\}$ .

This is done in [2]. Letting  $ab$  denote the abelian Lie algebra, we have the following dimensions of the closures of the orbits:

**Lemma 5.1.** *We have:*

$$\begin{aligned} \dim \bar{o}(ab) &= 0, & \dim \bar{o}(r_3) &= 5, & \dim \bar{o}(l_a) &= 5, & a \neq 1, \\ \dim \bar{o}(l_1) &= 3, & \dim \bar{o}(n_3) &= 3, & \dim \bar{o}(\text{sl}_2) &= 6 \end{aligned}$$

We use this information to find defining ideals of the closures of the orbits.

**Remark 5.2.** We do not prove radicality of the ideals. Gerhard Pfister has implemented the algorithm of Krick-Logar and Kemper for computing radicals in Singular [5]. This algorithm proves that the defining ideals below for the closures of the orbits of  $\mathfrak{sl}_2$ ,  $l_1$ ,  $l_{-1}$ ,  $n_3$ ,  $r_3$  and  $l_a$  for several choices of  $a$  are radical, but it remains to prove this for general  $a$ . However, these ideals, or rather their quotients are the objects of our study. The main result then proves to give a reasonable algebraic classifying structure. Of course, this indicates that all ideals in question are radical.

### 5.1 $\mathfrak{sl}_2$

$\mathfrak{sl}_2 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$  is an element in  $\underline{\text{Lie}}(3)^{(1)}$  which is of dimension 6. Thus  $\bar{o}(\mathfrak{sl}_2) = \underline{\text{Lie}}(3)^{(1)}$ . This means that a defining ideal for  $\bar{o}(\mathfrak{sl}_2)$  is

$$\mathfrak{a}_s = (x_{21} + x_{32}, x_{33} - x_{11}, x_{12} + x_{23})$$

### 5.2 $l_{-1}$

It is well known ([7]) that the intersection of the two components of  $\underline{\text{Lie}}(3)$  is the closure of the orbit of  $l_{-1}$ . This is so because  $\bar{o}(l_{-1}) \subseteq \underline{\text{Lie}}(3)^{(1)} \cap \underline{\text{Lie}}(3)^{(2)}$  and because the dimensions coincide. Letting

$$s = \begin{vmatrix} 2x_{13} & x_{23} - x_{12} & x_{33} + x_{11} \\ x_{23} - x_{12} & -2x_{22} & x_{21} - x_{32} \\ x_{33} + x_{11} & x_{21} - x_{32} & 2x_{31} \end{vmatrix}$$

a defining ideal of the closure of the orbit of  $l_{-1}$  is

$$\mathfrak{a}_{l_{-1}} = (s, x_{21} + x_{32}, x_{33} - x_{11}, x_{12} + x_{23})$$

### 5.3 $n_3$

$n_3$  is an element in  $\underline{\text{Lie}}(3)^{(2)}$ , and is characterized by the fact that its rank is 1. Letting  $s_{ij}$  be the  $ij$ -minor of the matrix  $(x_{ij})$  we find that a defining ideal of the closure of the orbit of  $n_3$  is

$$\mathfrak{a}_{n_3} = (x_{21} + x_{32}, x_{33} - x_{11}, x_{12} + x_{23}, s_{ij}), \quad 1 \leq i, j \leq 3$$

### 5.4 $l_a$ , $a \neq -1, 1$

From [4], it is well known that on the complement of the intersection of the two components, the expression

$$J_{i,j} = \frac{\text{Tr}(\text{ad}(x_i) \text{ad}(x_j))}{\text{Tr}(\text{ad}(x_i)) \text{Tr}(\text{ad}(x_j))}$$

takes the same value  $C = \frac{1+a^2}{(1+a)^2}$  on the isomorphism classes of  $l_a$ ,  $a \neq -1, 1$ . We compute and find that  $\text{GL}_3(k)$  takes any  $J_{ij}$  into any other  $J_{kl}$ ,  $i \neq j$ ,  $k \neq l$ , and any  $J_{ii}$  into any other  $J_{ll}$ . We also check for one choice that  $J_{ij} = J_{ii}$ ,  $i \neq j$ . As our defining ideal of the closure of the

