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Research Article

Quantizations of Group Actions

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Abstract We describe quantizations on monoidal categories of modules over finite groups. Those are given by quantizers which are elements of a group algebra. Over the complex numbers we find these explicitly. For modules over S_3 and A_4 we give explicit forms for all quantizations.

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1 Introduction

In [1,3], we found the quantizations of the monoidal categories of modules graded by finite abelian groups. Quantizations are natural isomorphisms of the tensor bifunctor $Q : \otimes \to \otimes$ that satisfy the coherence condition. By this condition, the quantizations are 2-cocycles, and under action by isomorphisms of the identity functor they are representatives of the second cohomology of the group.

With these explicit descriptions of quantizations, we showed that classical non-commutative algebras like the quaternions and the octonions are obtained by quantizing a required number of copies of $\mathbb{R}$. Moreover, we obtained new classes of non-commutative algebras. The resulting non-commutativity is governed by a braiding which is the quantization of the twist.

In this paper, we will investigate the situation for quantizations of modules with action of finite groups that are not necessarily abelian. Further the definition of quantizations is widened as they may be natural transformations, not only natural isomorphisms.

Quantizations Q of the monoidal category of modules over finite groups G are realized by elements q_Q in the group algebra $\mathbb{C}[G \times G]$ called quantizers. These satisfy a form of the coherence condition, normalization, and invariance with respect to G-action.

Let $\hat{G}$ be the dual of G. We use the Fourier transform to reconsider these quantizers as sequences of operators $\hat{q}_{\alpha,\beta}$ in $\mathrm{End}(E_\alpha \otimes E_\beta)$, where E_α, E_β are irreducible representations corresponding to $\alpha, \beta \in \hat{G}$. The coherence condition for the operators $\hat{q}_{\alpha,\beta}$ gives a system of quadratic matrix equations.

There is an equivalence relation on these quantizers by the action by natural isomorphisms of the identity functor. Taking the orbits of this action, we arrive at the final expressions of the quantizers.

We apply the inverse of the Fourier transform to move back to the group algebra where we now have the quantizers q_Q in $\mathbb{C}[G \times G]$ realizing the non-trivial quantizations in the category.

To sum up, the procedure is as follows:

$$
\begin{tikzcd}
\text{Quantizations } Q \arrow[d] & \\
\text{Quantizers } q_Q \in \mathbb{C}[G \times G] \arrow[r, "\text{Fourier}"] & \hat{q}_Q \in \oplus_{\alpha,\beta\in\hat{G}} \mathrm{End}\left(E_\alpha \otimes E_\beta\right) \arrow[d, "\text{Calculations}"'] \\
& \text{Explicit sequences } \left\{\hat{q}_{\alpha,\beta} \in \mathrm{End}\left(E_\alpha \otimes E_\beta\right)\right\}_{\alpha,\beta\in\hat{G}} \arrow[ul, "\text{Fourier}^{-1}"']
\end{tikzcd}
$$

We illustrate this method for abelian groups (see, e.g., [2]) and the permutation groups S_3 and A_4. For S_3 there is a 1-parameter family of quantizations. For A_4 we have a larger selection with 2 parameters producing quantizations.

2 Quantizations in braided monoidal categories

A braided monoidal category C is a category equipped with a tensor product $\otimes$ and two natural isomorphisms: an associativity constraint $\mathrm{assoc}_C : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ and a braiding $\sigma_C : X \otimes Y \to Y \otimes X$, for objects X, Y, Z in C, that both satisfy MacLane coherence conditions; see [5].

Let C, D be braided monoidal categories and $\Phi : C \to D$ a functor. A *quantization* Q of the functor Φ is a natural transformation of the tensor bifunctor

$$
\begin{gathered}
Q : \otimes_D \circ (\Phi \times \Phi) \longrightarrow \Phi \circ \otimes_C, \\
Q = Q_{X,Y} : \Phi(X) \otimes_D \Phi(Y) \longrightarrow \Phi(X \otimes_C Y)
\end{gathered}
$$

that satisfies the following coherence condition:

$$
\begin{tikzcd}
\Phi(X) \otimes \big(\Phi(Y) \otimes \Phi(Z)\big) \arrow[r, "1\otimes Q"] \arrow[d, "\mathrm{assoc}_D"] & \Phi(X) \otimes \Phi(Y \otimes Z) \arrow[r, "Q"] & \Phi\big(X \otimes (Y \otimes Z)\big) \arrow[d, "\Phi(\mathrm{assoc}_C)"] \\
\big(\Phi(X) \otimes \Phi(Y)\big) \otimes \Phi(Z) \arrow[r, "Q\otimes 1"] & \Phi(X \otimes Y) \otimes \Phi(Z) \arrow[r, "Q"] & \Phi\big((X \otimes Y) \otimes Z\big)
\end{tikzcd}
\tag{2.1}
$$

for $X, Y \in \mathrm{Obj}(C)$ (see [4] for details).

Note that we do not require the quantizations to be natural isomorphisms, only natural transformations (cf. [4]).

We denote the set of quantizations of Φ by $q(\Phi)$.

Let $\lambda : \Phi \to \Phi$ be a unit preserving natural isomorphism of the functor Φ. Then, we define an action of $\lambda : Q \mapsto \lambda(Q)$ on the quantizations by requiring that the diagram

$$
\begin{tikzcd}
\Phi(X) \otimes \Phi(Y) \arrow[r, "Q_{X,Y}"] \arrow[d, "\lambda_X \otimes \lambda_Y"] & \Phi(X \otimes Y) \arrow[d, "\lambda_{X\otimes Y}"] \\
\Phi(X) \otimes \Phi(Y) \arrow[r, "\lambda(Q_{X,Y})"] & \Phi(X \otimes Y)
\end{tikzcd}
\tag{2.2}
$$

commutes.

The orbits of the above action we denote by $\mathfrak{Q}(\Phi)$.

If a quantization is a natural isomorphism, we will call it a *regular* quantization and use the notations $\mathbf{q}^\circ(\Phi)$ and $\mathfrak{Q}^\circ(\Phi)$.

If the functor Φ is the identity $Id_C : C \to C$, we call Q a *quantization of the category* C and use the notations $\mathbf{q}(C) = \mathbf{q}(Id_C)$ and $\mathfrak{Q}(C) = \mathfrak{Q}(Id_C)$.

For the following result, see also [4].

Let $\Phi : B \to C, \Psi : C \to D$ be functors between the monoidal categories B, C, D and let Q^Φ, Q^Ψ be quantizations of Φ and Ψ. The following formula defines the quantization

$$
Q^{\Phi\circ\Psi} = \Psi\big(Q^{\Phi}_{X,Y}\big) \circ Q^{\Psi}_{\Phi(X),\Phi(Y)}
\tag{2.3}
$$

of the composition $\Psi \circ \Phi$.

We call $Q^{\Phi\circ\Psi}$ composition of quantizations. Then the composition defines an associative multiplication

$$
\mathbf{q}(\Phi) \times \mathbf{q}(\Psi) \longrightarrow \mathbf{q}(\Psi \circ \Phi)
$$

on the sets of quantizations.

In particular, the composition (2.3), where $\Phi = \Psi = Id_C$, defines a multiplication

$$
\mathbf{q}(C) \times \mathbf{q}(C) \longrightarrow \mathbf{q}(C)
$$

and gives a monoid structure on the set of quantizations.

Moreover, the regular quantizations form a group in this monoid.

The monoid $\mathbf{q}(C)$ acts on a variety of objects (see, e.g., [4]). We list some examples here.

Braidings. Let σ be a braiding in the category C. If $Q \in \mathbf{q}^\circ(C)$, then we define an action of Q on braidings by requiring that the following diagram

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{Q_{X,Y}} & X \otimes Y \\ \sigma_Q \downarrow & & \downarrow \sigma \\ Y \otimes X & \xrightarrow{Q_{Y,X}} & Y \otimes X \end{array}$$

commutes, where

$$Q : \sigma \longmapsto \sigma_Q = Q_{Y,X}^{-1} \circ \sigma \circ Q_{X,Y}.$$

Then σ_Q is a braiding too.

Algebras. Let A be an associative algebra in the category C with multiplication $\mu : A \otimes A \to A$. Let $Q \in \mathbf{q}(C)$. Then we define a new multiplication μ^Q on A by requiring that the following diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "Q_{A,A}"] \arrow[dr, "\mu^Q"'] & A \otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd}$$

commutes.

Then (A, μ^Q) is an algebra in the category C too. We call it the *quantized algebra*.