orbit of  $l_a$ ,  $a \neq -1, 1$  is invariant, we see that for  $C \neq \frac{1}{2}$ ,  $J = C$  contains Lie-algebras from only one orbit, the orbit of  $l_a$ , where  $C = \frac{1+a^2}{(1+a)^2}$ , and thus this algebraic set must be the closure of the orbit. The different expressions for  $J$  are the following fractions:

$$\begin{aligned} J_{(1,1)} &= \frac{(C_{13}^3)^2 + (C_{12}^2)^2 + 2C_{13}^2 C_{12}^3}{(C_{12}^2 + C_{13}^3)^2} \\ J_{(1,2)} &= \frac{C_{12}^3 (C_{23}^2 - C_{13}^1) - C_{12}^1 C_{12}^2 + C_{23}^3 C_{13}^3}{(C_{12}^2 + C_{13}^3)(C_{23}^3 - C_{12}^1)} \\ J_{(1,3)} &= \frac{C_{13}^2 (C_{23}^3 + C_{12}^1) + C_{13}^1 C_{13}^3 + C_{23}^2 C_{12}^2}{(C_{12}^2 + C_{13}^3)(C_{13}^1 + C_{23}^2)} \\ J_{(2,2)} &= \frac{(C_{23}^3)^2 + (C_{12}^1)^2 - 2C_{23}^1 C_{12}^3}{(C_{23}^3 - C_{12}^1)^2} \\ J_{(2,3)} &= \frac{C_{23}^1 (C_{12}^2 - C_{13}^3) - C_{23}^2 C_{23}^3 + C_{13}^1 C_{12}^1}{(C_{12}^1 - C_{23}^3)(C_{13}^1 + C_{23}^2)} \\ J_{(3,3)} &= \frac{(C_{23}^2)^2 + (C_{13}^1)^2 + 2C_{23}^1 C_{13}^2}{(C_{13}^1 + C_{23}^2)^2} \end{aligned}$$

This results in the following: For  $a \neq -1, 1$ ,  $\bar{o}(l_a)$  is given by the ideal generated by the following polynomials:

$$\begin{aligned} j_1(a) &= x_{32}^2 + x_{21}^2 + 2x_{22}x_{31} - C(x_{21} + x_{32})^2 \\ j_2(a) &= x_{31}(x_{23} - x_{12}) - x_{11}x_{21} + x_{33}x_{32} - C(x_{21} + x_{32})(x_{33} - x_{11}) \\ j_3(a) &= x_{22}(x_{33} + x_{11}) + x_{12}x_{32} + x_{23}x_{21} - C(x_{21} + x_{32})(x_{12} + x_{23}) \\ j_4(a) &= x_{33}^2 + x_{11}^2 - 2x_{13}x_{31} - C(x_{33} - x_{11})^2 \\ j_5(a) &= x_{13}(x_{21} - x_{32}) - x_{23}x_{33} + x_{12}x_{11} - C(x_{11} - x_{33})(x_{12} + x_{23}) \\ j_6(a) &= x_{23}^2 + x_{12}^2 + 2x_{13}x_{22} - C(x_{12} + x_{23})^2 \end{aligned}$$

So  $\mathfrak{a}_a = (j_1(a), \dots, j_6(a))$ ,  $a \neq -1, 1$ .

## 5.5 $l_1$

We see that both  $r_3$  and  $l_1$  is contained in the ideal given by  $J = \frac{1}{2}$ , for instance

$$J_{(1,1)} = \frac{1^2 + 1^2 + 2 \cdot 0 \cdot 0}{(1+1)^2} = \frac{1^2 + 1^2 + 2 \cdot 1 \cdot 0}{(1+1)^2} = \frac{1}{2}$$

Thus both of the closures of the orbits are in the zero set of the ideal  $(j_1(1), \dots, j_6(1))$ .