Modules. Let E be a left module over the algebra A in the category C with multiplication $\nu : A \otimes E \to E$. Let $Q \in \mathbf{q}(C)$. We define a quantized multiplication ν^Q on E by requiring that the following diagram

$$\begin{tikzcd} A \otimes E \arrow[r, "Q_{A,E}"] \arrow[dr, "\nu^Q"'] & A \otimes E \arrow[d, "\nu"] \\ & E \end{tikzcd}$$

commutes.

Then (E, ν^Q) is a module over the quantized algebra (A, μ^Q). We call it the *quantized module*.

Right modules are quantized in a similar way.

Coalgebras. Let A^* be a coalgebra in the category C with comultiplication $\mu^* : A^* \to A^* \otimes A^*$. Let $Q \in \mathbf{q}^\circ(C)$. We define a new comultiplication $(\mu^*)^Q$ on A^* by requiring that the following diagram

$$\begin{tikzcd} A^* \otimes A^* \arrow[r, "Q_{A^*,A^*}"] & A^* \otimes A^* \\ & A^* \arrow[u, "\mu^*"'] \arrow[ul, "(\mu^*)^Q"] \end{tikzcd}$$

commutes.

Then $(A^*, (\mu^*)^Q)$ is a coalgebra in the category C too. We call it the *quantized coalgebra*.

Bialgebras. Let B be a bialgebra in the category C with multiplication $\mu_B : B \otimes B \to B$ and comultiplication $\mu_B^* : B \to B \otimes B$. Let $Q \in \mathbf{q}^\circ(C)$. The *quantized bialgebra* is the same object $B_Q = B$ equipped with the quantized multiplication μ_B^Q and quantized comultiplication $(\mu_B^*)^Q$, quantized as above. This is also a bialgebra in the category.

3 Quantizations of G-modules

Let R be a commutative ring with unit and let G be a finite group. Denote by $\mathrm{Mod}_R(G)$ the monoidal category of finitely generated G-modules over R and let $\mathbf{q}(G)$ and $\mathfrak{Q}(G)$ be the sets of quantizations and orbits of this category.

Let X and Y be G-modules. Recall that the tensor product $X \otimes Y$ over R is a G-module with action

$$g(x \otimes y) = g(x) \otimes g(y)$$

for $g \in G, x \in X, y \in Y$.

Let $R[G]$ be the group algebra of G over R.

There is an isomorphism between the categories of G-modules and $R[G]$-modules, $\mathrm{Mod}_R(G) = \mathrm{Mod}_R(R[G])$ (see, e.g., [2]).

Hence,

$$q(G) = \mathbf{q}(\mathrm{Mod}_R(R[G])), \quad \mathfrak{Q}(G) = \mathfrak{Q}(\mathrm{Mod}_R(R[G])).$$

Theorem 1. *Any quantization $Q \in \mathbf{q}(G)$ of the category of G-modules has the form*

$$Q_{X,Y} : x \otimes y \mapsto q_Q \cdot (x \otimes y) = \sum_{g,h \in G} q_{g,h} gx \otimes hy,$$

where

$$q_Q = \sum_{g,h \in G} q_{g,h}(g,h)$$

are elements of the group algebra $R[G \times G]$, for $x \in X$, $y \in Y$, and $X, Y \in \mathrm{Obj}(\mathrm{Mod}_R(G))$.

Proof. We identify elements $x \in X$ for $X \in \mathrm{Obj}(\mathrm{Mod}_R(G))$ with morphisms

$$\phi_x : R[G] \longrightarrow X, \quad h \longmapsto h \cdot x.$$

Then, for any elements $x \in X, y \in Y$, $X, Y \in \mathrm{Obj}(\mathrm{Mod}_R(G))$ and a quantization $Q \in \mathbf{q}(G)$, the following diagram

$$\begin{tikzcd}
X \otimes Y \arrow[r, "Q_{X,Y}"] & X \otimes Y \\
R[G] \otimes R[G] \arrow[u, "\phi_x \otimes \phi_y"] \arrow[r, "Q_{R[G],R[G]}"] & R[G] \otimes R[G] \arrow[u, "\phi_{x \otimes y}"']
\end{tikzcd}$$

commutes. Therefore,

$$\begin{tikzcd}
x \otimes y \arrow[r, "Q_{X,Y}"] & Q_{X,Y}(x \otimes y) = q_Q \cdot (x \otimes y) \\
1 \otimes 1 \arrow[u, "\phi_x \otimes \phi_y"] \arrow[r, "Q_{R[G],R[G]}"] & q_Q \arrow[u, "\phi_{x \otimes y}"']
\end{tikzcd}$$

where $q_Q = Q_{R[G],R[G]}(1 \otimes 1) \in R[G \times G]$, and

$$Q_{X,Y}(x \otimes y) = q_Q \cdot (x \otimes y). \qquad \square$$

We call the elements q_Q *quantizers* and identify $\mathbf{q}(G)$ with the set $\{q_Q \in R[G \times G]\}$.

Theorem 2. *An element $q \in R[G \times G]$ defines a quantization on the category of G-modules if and only if it satisfies the following conditions:*

(i) *the coherence condition*

$$(1 \otimes \Delta)(q) \cdot (1 \otimes q) = (\Delta \otimes 1)(q) \cdot (q \otimes 1), \tag{3.1}$$

where Δ is the diagonal in $R[G] \times R[G]$;

(ii) *the normalization condition*

$$q \cdot (x \otimes 1) = q \cdot (1 \otimes x) = 1; \tag{3.2}$$

(iii) *the naturality condition*

$$q \cdot \Delta(g) = \Delta(g) \cdot q, \tag{3.3}$$

for $g \in G$.

Proof. The coherence condition follows from (2.1) where

$$Q_{XY,Z} : (X \otimes Y) \otimes Z \longrightarrow (X \otimes Y) \otimes Z$$

is represented as follows:

$$(x \otimes y) \otimes z \longrightarrow (\Delta \otimes 1)(q)\big((x \otimes y) \otimes z\big) = \sum_{g,h \in G} q_{g,h}(gx \otimes gy) \otimes hz,$$

and similarly for $Q_{X,YZ}$.

The two other conditions follow straightforwardly from the normalization and naturality conditions on the quantizations. □

See also [4] for similar settings.

We will from now on use the notion quantizer instead of quantization.

Remark that the conditions (3.1), (3.2), and (3.3) are some kind of 2-cyclic condition on $\mathbf{q}(G)$ (see, e.g., Section 6 on finite abelian groups).

Let $U(G)$ be the set of units of $R[G]$.

Theorem 3. *The set of quantizers of G-modules $\mathfrak{Q}(G)$ is the orbit space of the following $U(G)$-action on $\mathbf{q}(G)$:*

$$l(q) \stackrel{\text{def}}{=} \Delta(l) \cdot q \cdot l^{-\otimes 2}, \tag{3.4}$$

where $l \in U(G)$.

Proof. Representing as above elements $x \in X$ by morphisms $\phi_x : R[G] \to X$, we get the following commutative diagram with $\lambda_X : X \to X$:

$$\begin{tikzcd} X \arrow[r, "\lambda_X"] & X \\ R[G] \arrow[u, "\phi_x"] \arrow[r, "\lambda_{R[G]}"] & R[G] \arrow[u, "\phi_x"'] \end{tikzcd}$$

for any unit preserving natural isomorphism of the identity functor, $\lambda : Id_{\mathrm{Mod}_R(R[G])} \to Id_{\mathrm{Mod}_R(R[G])}$.

Therefore, λ is uniquely defined by elements $l \in \lambda_{R[G]}(1)$, and

$$\lambda_X(x) = \lambda_X\big(\phi_x(1)\big) = \phi_x\big(\lambda_{R[G]}(1)\big) = \phi_x(l) = l \cdot x.$$

Let $q \in \mathbf{q}(G)$. Then the action (2.2) gives

$$\lambda(q) = \Delta(l) \cdot q \cdot \big(l^{-1} \otimes l^{-1}\big)$$

with $l \in U(G)$. □

We say that two quantizers $p, q \in \mathbf{q}(G)$ are equivalent if $p = l(q)$ for some $l \in U(G)$.

4 The Fourier transform

In this section, we will use the Fourier transform to find the quantizers, under the assumption that $R=\mathbb{C}$.

Below we list necessary formulae from representation theory of groups (see, e.g., [6]).

Denote by $\hat{G}$ the dual of G. For each $\alpha\in\hat{G}$ we pick the corresponding irreducible representation on E_α, $\dim E_\alpha=d_\alpha$, and an explicit realization of this representation by a $d_\alpha\times d_\alpha$-matrix $D^\alpha(g)=|D^\alpha_{ij}(g)|:E_\alpha\to E_\alpha$, for each $g\in G$.

The elements $D^\alpha_{ij}(g)\cdot g\in\mathbb{C}[G]$ span the group algebra $\mathbb{C}[G]$, and $\mathbb{C}[G]$ is isomorphic as an algebra to a direct sum of matrix algebras by the Fourier transform

$$F:\mathbb{C}[G]\longrightarrow\oplus_{\alpha\in\hat{G}}End(E_\alpha),$$
$$\tilde{f}=\sum_{g\in G}f(g)\cdot g\longmapsto\hat{f}=\{\hat{f}_\alpha\}_{\alpha\in\hat{G}}.$$

We will consider $\hat{f}$ as a "function" on the dual group which at each point $\alpha\in\hat{G}$ takes values in $\mathrm{End}(E_\alpha)$:

$$\hat{f}_\alpha=\hat{f}(\alpha)\in\mathrm{End}\left(E_\alpha\right)$$

and

$$F(\tilde{f})(\alpha)=\hat{f}_\alpha=\sum_{g\in G}D^\alpha(g)f(g).$$

The inverse of the Fourier transform has the following form:

$$F^{-1}(\hat{f})=\frac{1}{|G|}\sum_{g\in G}\sum_{\alpha\in\hat{G}}d_\alpha\,\mathrm{Tr}\left(D^\alpha(g)^*\hat{f}_\alpha\right)\cdot g,$$

where the * denotes the adjoint.

As we have seen the quantizers are elements of $\mathbb{C}[G\times G]$.

The dual of $G\times G$ is $\widehat{G\times G}=\hat{G}\times\hat{G}$, and

$$E_{\alpha\beta}=E_\alpha\otimes E_\beta$$

for $(\alpha,\beta)\in\hat{G}\times\hat{G}$ with the action

$$D^{\alpha,\beta}(g,h)=D^\alpha(g)\otimes D^\beta(h).$$

In this case, the Fourier transform

$$F:\mathbb{C}[G]\otimes\mathbb{C}[G]=\mathbb{C}[G\times G]\longrightarrow\oplus_{\alpha,\beta\in\hat{G}}\mathrm{End}(E_\alpha\otimes E_\beta)$$

and its inverse have the following forms:

$$F(\tilde{f})(\alpha,\beta)=\hat{f}_{\alpha,\beta}=\sum_{g,h\in G}f(g,h)D^{\alpha,\beta}(g,h), \tag{4.1}$$

$$F^{-1}(\hat{f})=\frac{1}{|G|^2}\sum_{g,h\in G}\sum_{\alpha,\beta\in\hat{G}}d_{\alpha,\beta}\,\mathrm{Tr}\left(D^{\alpha,\beta}(g,h)^*\hat{f}_{\alpha,\beta}\right)\cdot(g,h), \tag{4.2}$$

where

$$f=\sum_{g,h\in G}f(g,h)(g,h)$$

and

$$\left\{\hat{f}_{\alpha,\beta} \in \mathrm{End}\left(E_\alpha \otimes E_\beta\right)\right\}_{\alpha,\beta\in\hat{G}}.$$

Let $\chi_\alpha(g) = \mathrm{Tr}(D^\alpha(g))$ be the character of the irreducible representation $E_\alpha, \alpha \in \hat{G}$.

Splitting of the tensor product of $E_\alpha \otimes E_\beta$ into a sum of irreducible representations, we get isomorphisms

$$\nu_{\alpha,\beta} : E_\alpha \otimes E_\beta \longrightarrow \oplus_{\gamma\in\hat{G}} c^\gamma_{\alpha\beta} E_\gamma, \tag{4.3}$$

where $c^\gamma_{\alpha\beta} \in \mathbb{N}$ are the *Clebsch-Gordan integers*.

These integers can be computed as follows:

$$\chi_\alpha \cdot \chi_\beta = \sum_\gamma c^\gamma_{\alpha\beta} \chi_\gamma.$$

Projections p_α of G-modules $E = \sum_{\alpha\in\hat{G}} c^\alpha E_\alpha$ onto its irreducible components $c_\alpha E_\alpha = E_{(\alpha)}$ are the following:

$$p_\alpha = \frac{d_\alpha}{|G|} \sum_{g\in G} \chi_\alpha(g) D^\alpha\left(g^{-1}\right),$$

where $E_{(\alpha)} \simeq \mathbb{C}^{c^\alpha} \otimes E_\alpha$. They satisfy orthogonality conditions

$$\sum_{\alpha\in\hat{G}} p_\alpha = 1, \quad p_\alpha^2 = p_\alpha, \quad p_\alpha p_\beta = 0.$$

For the tensor products (4.3), the projectors take the form

$$p_\gamma = \sum_{g\in G} \chi_\gamma(g) D^\alpha\left(g^{-1}\right) \otimes D^\beta\left(g^{-1}\right).$$

The matrix representation is $D^\alpha(g) \otimes D^\beta(h) = |D^{\alpha\beta}_\gamma(g,h)|$, $g,h \in G$, where

$$\mathbb{C}^{c^\gamma_{\alpha\beta}} \otimes E_\gamma \xrightarrow{D^{\alpha\beta}_\gamma(g,h)} \mathbb{C}^{c^\gamma_{\alpha\beta}} \otimes E_\gamma.$$

5 The Fourier transform on quantizers

We now rewrite the coherence condition (2.1) for quantizers in terms of their Fourier transforms.

Let $q = \sum_{g,h\in G} q_{g,h}(g,h)$ be a quantizer and $F(q) = \hat{q} = \sum_{\alpha,\beta\in\hat{G}} \hat{q}_{\alpha,\beta}$, where $\hat{q}_{\alpha,\beta} \in \mathrm{End}(E_\alpha \otimes E_\beta)$. Then

$$\hat{q}(\alpha,\beta) = \hat{q}_{\alpha,\beta} = \sum_{g,h\in G} q(g,h) D^{\alpha,\beta}(g,h).$$

The operators $\hat{q}_{\alpha,\beta} : E_\alpha \otimes E_\beta \to E_\alpha \otimes E_\beta$ are G-morphisms.

Therefore, due to isomorphisms $\nu_{\alpha,\beta}$, each $\hat{q}_{\alpha,\beta}$ is a direct sum

$$\hat{q}_{\alpha,\beta} = \oplus_{\gamma\in\hat{G}} \hat{q}^\gamma_{\alpha,\beta} \tag{5.1}$$

of operators $\hat{q}^\gamma_{\alpha,\beta} : c^\gamma_{\alpha\beta} E_\gamma \to c^\gamma_{\alpha\beta} E_\gamma$.

Note that the operators $\hat{q}^\gamma_{\alpha,\beta}$ are given by $c^\gamma_{\alpha\beta} \times c^\gamma_{\alpha\beta}$-matrices.

Rewriting the coherence condition in terms of these operators, we get the following result.

Theorem 4. *Let q be a quantizer on the monoidal category $\mathrm{Mod}_{\mathbb{C}}(G)$. Then the coherence condition diagram (2.1) under the Fourier transform takes the following form:*