Considering the invariant ideal

$$(x_{13}, x_{22}, x_{31}, x_{23} - x_{12}, x_{33} + x_{11}, x_{21} - x_{32})$$

we see that this ideal contains  $l_1$ , but not  $r_3$  because  $x_{22} \neq 0$  there. This implies that

$$\mathfrak{a}_{l_1} = (j_1(1), \dots, j_6(1), x_{13}, x_{22}, x_{31}, x_{23} - x_{12}, x_{33} + x_{11}, x_{21} - x_{32})$$

and it follows that

$$\mathfrak{a}_{l_1} = (x_{13}, x_{22}, x_{31}, x_{23} - x_{12}, x_{33} + x_{11}, x_{21} - x_{32})$$



### 5.6 $r_3$

Because the orbit of  $r_3$  has to be the open set  $Z((j_1(1), \dots, j_6(1))) - Z(\mathfrak{a}_{l_1})$ , it follows that its closure is the zero set of  $\mathfrak{a}_{r_3} = (j_1(1), \dots, j_6(1))$ . Thus the objects of our study are given as  $A\sharp G$ -modules, where  $A = \text{Lie}(3)$ ,  $G = \text{GL}_3(k)$  with its given action on  $A$  in this case, and where  $A\sharp G$  is the skew group algebra.

## 6 Tangent space dimensions

Let  $A$  be a finitely generated  $k$ -algebra with an action of a linearly reductive group  $G$ . In this paper, the interest is classification of the orbits in  $\text{Spec}(A)$  under the action of  $G$ . If the closures of two different orbits are different, we can consider classification of the closures of the orbits as well as the orbits themselves. If  $\nabla : G \rightarrow \text{Aut}_k(A)$  is the action of  $G$  on  $A$ , and if the ideal of the closure of the orbit of  $x \in \text{Spec}(A)$  is  $\mathfrak{a}_x$ , then  $A/\mathfrak{a}_x$  has an induced  $G$ -action  $\nabla : G \rightarrow \text{Aut}_k(A/\mathfrak{a}_x)$ . Thus the obstruction theory above has to be generalized to the category of  $A - G$ -modules which is the category of  $A$  modules  $M$  with  $G$ -action such that the two operations commute, i.e.  $\nabla_g(ma) = \nabla_g(m)\nabla_g(a)$ . This is just the theory in [9] on  $A\sharp G$ -modules.

This obstruction theory makes it possible to generalize the algorithm from [12]. This is done in [14] and will be published elsewhere. In this paper it turns out that the liftings are unobstructed, and so we will only need the tangent spaces. We need the following fact from [6]:

**Lemma 6.1.** *Let  $M, N$  be two  $A - G$ -modules where  $G$  is reductive. Then*

$$\text{Ext}_{A-G}^i(M, N) \cong \text{Ext}_A^i(M, N)^G$$

If a moduli space for 3-dimensional Lie algebras exists, it should have the expected tangent space dimensions corresponding to the correct cohomology. If a commutative moduli space  $\mathbb{M}$  exists, the tangent space in a point  $\mathfrak{g} \in \mathbb{M}$  is the Chevalley-Eilenberg-MacLane cohomology  $H_{CE}^p$ . The commutative theory is well known, see [1], and it has been proved that there exists no commutative algebraic moduli. In the construction of the noncommutative moduli, the tangent spaces between the modules are studied. For Lie-algebras the correct cohomology giving these tangent spaces is not known. However there exists such a cohomology in the category of  $A$ -modules, namely  $\text{Ext}_A^1(M, N)$ . This is the reason for passing from Lie algebras to  $A$ -modules, or in fact  $A - G$ -modules in our situation.