$$
\begin{tikzcd}
\sum_{\eta,\zeta} c^{\zeta}_{\alpha\eta} c^{\eta}_{\beta\gamma} E_{\zeta} \arrow[r] & \sum_{\eta,\zeta} c^{\zeta}_{\alpha\eta} c^{\eta}_{\beta\gamma} E_{\zeta} \arrow[r, "\sum_{\zeta,\eta} \hat{q}^{\zeta}_{\alpha,\eta}"] & \sum_{\eta,\zeta} c^{\zeta}_{\alpha\eta} c^{\eta}_{\beta\gamma} E_{\zeta} \\
E_{\alpha} \otimes (\sum_{\eta} c^{\eta}_{\beta\gamma} E_{\eta}) \arrow[u, "\nu_{\alpha\eta}"] \arrow[r, "1\otimes\sum_{\eta} \hat{q}^{\eta}_{\beta,\gamma}"] & E_{\alpha} \otimes (\sum_{\eta} c^{\eta}_{\beta\gamma} E_{\eta}) \arrow[u, "\nu_{\alpha\eta}"] \arrow[r, "\sum_{\eta} \hat{q}_{\alpha,\eta}"] & E_{\alpha} \otimes (\sum_{\eta} c^{\eta}_{\beta\gamma} E_{\eta}) \arrow[u, "\nu_{\alpha\eta}"] \\
E_{\alpha} \otimes (E_{\beta} \otimes E_{\gamma}) \arrow[u, "\nu_{\beta\gamma}"] \arrow[r, "1\otimes \hat{q}_{\beta,\gamma}"] \arrow[d, "\mathrm{assoc}"] & E_{\alpha} \otimes (E_{\beta} \otimes E_{\gamma}) \arrow[u, "\nu_{\beta\gamma}"] \arrow[r, "\hat{q}_{\alpha,\beta\gamma}"] & E_{\alpha} \otimes (E_{\beta} \otimes E_{\gamma}) \arrow[u, "\nu_{\beta\gamma}"] \arrow[d, "\mathrm{assoc}"] \\
(E_{\alpha} \otimes E_{\beta}) \otimes E_{\gamma} \arrow[r, "\hat{q}_{\alpha,\beta}\otimes 1"] \arrow[d, "\nu_{\alpha\beta}"] & (E_{\alpha} \otimes E_{\beta}) \otimes E_{\gamma} \arrow[r, "\hat{q}_{\alpha\beta,\gamma}"] \arrow[d, "\nu_{\alpha\beta}"] & (E_{\alpha} \otimes E_{\beta}) \otimes E_{\gamma} \arrow[d, "\nu_{\alpha\beta}"] \\
(\sum_{\xi} c^{\xi}_{\alpha\beta} E_{\xi}) \otimes E_{\gamma} \arrow[r, "\sum_{\xi} \hat{q}^{\xi}_{\alpha,\beta}\otimes 1"] \arrow[d, "\nu_{\xi\gamma}"] & (\sum_{\xi} c^{\xi}_{\alpha\beta} E_{\xi}) \otimes E_{\gamma} \arrow[r, "\sum_{\xi} \hat{q}_{\xi,\gamma}"] \arrow[d, "\nu_{\xi\gamma}"] & (\sum_{\xi} c^{\xi}_{\alpha\beta} E_{\xi}) \otimes E_{\gamma} \arrow[d, "\nu_{\xi\gamma}"] \\
\sum_{\xi,\zeta} c^{\zeta}_{\xi\gamma} c^{\xi}_{\alpha\beta} E_{\zeta} \arrow[r] & \sum_{\xi,\zeta} c^{\zeta}_{\xi\gamma} c^{\xi}_{\alpha\beta} E_{\zeta} \arrow[r, "\sum_{\zeta,\xi} \hat{q}^{\zeta}_{\xi,\gamma}"] & \sum_{\xi,\zeta} c^{\zeta}_{\xi\gamma} c^{\xi}_{\alpha\beta} E_{\zeta}
\end{tikzcd}
$$

where $\hat{q}_{\alpha,\beta\gamma} \in \mathrm{End}(E_{\alpha} \otimes (E_{\beta} \otimes E_{\gamma}))$ and $\hat{q}_{\alpha\beta,\gamma} \in \mathrm{End}((E_{\alpha} \otimes E_{\beta}) \otimes E_{\gamma})$.

Assuming that our category is strict, we get the following conditions for the quantizers.

Theorem 5. *The set of operators $\hat{q}^{\gamma}_{\alpha,\beta} \in \mathrm{End}(c^{\gamma}_{\alpha\beta} E_{\gamma}, c^{\gamma}_{\alpha\beta} E_{\gamma})$ defines a quantizer*

$$q = \frac{1}{|G|^2} \sum_{g,h\in G} \sum_{\alpha,\beta\in\hat{G}} d_{\alpha,\beta} \oplus_{\gamma\in\hat{G}} \mathrm{Tr}(D^{\alpha,\beta}_{\gamma}(g,h)^* \hat{q}^{\gamma}_{\alpha,\beta}) \cdot (g,h) \in \mathbb{C}[G \times G]$$

if and only if these operators are solutions of the following system of quadratic equations:

$$\sum_{\eta,\zeta\in\hat{G}} \hat{q}^{\zeta}_{\alpha,\eta} \hat{q}^{\eta}_{\beta,\gamma} = \sum_{\zeta,\xi\in\hat{G}} \hat{q}^{\zeta}_{\xi,\gamma} \hat{q}^{\xi}_{\alpha,\beta}, \quad \hat{q}^{\alpha}_{\alpha,0} = \hat{q}^{\alpha}_{0,\alpha} = 1$$

for all $\alpha, \beta, \gamma \in \hat{G}$.

Proof. The first condition follows from Theorem 4.

The second condition is the normalization condition, where $\hat{q}_{0,\alpha} = \hat{q}^{\alpha}_{0,\alpha} = 1 : 1 \otimes E_{\alpha} = E_{\alpha} \to 1 \otimes E_{\alpha} = E_{\alpha}$ and $\hat{q}_{\alpha,0} = \hat{q}^{\alpha}_{\alpha,0} = 1 : E_{\alpha} \otimes 1 = E_{\alpha} \to E_{\alpha} \otimes 1 = E_{\alpha}$ □

Let $F(l) = \sum_{\alpha\in\hat{G}} \hat{l}_{\alpha}$ be the Fourier transform of $l \in U(G)$, where $\hat{l}_{\alpha} \in \mathbb{C}^*$ due to the Shur lemma, and $l_0 = 1$. Then action (3.4) can be rewritten as follows:

$$l(\hat{q}_{\alpha,\beta}) = \oplus_{\gamma\in\hat{G}} \hat{l}_{\gamma} \hat{l}^{-1}_{\alpha} \hat{l}^{-1}_{\beta} \hat{q}_{\alpha,\beta}, \tag{5.2}$$

where $\hat{l}_{\alpha}, \hat{l}_{\beta}, \hat{l}_{\gamma} \in \mathbb{C}^*$.

6 Finite abelian groups

Let G be a finite abelian group and $R = \mathbb{C}$.

In [1,2], we investigated regular quantizations of modules with action and coaction by finite abelian groups. In this section, we will revisit this case by using the Fourier transform.

By theorem 1 the quantizations of G-modules have the form $x \otimes y \mapsto q \cdot (x \otimes y)$ for elements $x \in X, y \in Y$ in G-modules X and Y where $q = \sum_{g,h\in G} q_{g,h}(g,h) \in \mathbb{C}[G \times G]$.

Let $\hat{G}$ be the dual of G. All irreducible representations of G are 1-dimensional and identified with characters $\alpha \in Hom(G, \mathbb{C}^*) = \hat{G}$.

The Fourier transform has the form

$$F(f)(\alpha) = \hat{f}_\alpha = \sum_{g \in G} f(g)\alpha(g^{-1})$$

for $f = \sum_{g \in G} f(g)g \in \mathbb{C}[G]$. The inverse of this Fourier transform is

$$F^{-1}(\hat{f}_\alpha) = \frac{1}{|G|} \sum_{g \in G} \hat{f}_\alpha \alpha(g)g.$$

Then

$$F(q)(\alpha, \beta) = \hat{q}_{\alpha,\beta} = \sum_{g,h \in G} q_{g,h} \alpha(g^{-1})\beta(h^{-1})$$

is the operator

$$\hat{q}_{\alpha,\beta} : E_\alpha \otimes E_\beta \to E_\alpha \otimes E_\beta,$$

where $\alpha, \beta \in \hat{G}$.

Clearly $\hat{q}_{\alpha,\beta} \in \mathbb{C}^*$.

Corresponding to Theorem 5 we thus have the following conditions on $\hat{q}_{\alpha,\beta}$:

$$\hat{q}_{\alpha\cdot\beta,\gamma}\hat{q}_{\alpha,\beta} = \hat{q}_{\alpha,\beta\cdot\gamma}\hat{q}_{\beta,\gamma}, \quad \hat{q}_{0,\alpha} = \hat{q}_{\alpha,0} = 1$$

for all $\alpha, \beta, \gamma \in \hat{G}$, where the first condition is given by the coherence condition and the second is the normalization condition.

Denote by $\mathbf{q}(\hat{G})$ the group of all functions satisfying these conditions. We see that they are 2-cocycles.

Hence $\mathbf{q}(\hat{G})$ is represented by the multiplicative 2-cocycles $\hat{q}$ on $\hat{G}$ with coefficients in $\mathbb{C}^*$, where

$$\hat{q}(\alpha, \beta) = \hat{q}_{\alpha,\beta}.$$

The $\mathbb{C}^*$-action (5.2) on the operators $\hat{q}_{\alpha,\beta}$ has the form

$$\hat{l}_\alpha^{-1}\hat{l}_\beta^{-1}\hat{q}_{\alpha,\beta}\hat{l}_{\alpha\cdot\beta},$$

where $\hat{l}_\alpha, \hat{l}_\beta, \hat{l}_{\alpha\cdot\beta} \in \mathbb{C}^*$.

Summing up, we get the following result.