As explained in [9] the tangent space of the non commutative moduli of a family of  $A - G$ -modules is given by  $\text{Ext}_{A-G}^1(V, W)$ , where  $V, W$  runs through all possible selections of pairs of  $A - G$ -modules. In our situation,  $A = \text{Lie}(3)$ ,  $G = \text{GL}_3(k)$  with the given action on  $\text{Spec}(A) = \underline{\text{Lie}}(3)$ . The  $A - G$ -modules we consider are

$$\begin{aligned} \underline{\text{sl}}_2 = \underline{s} &= A/\mathfrak{a}_s, & \underline{r}_3 &= A/\mathfrak{a}_{r_3}, & \underline{l}_{-1} &= A/\mathfrak{a}_{l_{-1}}, & \underline{n}_3 &= A/\mathfrak{a}_{n_3} \\ \underline{l}_a &= A/\mathfrak{a}_{l_a}, & a &\neq -1 \end{aligned}$$

To be able to compute the tangent spaces above, we need the following general fact which was proved in [14]. We assume that  $G$  is a linear reductive group acting on a  $k$ -algebra  $A$  of finite type,  $k$  algebraically closed.

**Lemma 6.2.** *Let  $\mathfrak{a} \subseteq \mathfrak{b}$  be two  $G$ -invariant ideals in  $A$ . Then*

$$\mathrm{Ext}_{A-G}^1(A/\mathfrak{a}, A/\mathfrak{b}) \cong \mathrm{Hom}_A(\mathfrak{a}/\mathfrak{a}^2, A/\mathfrak{b})^G$$

and the action of  $g \in G$  is given by

$$\tilde{\nabla}_g : \mathrm{Hom}_A(\mathfrak{a}/\mathfrak{a}^2, A/\mathfrak{b}) \longrightarrow \mathrm{Hom}_A(\mathfrak{a}/\mathfrak{a}^2, A/\mathfrak{b})$$

where  $\tilde{\nabla}_g(\phi) = \nabla_g \circ \phi \circ \nabla_{g^{-1}}$  and  $\nabla_g : \mathfrak{a}/\mathfrak{a}^2 \longrightarrow \mathfrak{a}/\mathfrak{a}^2$  and  $\nabla_{g^{-1}} : A/\mathfrak{b} \longrightarrow A/\mathfrak{b}$  are the actions induced from  $\nabla_g : A \longrightarrow A$ .

Also notice the following:

**Lemma 6.3.** *For all selections of pairs of ideals  $\mathfrak{a}, \mathfrak{b}$  among  $\mathfrak{a}_{\mathrm{sl}_2}, \mathfrak{a}_{l_{-1}}, \mathfrak{a}_{n_3}, \mathfrak{a}_{l_a}, a \neq -1, 1, \mathfrak{a}_{l_1}, \mathfrak{a}_{r_3}$  we find that if  $\mathfrak{a} \not\subseteq \mathfrak{b}$  then  $\mathrm{Ext}_A^1(A/\mathfrak{a}, A/\mathfrak{b}) = 0$ .*

**Proof.** Using Singular [5], we can find free resolutions and compute that  $\mathrm{Ext}_A^1(A/\mathfrak{a}, A/\mathfrak{b}) = 0$ , Hence  $\mathrm{Ext}_A^1(A/\mathfrak{a}, A/\mathfrak{b})^G = 0$ .  $\square$

With these preliminaries, we are left with some straight forward calculations, all based on the same technique: Consider the standard elementary  $3 \times 3$ -matrices  $E_{ij}$ ,  $1 \leq i < j \leq 3$ , (interchange rows  $i$  and  $j$ ),  $E_i(c)$ ,  $1 \leq i \leq 3$ ,  $c \neq 0$  (multiply the  $i$ 'th. row with  $c$ ),  $E_{ij}(c)$ ,  $1 \leq i \neq j \leq 3$ , (add  $c$  times the  $i$ 'th. row to the  $j$ 'th). Then these elements generate  $\mathrm{GL}_3(k) = G$  and the invariants under  $G$  are the elements invariant under these generators. We compute the action of each elementary matrix in the generator set of  $G$  on the variables  $x_{ij}$  by the rule

$$\nabla_g(x_{ij}) = g^{-1} \cdot (x_{ij}) \cdot \mathrm{Coef}(g)$$

Then we find the induced action on the generators of the ideals and use lemma 6.2 to compute the invariant homomorphisms.