Theorem 6. *Let G be a finite abelian group. Then the group of regular quantizations $\mathfrak{Q}^\circ(\hat{G})$ is isomorphic to the 2nd multiplicative cohomology group $H^2(\hat{G}, \mathbb{C}^*)$.*

Moreover, any 2-cocycle $z \in Z^2(\hat{G}, \mathbb{C}^*)$ defines a quantizer q_z in the following way:

$$q_z = \frac{1}{|G|^2} \sum_{g,h \in G} \sum_{\alpha,\beta \in \hat{G}} z(\alpha, \beta)\alpha(g)\beta(h) \cdot (g, h) \in \mathbb{C}[G \times G].$$

7 Quantizations of S_3-modules

We consider the symmetric group $G = S_3$. Let the representatives of the orbits of the adjoint action be $()$, $(1,2)$, and $(1,2,3)$ and let χ_0, χ_1, and χ_2 be the characters of the irreducible representations corresponding to these orbits. These irreducible representations are the trivial, sign, and standard representations on modules E_0, E_1, and E_2 with matrix realizations D^0, D^1, and D^2, respectively.

Theorem 7. *For S_3-modules over $\mathbb{C}$ the set of quantizers $\mathfrak{Q}(S_3)$ consists of the following:*

(i) *the trivial quantizer $q = 1$;*

(ii) *the 1-parameter family of quantizers*

$$q_{(\mathrm{a})}=1+p\sum_{g,h\in S_3}\mathrm{Tr}\big(D_0^{2,2}(g,h)^*\big)+\mathrm{Tr}\big(D_1^{2,2}(g,h)^*\big)\cdot(g,h),$$

where $p\in\mathbb{C}$*;*

(iii) *the discrete set of discrete quantizers*

$$q_{(\mathrm{b})}=1+\sum_{g,h\in S_3}\mathrm{Tr}\left(D_1^{2,2}(g,h)^*\right)+\mathrm{Tr}\left(D_2^{2,2}(g,h)^*\right)\cdot(g,h),\quad q_{(\mathrm{c})}=1+\sum_{g,h\in S_3}\mathrm{Tr}\left(D_2^{2,2}(g,h)^*\right)\cdot(g,h),$$

$$q_{(\mathrm{d})}=1+\sum_{g,h\in S_3}\mathrm{Tr}\left(D_1^{2,2}(g,h)^*\right)\cdot(g,h),\quad q_{(\mathrm{e})}=1+\sum_{g,h\in S_3}\mathrm{Tr}\left(D_0^{2,2}(g,h)^*\right)+\mathrm{Tr}\left(D_1^{2,2}(g,h)^*\right)\cdot(g,h).$$

The operators $D_i^{2,2}:E_i\to E_i, i=0,1,2$*, are the components of* $D^{2,2}$ *corresponding to the decomposition of the tensor product* $E_2\otimes E_2=E_0\oplus E_1\oplus E_2$*.*

Proof. The multiplication table for the characters of S_3 is

$\cdot$	χ_0	χ_1	χ_2
χ_0	χ_0	χ_1	χ_2
χ_1	χ_1	χ_0	χ_2
χ_2	χ_2	χ_2	$\chi_0+\chi_1+\chi_2$

,

and by (4.3) we get the multiplication table for irreducible representations

$\otimes$	E_0	E_1	E_2
E_0	E_0	E_1	E_2
E_1	E_1	E_0	E_2
E_2	E_2	E_2	$E_0\oplus E_1\oplus E_2$

,

where the irreducible representations E_0, E_1, and E_2 are 1, 1, and 2 dimensional, respectively.

By (5.1) the quantizers $\hat{q}_{ij}$ in $\mathrm{End}(E_i\otimes E_j)$ are decomposed as follows:

$$\hat{q}_{11}=\hat{q}_{11}^0,\quad \hat{q}_{12}=\hat{q}_{12}^2,\quad \hat{q}_{21}=\hat{q}_{21}^2,\quad \hat{q}_{22}=\hat{q}_{22}^0\oplus\hat{q}_{22}^1\oplus\hat{q}_{22}^2.$$

By normalization condition

$$\hat{q}_{00}=\hat{q}_{01}=\hat{q}_{10}=\hat{q}_{02}=\hat{q}_{20}=1.$$

Theorem 5 for triple tensor products of all combinations of E_0,E_1,E_2 gives the following relations (see the appendix for the details of the calculations):

$$\hat{q}_{12}=\hat{q}_{21},\tag{7.1}$$

$$\hat{q}_{11}=\left(\hat{q}_{12}\right)^2,\tag{7.2}$$

$$\hat{q}_{22}^0=\hat{q}_{12}\hat{q}_{22}^1.\tag{7.3}$$

The action of the group $U(S_3)$ has the following form:

$$\hat{q}_{11}\longrightarrow\frac{\hat{l}_0}{(\hat{l}_1)^2}\cdot\hat{q}_{11}=\frac{1}{(\hat{l}_1)^2}\cdot\hat{q}_{11},\quad \hat{q}_{12}=\hat{q}_{21}\longrightarrow\frac{\hat{l}_2}{\hat{l}_1\hat{l}_2}\cdot\hat{q}_{12}=\frac{1}{\hat{l}_1}\cdot\hat{q}_{12},$$

$$\hat{q}_{22}^0\longrightarrow\frac{\hat{l}_0}{(\hat{l}_2)^2}\cdot\hat{q}_{22}^0=\frac{1}{(\hat{l}_2)^2}\cdot\hat{q}_{22}^0,\quad \hat{q}_{22}^1\longrightarrow\frac{\hat{l}_1}{(\hat{l}_2)^2}\cdot\hat{q}_{22}^1,\quad \hat{q}_{22}^2\longrightarrow\frac{\hat{l}_2}{(\hat{l}_2)^2}\cdot\hat{q}_{22}^2=\frac{1}{\hat{l}_2}\cdot\hat{q}_{22}^2,$$

where $\hat{l}_0=1$ and $\hat{l}_1,\hat{l}_2\in\mathbb{C}^*$.

If the quantizers all are nonzero, we may choose $\hat{l}_1,\hat{l}_2$ such that $\hat{q}_{12},\hat{q}_{22}^2\to 1$, by (7.1), (7.2) then also $\hat{q}_{11},\hat{q}_{21}\to 1$ and by (7.3) $\hat{q}_{22}^0=\hat{q}_{22}^1$. We then have the following sequence of quantizers depending on one parameter $\lambda\in\mathbb{C}$:

	$\hat{q}_{11}$	$\hat{q}_{12}$	$\hat{q}_{21}$	$\hat{q}_{22}^0$	$\hat{q}_{22}^1$	$\hat{q}_{22}^2$
(a)	1	1	1	λ	λ	1

.

Equivalently, the representatives can be chosen as follows:

$$\begin{array}{c|c|c|c|c|c|c|}
 & \hat{q}_{11} & \hat{q}_{12} & \hat{q}_{21} & \hat{q}_{22}^{0} & \hat{q}_{22}^{1} & \hat{q}_{22}^{2} \\ \hline
(\mathrm{a}') & \lambda^2 & \lambda & \lambda & \lambda & 1 & 1 \\ \hline
(\mathrm{a}'') & \lambda^{-2} & \lambda^{-1} & \lambda^{-1} & 1 & \lambda & 1 \\ \hline
(\mathrm{a}''') & 1 & 1 & 1 & 1 & 1 & \lambda \\ \hline
\end{array}. \tag{7.4}$$

If one or both of the quantizers $\hat{q}_{12}, \hat{q}_{22}^2$ are equal to zero, then the rest will either be equal to 0 or map to 1 by choosing l_1, l_2 appropriately.

By the conditions (7.1)–(7.3), the quantizers vary as follows:

$$\begin{array}{|c|c|c|c|c|c|c|}
 & \hat{q}_{11} & \hat{q}_{12} & \hat{q}_{21} & \hat{q}_{22}^{0} & \hat{q}_{22}^{1} & \hat{q}_{22}^{2} \\ \hline
(\mathrm{b}) & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline
(\mathrm{c}) & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
(\mathrm{d}) & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline
(\mathrm{e}) & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
(\mathrm{f}) & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
(\mathrm{g}) & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline
\end{array}.$$