We will give two computations as examples, both of importance, and state the rest.

### 6.1 Example. Computation of $\mathrm{Ext}_{A-G}^1(l_{-1}, l_{-1})$

We recall that  $\mathfrak{a}_{l_{-1}} = (s, x_{21} + x_{32}, x_{33} - x_{11}, x_{12} + x_{23})$ . Assume that  $\phi = (h, f_1, f_2, f_3) : \mathfrak{a}_{l_{-1}}/(\mathfrak{a}_{l_{-1}})^2 \longrightarrow A/\mathfrak{a}_{l_{-1}}$  is invariant.

For  $g = c \cdot \mathrm{Id} = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$ , we find  $\mathrm{Coef}(g) = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$ , and so the action of this element on a general point is given by

$$g^{-1} \cdot \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \cdot \mathrm{Coef}(g) = \frac{1}{c} \cdot c^2 \cdot \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = c \cdot \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

Thus, if  $\phi$  is invariant, the composition

$$\mathfrak{a}_{l_{-1}}/(\mathfrak{a}_{l_{-1}})^2 \xrightarrow{\nabla_g} \mathfrak{a}_{l_{-1}}/(\mathfrak{a}_{l_{-1}})^2 \xrightarrow{\phi} A/\mathfrak{a}_{l_{-1}} \xrightarrow{\nabla_{g^{-1}}} A/\mathfrak{a}_{l_{-1}}$$

leaves  $\phi$  unaltered. In particular  $\nabla_{g^{-1}}(\phi(\nabla_g(s))) = \phi(s) \iff \nabla_{g^{-1}}(c^3 \cdot h) = h$ , which implies that  $h$  is homogeneous of degree 3. Furthermore, we find  $\nabla_{E_{ij}}(s) = -s$ ,  $1 \leq i < j \leq 3$ ,

$\nabla_{E_i(c)}(s) = cs$ ,  $1 \leq i \leq 3$ ,  $\nabla_{E_{ij}(c)}(s) = s$ ,  $1 \leq i \neq j \leq 3$ . Thus we are looking for a homogeneous polynomial  $h$  of degree 3 with the same properties as  $s$  above. It is easy to check that

$$h = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

fulfills the conditions, and it is straight forward to check that it is the only such (in the same way as in the next example). We find that  $h \in \mathfrak{a}_{\underline{1}}$ , that is  $h = 0$ . Using similar techniques, we find that  $f_1 = f_2 = f_3 = 0$ . So  $\text{Ext}_{A-G}^1(\underline{l}_{-1}, \underline{l}_{-1})$  is of  $k$ -dimension 0.

## 6.2 Example. Computation of $\text{Ext}_{A-G}^1(\underline{l}_a, \underline{l}_a)$ , $a \neq 1, -1$

Recall that for  $a \neq 1, -1$ ,  $\mathfrak{a}_{l_a} = (j_1(a), \dots, j_6(a))$ . We find the following actions on  $(j_1, \dots, j_6)$ :

$$\begin{aligned} & (j_1, j_2, j_3, j_4 j_5, j_6) \\ E_{12} & (j_4, j_2, -j_5, j_1, -j_3, j_6) \\ E_{13} & (j_6, j_5, j_3, j_4, j_2, j_1) \\ E_{23} & (j_1, -j_3, -j_2, j_6, j_5, j_4) \\ E_1(c) & (c^2 j_1, c j_2, c j_3, j_4, j_5, j_6) \\ E_2(c) & (j_1, c j_2, j_3, c^2 j_4, c j_5, j_6) \\ E_3(c) & (j_1, j_2, c j_3, j_4, c j_5, c^2 j_6) \\ E_{12}(c) & (j_1 + 2c j_2 + c^2 j_4, j_2 + c j_4, j_3 - c j_5, j_4, j_5, j_6) \\ E_{13}(c) & (j_1 - 2c j_3 + c^2 j_6, j_2 + c j_5, j_3 - c j_6, j_4, j_5, j_6) \\ E_{23}(c) & (j_1, j_2 - c j_3, j_3, j_4 + 2c j_5 + c^2 j_6, j_5 + c j_6, j_6) \\ E_{21}(c) & (j_1, j_2 + c j_1, j_3, j_4 + 2c j_2 + c^2 j_1, j_5 - c j_3, j_6) \\ E_{31}(c) & (j_1, j_2, j_3 - c j_1, j_4, j_5 + c j_2, j_6 - 2c j_3 + c^2 j_1) \\ E_{32}(c) & (j_1, j_2, j_3 - c j_2, j_4, j_5 + c j_4, j_6 + 2c j_5 + c^2 j_4). \end{aligned}$$

In addition we can see that if  $E = c \cdot \text{Id}$ , then  $\nabla_E(x_{ij}) = (cx_{ij})$  also implying that  $\nabla_E(j_i) = c^2 j_i$ ,  $1 \leq i \leq 6$ .

Now  $\phi : \mathfrak{a}_{l_a}/\mathfrak{a}_{l_a}^2 \longrightarrow A/\mathfrak{a}_{l_a}$  is determined by its image of  $j_1, \dots, j_6$ , that is  $\phi(j_i) = h_i$ ,  $1 \leq i \leq 6$ . That  $\phi$  is invariant means that it is invariant under the composition

$$\mathfrak{a}_{l_a}/\mathfrak{a}_{l_a}^2 \xrightarrow{\nabla_g} \mathfrak{a}_{l_a}/\mathfrak{a}_{l_a}^2 \xrightarrow{\phi} A/\mathfrak{a}_{l_a} \xrightarrow{\nabla_{g^{-1}}} A/\mathfrak{a}_{l_a}$$

for all  $g \in G$ , that is for all the generators of  $G$ . First, let  $g = c \cdot \text{Id}$ . Then the invariance of  $\phi$  means that  $\nabla_{g^{-1}}(c^2 h_i) = h_i$  for  $1 \leq i \leq 6$ . This forces each  $h_i$  to be homogeneous of degree 2. Then  $g = E_i(c)$ ,  $1 \leq i \neq j \leq 3$  tells us which degree 2 monomials that are possible. Finally, investigating the action of  $g = E_{ij}(c)$ ,  $1 \leq i \neq j \leq 3$  gives

$$\begin{aligned} \phi = \alpha & ((x_{21} + x_{32})^2, (x_{21} + x_{32})(x_{33} - x_{11}), (x_{21} + x_{32})(x_{12} + x_{23}), \\ & (x_{33} - x_{11})^2, (x_{11} - x_{33})(x_{12} + x_{23}), (x_{12} + x_{23})^2). \end{aligned}$$

Then we have to prove that this invariant morphism (with  $\alpha = 1$ ) is an element in  $\text{Ext}_A^1(\underline{l}_a, \underline{l}_a)$ , which means that it really is an  $A$ -module homomorphism  $\phi : \mathfrak{a}_{l_a}/\mathfrak{a}_{l_a}^2 \longrightarrow A/\mathfrak{a}_{l_a}$ . This follows

from the fact that  $\phi \cdot S = 0$ , where  $S$  is the syzygy-module of  $\mathfrak{a}_a$ . Thus  $\{\phi\}$  is a  $k$ -vector space basis for the 1-dimensional space  $\text{Ext}_{A-G}^1(l_a, l_a)$ .

Notice that exactly the same computation yields the computation of  $\text{Ext}_{A-G}^1(r_3, r_3)$  and for  $\text{Ext}_{A-G}^1(r_3, l_1)$ . Thus  $\{\phi\}$  is also a basis for the 1-dimensional space  $\text{Ext}_{A-G}^1(r_3, r_3)$  and for  $\text{Ext}_{A-G}^1(r_3, l_1)$ .

The result of this computation is the following:

**section 6.4.** For  $a \neq 1, -1$ ,  $\text{Ext}_{A-G}^1(l_a, l_a) = 1$ . Also  $\text{Ext}_{A-G}^1(r_3, r_3) = 1$ ,  $\text{Ext}_{A-G}^1(r_3, l_1) = 1$ . For all other possible selections of pairs of orbits  $V, W$ ,  $\text{Ext}_{A-G}^1(V, W) = 0$ .