We now apply the inverse Fourier transform (4.2) to the quantizers (a)–(g) and get the corresponding element q in the group algebra,

$$\begin{aligned}
q &= \frac{1}{|S_3|^2} \sum_{g,h\in S_3} F^{-1}(\hat{q})(g,h) = \frac{1}{|S_3|^2} \sum_{g,h\in S_3} \sum_{\chi_i,\chi_j\in \hat{S}_3} d_{i,j} \operatorname{Tr}\left(D^{i,j}(g,h)^* \hat{q}_{ij}\right) \cdot (g,h) \\
&= \frac{1}{|S_3|^2} \sum_{g,h\in S_3} \left(1 + \operatorname{sign}(h) + \operatorname{sign}(g) + \hat{q}_{11} \operatorname{sign}(g)\operatorname{sign}(h)\right) \cdot (g,h) \\
&\quad + \frac{2}{|S_3|^2} \sum_{g,h\in S_3} \left(\operatorname{Tr}(D^2(h)^*) + \operatorname{Tr}\left(D^2(g)^*\right)\right) \cdot (g,h) \\
&\quad + \frac{2}{|S_3|^2} \sum_{g,h\in S_3} \left(\hat{q}_{12} \operatorname{sign}(g) \operatorname{Tr}\left(D^2(h)^*\right) + \hat{q}_{21} \operatorname{Tr}(D^2(g)^*) \operatorname{sign}(h)\right) \cdot (g,h) \\
&\quad + \frac{4}{|S_3|^2} \sum_{g,h\in S_3} \operatorname{Tr}\left(D^{2,2}(g,h)^* \hat{q}_{22}\right) \cdot (g,h) \\
&= 1 + \frac{4}{|S_3|^2} \sum_{g,h\in S_3} \operatorname{Tr}\left(D_0^{2,2}(g,h)^* \hat{q}_{22}^0\right) + \operatorname{Tr}\left(D_1^{2,2}(g,h)^* \hat{q}_{22}^1\right) + \operatorname{Tr}\left(D_2^{2,2}(g,h)^* \hat{q}_{22}^2\right) \cdot (g,h),
\end{aligned}$$

where $\operatorname{Tr}(D^{i,j}(g,h)^* \hat{q}_{ij}) = \hat{q}_{ij} \operatorname{Tr}(D^i(g)^* \otimes D^j(h)^*) = \hat{q}_{ij} \operatorname{Tr}(D^i(g)^*) \operatorname{Tr}(D^j(h)^*)$. Since $\operatorname{Tr}(\text{trivial}(g)^*)$ is 1 for all $g \in S_3$; $\operatorname{Tr}(\operatorname{sign}(g)^*)$ is 1 for (), (1,2,3), and (1,3,2) and -1 for (1,2), (1,3), and (2,3); and $\operatorname{Tr}(D^2(g)^*)$ is 2 for (), -1 for (1,2,3) and (1,3,2) and 0 for (1,2), (1,3), and (2,3), most of the sum cancels out.

Then for option (a) we let $\lambda = 1 + \frac{|S_3|^2}{4} p$ for $p \in \mathbb{C}$ and get the following class of quantizers:

$$q_{(\mathrm{a})} = 1 + p \sum_{g,h\in S_3} \operatorname{Tr}\left(D_0^{2,2}(g,h)^*\right) + \operatorname{Tr}\left(D_1^{2,2}(g,h)^*\right) \cdot (g,h).$$

In addition to this case, the combinations (b)-(e) give the following quantizers:

$$q_{(\mathrm{b})} = 1 + \sum_{g,h\in S_3} \operatorname{Tr}\left(D_1^{2,2}(g,h)^*\right) + \operatorname{Tr}\left(D_2^{2,2}(g,h)^*\right) \cdot (g,h), \quad q_{(\mathrm{c})} = 1 + \sum_{g,h\in S_3} \operatorname{Tr}\left(D_2^{2,2}(g,h)^*\right) \cdot (g,h),$$

$$q_{(\mathrm{d})} = 1 + \sum_{g,h\in S_3} \operatorname{Tr}\left(D_1^{2,2}(g,h)^*\right) \cdot (g,h), \quad q_{(\mathrm{e})} = 1 + \sum_{g,h\in S_3} \operatorname{Tr}\left(D_0^{2,2}(g,h)^*\right) + \operatorname{Tr}\left(D_1^{2,2}(g,h)^*\right) \cdot (g,h).$$

The combinations (f) and (g) give the trivial quantizer. $\square$

Remark that from (7.4) the quantizer $q_{(\mathrm{a})}$ has the following equivalent forms:

$$q_{(\mathrm{a}')} = 1 + p \sum_{g,h\in S_3} \mathrm{Tr}\left(D_0^{2,2}(g,h)^*\right)\cdot(g,h), \quad q_{(\mathrm{a}'')} = 1 + p \sum_{g,h\in S_3} \mathrm{Tr}\left(D_1^{2,2}(g,h)^*\right)\cdot(g,h),$$
$$q_{(\mathrm{a}''')} = 1 + p \sum_{g,h\in S_3} \mathrm{Tr}\left(D_2^{2,2}(g,h)^*\right)\cdot(g,h).$$

8 Quantizations of A_4

Let $G = A_4$ be the alternating group. Elements $(\,)$, $(12)(34)$, (123), (132) represent the orbits of the adjoint G-action and we let χ_0, χ_1, χ_2, and χ_3 be the characters of the irreducible representations corresponding to these orbits.

These irreducible representations are the trivial representation, the first and second nontrivial one-dimensional representations, and the three-dimensional irreducible representation on modules E_0, E_1, E_2, and E_3 with matrix realizations D^0, D^1, D^2, and D^3, respectively.

Theorem 8. *For A_4-modules, the set of quantizers $\mathbb{Q}(A_4)$ consists of the following:*

(i) *the trivial quantizer $q = 1$,*
(ii) $q_{(\mathrm{a})} = 1 + \sum_{g,h\in A_4} \mathrm{Tr}(D_3^{3,3}(g,h)^* M)\cdot(g,h)$,

where $M = \left[\begin{smallmatrix}\lambda & 0\\ 0 & \kappa\end{smallmatrix}\right], \left[\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right]$, $\lambda,\kappa\in\mathbb{C}$,

(iii) $q_{(\mathrm{b})} = 1 + \sum_{g,h\in A_4} \mathrm{Tr}(D_3^{3,3}(g,h)^* P)\cdot(g,h)$,
(iv) $q_{(\mathrm{c})} = 1 + \sum_{g,h\in A_4} (\mathrm{Tr}(D_1^{3,3}(g,h)^*) + \mathrm{Tr}(D_3^{3,3}(g,h)^* P))\cdot(g,h)$,
(v) $q_{(\mathrm{d})} = 1 + \sum_{g,h\in A_4} (\mathrm{Tr}(D_2^{3,3}(g,h)^*) + \mathrm{Tr}(D_3^{3,3}(g,h)^* P))\cdot(g,h)$,
(vi) $q_{(\mathrm{e})} = 1 + \sum_{g,h\in A_4} (\mathrm{Tr}(D_1^{3,3}(g,h)^*) + \mathrm{Tr}(D_2^{3,3}(g,h)^*) + \mathrm{Tr}(D_3^{3,3}(g,h)^* P))\cdot(g,h)$,

where P are 2×2-matrices of the form $\left[\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1 & 0\\ 0 & \lambda\end{smallmatrix}\right]$, $\lambda\in\mathbb{C}$.

The operators $D_i^{3,3}$, $i = 0,1,2,3$, are components of $D^{3,3}$ corresponding to the decomposition $E_3\otimes E_3 = E_0\oplus E_1\oplus E_2\oplus 2E_3$.

Proof. The multiplication table for the characters of A_4 has the form

$\cdot$	χ_0	χ_1	χ_2	χ_3
χ_0	χ_0	χ_1	χ_2	χ_3
χ_1	χ_1	χ_2	χ_0	χ_3
χ_2	χ_2	χ_0	χ_1	χ_3
χ_3	χ_3	χ_3	χ_3	$\chi_0+\chi_1+\chi_2+2\chi_3$

and by (4.3) we get the multiplication table for irreducible representations

$\otimes$	E_0	E_1	E_2	E_3
E_0	E_0	E_1	E_2	E_3
E_1	E_1	E_2	E_0	E_3
E_2	E_2	E_0	E_1	E_3
E_3	E_3	E_3	E_3	$E_0\oplus E_1\oplus E_2\oplus 2E_3$

.

Recall that the irreducible representations E_0, E_1, E_2, E_3 are 1, 1, 1, and 3 dimensional, respectively.

By (5.1), the quantizers $\hat{q}_{ij}$ in $\mathrm{End}(E_i\otimes E_j)$ split as follows:

$$\hat{q}_{11} = \hat{q}_{11}^2, \quad \hat{q}_{12} = \hat{q}_{12}^0, \quad \hat{q}_{21} = \hat{q}_{21}^0, \quad \hat{q}_{13} = \hat{q}_{13}^3, \quad \hat{q}_{31} = \hat{q}_{31}^3,$$
$$\hat{q}_{22} = \hat{q}_{22}^1, \quad \hat{q}_{23} = \hat{q}_{23}^3, \quad \hat{q}_{32} = \hat{q}_{23}^3, \quad \hat{q}_{33} = \hat{q}_{33}^0 \oplus \hat{q}_{33}^1 \oplus \hat{q}_{33}^2 \oplus \left[\hat{q}_{33}^3\right],$$

where we use the notation $[\hat{q}_{33}^3]$ to keep in mind that this is a 2×2-matrix acting on $2E_3$.

Further, by normalization condition

$$\hat{q}_{00} = \hat{q}_{01} = \hat{q}_{10} = \hat{q}_{02} = \hat{q}_{20} = \hat{q}_{03} = \hat{q}_{30} = 1.$$

Theorem 5 for triple tensor products of all combinations of E_0, E_1, E_2, E_3 give the following relations (see the appendix for details of the calculations)

$$\begin{aligned} &\hat{q}_{12} = \hat{q}_{21} = \hat{q}_{11}\hat{q}_{22}, \quad \hat{q}_{13} = \hat{q}_{31}, \quad \hat{q}_{23} = \hat{q}_{32}, \quad (\hat{q}_{13})^2 = \hat{q}_{11}\hat{q}_{23}, \quad (\hat{q}_{23})^2 = \hat{q}_{22}\hat{q}_{13}, \\ &\hat{q}_{33}^0 = \hat{q}_{13}\hat{q}_{33}^1 = \hat{q}_{23}\hat{q}_{33}^2, \quad \hat{q}_{33}^0\hat{q}_{23} = \hat{q}_{33}^1\hat{q}_{12}, \quad \hat{q}_{33}^0\hat{q}_{13} = \hat{q}_{33}^2\hat{q}_{12}, \quad \hat{q}_{33}^1\hat{q}_{11} = \hat{q}_{33}^2\hat{q}_{13}, \quad \hat{q}_{33}^1\hat{q}_{23} = \hat{q}_{33}^2\hat{q}_{22}. \end{aligned} \tag{8.1}$$

Moreover, the action of the group $U(A_4)$ has the form

$$\begin{aligned} &\hat{q}_{11} \longrightarrow \frac{\hat{l}_2}{(\hat{l}_1)^2}\hat{q}_{11}, \quad \hat{q}_{12} = \hat{q}_{21} \longrightarrow \frac{\hat{l}_0}{\hat{l}_1\hat{l}_2}\hat{q}_{21} = \frac{1}{\hat{l}_1\hat{l}_2}\hat{q}_{21}, \quad \hat{q}_{22} \longrightarrow \frac{\hat{l}_1}{(\hat{l}_2)^2}\hat{q}_{22}, \\ &\hat{q}_{13} = \hat{q}_{31} \longrightarrow \frac{\hat{l}_3}{\hat{l}_1\hat{l}_3}\hat{q}_{31} = \frac{1}{\hat{l}_1}\hat{q}_{31}, \quad \hat{q}_{23} = \hat{q}_{32} \longrightarrow \frac{\hat{l}_3}{\hat{l}_3\hat{l}_2}\hat{q}_{32} = \frac{1}{\hat{l}_2}\hat{q}_{32}, \\ &\hat{q}_{33}^0 \longrightarrow \frac{\hat{l}_0}{(\hat{l}_3)^2}\hat{q}_{33}^0 = \frac{1}{(\hat{l}_3)^2}\hat{q}_{33}^0, \quad \hat{q}_{33}^1 \longrightarrow \frac{\hat{l}_1}{(\hat{l}_3)^2}\hat{q}_{33}^1, \\ &\hat{q}_{33}^2 \longrightarrow \frac{\hat{l}_2}{(\hat{l}_3)^2}\hat{q}_{33}^2, \quad [\hat{q}_{33}^3] \longrightarrow \frac{1}{\hat{l}_3}[\hat{q}_{33}^3]. \end{aligned}$$

where $\hat{l}_0 = 1$ and $\hat{l}_1, \hat{l}_2, \hat{l}_3 \in \mathbb{C}^*$.

Assume that the quantizers are non-zero. Then we may choose $\hat{l}_1$, $\hat{l}_2$, $\hat{l}_3$ in such a way that all

$$\hat{q}_{11}, \hat{q}_{22}, \hat{q}_{12}, \hat{q}_{21}, \hat{q}_{13}, \hat{q}_{31}, \hat{q}_{23}, \hat{q}_{32}, \hat{q}_{33}^0, \hat{q}_{33}^1, \hat{q}_{33}^2$$

are equal to 1.

What remains is the 2×2-matrix $[\hat{q}_{33}^3]$ which by choosing of basis can be transformed to one of the following forms:

$$[q_{33}^3] = M = \begin{bmatrix} \lambda & 0 \\ 0 & \kappa \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

where $\lambda, \kappa \in \mathbb{C}$.

Hence we have the sequence

	$\hat{q}_{11}$	$\hat{q}_{22}$	$\hat{q}_{12}$	$\hat{q}_{21}$	$\hat{q}_{13}$	$\hat{q}_{31}$	$\hat{q}_{23}$	$\hat{q}_{32}$	$\hat{q}_{33}^0$	$\hat{q}_{33}^1$	$\hat{q}_{33}^2$	$[\hat{q}_{33}^3]$
(a)	1	1	1	1	1	1	1	1	1	1	1	M

Assume now one or more of the quantizers $\hat{q}_{11}$, $\hat{q}_{22}$, $\hat{q}_{12}$, $\hat{q}_{21}$, $\hat{q}_{13}$, $\hat{q}_{31}$, $\hat{q}_{23}$, $\hat{q}_{32}$, $\hat{q}_{33}^0$, $\hat{q}_{33}^1$, $\hat{q}_{33}^2$ may be equal to zero. The rest will then map to 1 by choosing $\hat{l}_1$, $\hat{l}_2$ properly.

By choosing $\hat{l}_3$ we reduce the matrix M to one of the following matrices P:

$$[\hat{q}_{33}^3] \longrightarrow P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}.$$

Finally by the conditions, (8.1) give the following possible sequences:

	$\hat{q}_{11}$	$\hat{q}_{22}$	$\hat{q}_{12}$	$\hat{q}_{21}$	$\hat{q}_{13}$	$\hat{q}_{31}$	$\hat{q}_{23}$	$\hat{q}_{32}$	$\hat{q}_{33}^0$	$\hat{q}_{33}^1$	$\hat{q}_{33}^2$	$[\hat{q}_{33}^3]$
(b)	1	1	1	1	1	1	1	1	0	0	0	P
(c)	0	0	0	0	0	0	0	0	0	1	0	P
(d)	0	0	0	0	0	0	0	0	0	0	1	P
(e)	0	0	0	0	0	0	0	0	0	1	1	P
(f)	0	0	0	0	0	0	0	0	0	0	0	P
(g)	0	1	0	0	0	0	0	0	0	1	0	P
(h)	0	1	0	0	0	0	0	0	0	0	0	P
(i)	1	0	0	0	0	0	0	0	0	0	1	P
(j)	1	0	0	0	0	0	0	0	0	0	0	P
(k)	1	1	1	1	0	0	0	0	0	0	0	P

Applying the inverse Fourier transform (4.2) to the sequences, we get the corresponding element in the group algebra,

$$\begin{aligned} q &= \sum_{g,h\in A_4} F^{-1}(\hat{q})(g,h) = \sum_{g,h\in A_4} \sum_{\chi_i,\chi_j\in\hat{A}_4} d_{i,j}\,\mathrm{Tr}\left(D^{i,j}(g,h)^*\hat{q}_{ij}\right)\cdot(g,h) \\ &= 1 + \frac{9}{|A_4|^2}\sum_{g,h\in A_4}\left(\mathrm{Tr}\left(D_0^{3,3}(g,h)^*\hat{q}_{33}^0\right) + \mathrm{Tr}\left(D_1^{3,3}(g,h)^*\hat{q}_{33}^1\right)\right)\cdot(g,h) \\ &\quad + \frac{9}{|A_4|^2}\sum_{g,h\in A_4}\left(\mathrm{Tr}\left(D_2^{3,3}(g,h)^*\hat{q}_{33}^2\right) + \mathrm{Tr}\left(D_3^{3,3}(g,h)^*\left[\hat{q}_{33}^3\right]\right)\right)\cdot(g,h), \end{aligned}$$

where, as mentioned, most of the sum cancels out.

Here $D_i^{3,3}$, $i=0,1,2,3$, are components of $D^{3,3}$ operating on the decomposition $E_3\otimes E_3 = E_0\oplus E_1\oplus E_2\oplus 2E_3$.