## 7 Noncommutative moduli

### 7.1 The formal noncommutative moduli

In the present situation we consider the closures of the orbits

$$\text{cl}(o(r_3)) \supseteq \text{cl}(o(l_1)) \iff \mathfrak{a}_{r_3} \subseteq \mathfrak{a}_{l_1}$$

To ease the tracking of the algorithm for computing the local formal moduli, we set

$$V_1 = A/\mathfrak{a}_{r_3}, \quad V_2 = A/\mathfrak{a}_{l_1}$$

and we use the notation

$$\mathfrak{a}_{r_3} = (j_1, \dots, j_6), \quad \mathfrak{a}_{l_1} = (x_{13}, x_{22}, x_{31}, x_{23} - x_{12}, x_{11} + x_{33}, x_{21} - x_{32}) = (h_1, \dots, h_6)$$

To compute the obstructions in the Yoneda complex we choose free resolutions of  $V_1$  and  $V_2$  in the following way, where we have added the basis for  $\text{Ext}_{A-G}^1(V_1, V_i)$ ,  $i = 1, 2$ :

$$\begin{array}{ccccccc} 0 & \longleftarrow & V_1 & \longleftarrow & A \xleftarrow{d_1^1=(j_1 \dots j_6)} A^6 & \xleftarrow{d_2^1} & A^{20} \longleftarrow \dots \\ & & & & \xi_1^{11} \swarrow & & \xi_2^{11} \swarrow \\ 0 & \longleftarrow & V_1 & \longleftarrow & A \xleftarrow{\xi_1^{12}} A^6 & \xleftarrow{\xi_2^{12}} & A^{20} \longleftarrow \dots \\ & & & & \xi_1^{12} \swarrow & & \xi_2^{12} \swarrow \\ 0 & \longleftarrow & V_2 & \longleftarrow & A \xleftarrow{d_1^2=(h_1 \dots h_6)} A^6 & \xleftarrow{d_2^2} & A^{21} \longleftarrow \dots \end{array}$$

where

$$\xi_1^{11} = \xi_1^{12} = ((x_{21} + x_{32})^2, (x_{21} + x_{32})(x_{33} - x_{11}), (x_{21} + x_{32})(x_{12} + x_{23}), (x_{33} - x_{11})^2, \\ (x_{11} - x_{33})(x_{12} + x_{23}), (x_{12} + x_{23})^2)$$

and where  $\xi_2^{11}$  and  $\xi_2^{12}$  are given by the condition

$$d_2^1 \circ \xi_1^{11} + \xi_2^{11} \circ d_1^1 = d_2^1 \circ \xi_1^{12} + \xi_2^{12} \circ d_1^1 = 0$$

Then the cup products (second order generalized Massey products) are given by their first term in the Yoneda complex. That is

$$\xi_1^{11} \circ \xi_2^{11} = \xi_1^{12} \circ \xi_2^{12} = 0$$

where the last equality is "strict", meaning that the product of the matrices are zero in  $A$ . Notice that the obstruction calculus for  $A - G$ -modules and  $A$ -modules are compatible. This tells us that the infinitesimal family defines a lifting to

$$\hat{H}_{V_1, V_2} = \begin{pmatrix} k[[t]] & \langle\langle u \rangle\rangle \\ 0 & k \end{pmatrix}$$

hence this is a pro-representing hull.

## 7.2 The noncommutative moduli Lie(3)

Consider the family of 3-dimensional Lie-algebras given as the zero set of  $(j_1(a), \dots, j_6(a))$  where

$$\begin{aligned} j_1(a) &= x_{32}^2 + x_{21}^2 + 2x_{22}x_{31} - C(x_{21} + x_{32})^2 \\ j_2(a) &= x_{31}(x_{23} - x_{12}) - x_{11}x_{21} + x_{33}x_{32} - C(x_{21} + x_{32})(x_{33} - x_{11}) \\ j_3(a) &= x_{22}(x_{33} + x_{11}) + x_{12}x_{32} + x_{23}x_{21} - C(x_{21} + x_{32})(x_{12} + x_{23}) \\ j_4(a) &= x_{33}^2 + x_{11}^2 - 2x_{13}x_{31} - C(x_{33} - x_{11})^2 \\ j_5(a) &= x_{13}(x_{21} - x_{32}) - x_{23}x_{33} + x_{12}x_{11} - C(x_{11} - x_{33})(x_{12} + x_{23}) \\ j_6(a) &= x_{23}^2 + x_{12}^2 + 2x_{13}x_{22} - C(x_{12} + x_{23})^2 \end{aligned}$$

whwre  $C = \frac{1+a^2}{(1+a)^2}$ . This family contains each isomorphism class  $l_a$ ,  $a \neq 1, -1$ , and  $r_3$  exactly once. Renaming this family to  $\mathfrak{g}(C)$  we have

$$\mathfrak{g}(C) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & a & 0 \end{pmatrix}, \quad C \neq \frac{1}{2}, \quad \text{and} \quad \mathfrak{g}\left(\frac{1}{2}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Given a  $k$ -algebra  $A$ . An  $A$ -module  $V$  is given by a morphism  $\phi : A \rightarrow \text{End}_k(V)$ . Using this, the obstruction theory and formation of moduli can be done in the Hochschild cohomology. In particular the tangent space of the deformation functor in this case is  $\text{HH}^1(A, \text{Hom}_k(V_i, V_j)) \cong \text{Der}_k(V) / \text{Triv}$ . See ([15]) for a standard example. This is used to suggest an algebraization of the local formal moduli where the maximal ideals on the diagonal represents the geometric point, i.e. the quotients of the maximal ideals on the diagonal are the simple modules.

The tangent space dimensions then suggests an algebraization of  $\hat{H}(\mathfrak{g}(C), l_1)$  to be

$$\begin{pmatrix} k[[t_{11}]] & \langle t_{12} \rangle / (t_{11} - \frac{1}{2})t_{12} \\ 0 & k \end{pmatrix}$$

This can not be the case because then

$$\hat{H}\left(\mathfrak{g}\left(\frac{1}{2}\right), k\right) = \begin{pmatrix} k[[t_{11}]] & \langle\langle t_{12} \rangle\rangle / t_{11}t_{12} \\ 0 & k \end{pmatrix}$$

which is contradicting any possible definition of moduli. This is explained the following way:  $\mathfrak{g}(\frac{1}{2})$  deforms to  $\begin{pmatrix} 0 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & \lambda & 0 \end{pmatrix}$  and the closure of the orbit of this Lie-algebra contains  $\begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}$ . These points are however collapsed to one point when dividing out with the group action. Thus the correct algebraization is

$$H = \begin{pmatrix} k[C] & k[C]_{(C-\frac{1}{2})} \langle t_{12} \rangle \\ 0 & k \end{pmatrix}$$

where the entry  $k[C]_{(C-\frac{1}{2})} \langle t_{12} \rangle$  means the cyclic left  $k[C]$ -module generated by  $t_{12}$  with coefficients in  $k[C]_{(C-\frac{1}{2})}$ , the localization of  $k[C]$  in the maximal ideal  $(C - \frac{1}{2})$ . We have proved following:

**section 7.1.** *The noncommutative moduli  $\underline{\text{Lie}}(3)$  is*

$$\mathbb{M} = \begin{pmatrix} k[C] & k[C]_{(C-\frac{1}{2})} \langle t_{12} \rangle & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 0 & k \end{pmatrix}$$

where the two first rows correspond to the Lie-algebras  $\mathfrak{g}(C)$  and  $l_1$  respectively, and where the four last rows corresponds to  $\underline{sl}_2(k)$ ,  $\underline{n}_3$ ,  $l_{-1}$ , and  $ab = \mathfrak{g}_0$  (respectively).

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