Writing $M+I$ instead of M in the sequence (a) we get a shorter form for q:

$$q_{(\mathrm{a})} = 1 + \frac{9}{|A_4|^2}\sum_{g,h\in A_4}\mathrm{Tr}\left(D_3^{3,3}(g,h)^*M\right)\cdot(g,h).$$

Further we get

$$\begin{aligned} q_{(\mathrm{b})} &= 1 + \frac{9}{|A_4|^2}\sum_{g,h\in A_4}\mathrm{Tr}\left(D_3^{3,3}(g,h)^*P\right)\cdot(g,h), \\ q_{(\mathrm{c})} &= 1 + \frac{9}{|A_4|^2}\sum_{g,h\in A_4}\left(\mathrm{Tr}(D_1^{3,3}(g,h)^*) + \mathrm{Tr}\left(D_3^{3,3}(g,h)^*P\right)\right)\cdot(g,h), \\ q_{(\mathrm{d})} &= 1 + \frac{9}{|A_4|^2}\sum_{g,h\in A_4}\left(\mathrm{Tr}(D_2^{3,3}(g,h)^*) + \mathrm{Tr}\left(D_3^{3,3}(g,h)^*P\right)\right)\cdot(g,h), \\ q_{(\mathrm{e})} &= 1 + \frac{9}{|A_4|^2}\sum_{g,h\in A_4}\left(\mathrm{Tr}(D_1^{3,3}(g,h)^*) + \mathrm{Tr}\left(D_2^{3,3}(g,h)^*\right) + \mathrm{Tr}\left(D_3^{3,3}(g,h)^*P\right)\right)\cdot(g,h). \end{aligned}$$

The combinations (f), (h), (j), and (k) give the same quantizer as (b), the combinations (g) and (c), (i) and (d) give the same quantizers.

We adjust the constants and have the result of the theorem. □

Appendix

Let $G = S_3$. By Theorem 5 the quantizers on tensor products of triples of irreducible representations satisfy

$$E_1\otimes E_1\otimes E_2 : \hat{q}_{11}\hat{q}_{02}E_2 = \hat{q}_{12}\hat{q}_{12}E_2, \tag{A.1a}$$

$$E_1\otimes E_2\otimes E_1 : \hat{q}_{12}\hat{q}_{21}E_2 = \hat{q}_{21}\hat{q}_{12}E_2, \tag{A.1b}$$

$$E_2\otimes E_1\otimes E_1 : \hat{q}_{21}\hat{q}_{21}E_2 = \hat{q}_{11}\hat{q}_{20}E_2, \tag{A.1c}$$

$$E_1\otimes E_2\otimes E_2 : \hat{q}_{12}\hat{q}_{22}^0E_0\oplus\hat{q}_{12}\hat{q}_{22}^1E_1\oplus\hat{q}_{12}\hat{q}_{22}^2E_2 = \hat{q}_{22}^1\hat{q}_{11}E_0\oplus\hat{q}_{22}^0\hat{q}_{10}E_1\oplus\hat{q}_{22}^2\hat{q}_{12}E_2, \tag{A.1d}$$

$$E_2\otimes E_1\otimes E_2 : \hat{q}_{21}\hat{q}_{22}^0E_0\oplus\hat{q}_{21}\hat{q}_{22}^1E_1\oplus\hat{q}_{21}\hat{q}_{22}^2E_2 = \hat{q}_{12}\hat{q}_{22}^0E_0\oplus\hat{q}_{12}\hat{q}_{22}^1E_1\oplus\hat{q}_{12}\hat{q}_{22}^2E_2, \tag{A.1e}$$

$$E_2\otimes E_2\otimes E_1 : \hat{q}_{22}^1\hat{q}_{11}E_0\oplus\hat{q}_{22}^0\hat{q}_{01}E_1\oplus\hat{q}_{21}\hat{q}_{22}^2E_2 = \hat{q}_{21}\hat{q}_{22}^0E_0\oplus\hat{q}_{21}\hat{q}_{22}^1E_1\oplus\hat{q}_{21}\hat{q}_{22}^2E_2, \tag{A.1f}$$

$$E_2\otimes E_2\otimes E_2 : \hat{q}_{22}^2\hat{q}_{22}^0E_0\oplus\hat{q}_{22}^2\hat{q}_{22}^1E_1 = \hat{q}_{22}^2\hat{q}_{22}^0E_0\oplus\hat{q}_{22}^2\hat{q}_{22}^1E_1, \tag{A.1g}$$

$$\hat{q}_{22}^0\hat{q}_{20}E_2\oplus\hat{q}_{22}^1\hat{q}_{21}E_2\oplus\hat{q}_{22}^2\hat{q}_{22}^2E_2 = \hat{q}_{22}^0\hat{q}_{02}E_2\oplus\hat{q}_{22}^1\hat{q}_{12}E_2\oplus\hat{q}_{22}^2\hat{q}_{22}^2E_2. \tag{A.1h}$$

Then by (A.1a)

$$\hat{q}_{11} = \hat{q}_{12}\hat{q}_{12},$$

by (A.1d)

$$\hat{q}_{22}^0 = \hat{q}_{12}\hat{q}_{22}^1,$$

and by for example (A.1e)

$$\hat{q}_{12} = \hat{q}_{21}.$$

Let $G = A_4$. By Theorem 5 the quantizers on tensor products of triples of irreducible representations satisfy

$$E_1 \otimes E_1 \otimes E_1 : \hat{q}_{11}\hat{q}_{12}E_0 = \hat{q}_{11}\hat{q}_{21}E_0, \tag{A.2a}$$
$$E_1 \otimes E_1 \otimes E_2 : \hat{q}_{12}\hat{q}_{10}E_1 = \hat{q}_{11}\hat{q}_{22}E_1, \tag{A.2b}$$
$$E_1 \otimes E_2 \otimes E_1 : \hat{q}_{21}\hat{q}_{10}E_1 = \hat{q}_{12}\hat{q}_{01}E_1, \tag{A.2c}$$
$$E_2 \otimes E_1 \otimes E_1 : \hat{q}_{11}\hat{q}_{22}E_1 = \hat{q}_{21}\hat{q}_{01}E_1, \tag{A.2d}$$
$$E_1 \otimes E_2 \otimes E_2 : \hat{q}_{22}\hat{q}_{11}E_2 = \hat{q}_{12}\hat{q}_{02}E_2, \tag{A.2e}$$
$$E_2 \otimes E_1 \otimes E_2 : \hat{q}_{12}\hat{q}_{20}E_2 = \hat{q}_{21}\hat{q}_{02}E_2, \tag{A.2f}$$
$$E_2 \otimes E_2 \otimes E_1 : \hat{q}_{21}\hat{q}_{20}E_2 = \hat{q}_{22}\hat{q}_{11}E_2, \tag{A.2g}$$
$$E_1 \otimes E_1 \otimes E_3 : \hat{q}_{13}\hat{q}_{13}E_3 = \hat{q}_{11}\hat{q}_{23}E_3, \tag{A.2h}$$
$$E_1 \otimes E_3 \otimes E_1 : \hat{q}_{31}\hat{q}_{13}E_3 = \hat{q}_{13}\hat{q}_{31}E_3, \tag{A.2i}$$
$$E_3 \otimes E_1 \otimes E_1 : \hat{q}_{11}\hat{q}_{32}E_3 = \hat{q}_{31}\hat{q}_{31}E_3, \tag{A.2j}$$
$$E_1 \otimes E_3 \otimes E_3 : \hat{q}^2_{33}\hat{q}_{12}E_0 \oplus \hat{q}^0_{33}\hat{q}_{10}E_1 \oplus \hat{q}^1_{33}\hat{q}_{11}E_2 \oplus [\hat{q}^3_{33}]\hat{q}_{13}2E_3 = \hat{q}_{13}(\hat{q}^0_{33}E_0 \oplus \hat{q}^1_{33}E_1 \oplus \hat{q}^2_{33}E_2 \oplus [\hat{q}^3_{33}]2E_3), \tag{A.2k}$$
$$E_3 \otimes E_1 \otimes E_3 : \hat{q}_{13}(\hat{q}^0_{33}E_0 \oplus \hat{q}^1_{33}E_1 \oplus \hat{q}^2_{33}E_2 \oplus [\hat{q}^3_{33}]2E_3) = \hat{q}_{31}(\hat{q}^0_{33}E_0 \oplus \hat{q}^1_{33}E_1 \oplus \hat{q}^2_{33}E_2 \oplus [\hat{q}^3_{33}]2E_3), \tag{A.2l}$$
$$E_3 \otimes E_3 \otimes E_1 : \hat{q}_{31}(\hat{q}^0_{33}E_0 \oplus \hat{q}^1_{33}E_1 \oplus \hat{q}^2_{33}E_2 \oplus [\hat{q}^3_{33}]2E_3) = \hat{q}^2_{33}\hat{q}_{21}E_0 \oplus \hat{q}^0_{33}\hat{q}_{01}E_1 \oplus \hat{q}^1_{33}\hat{q}_{11}E_2 \oplus [\hat{q}^3_{33}]\hat{q}_{31}2E_3, \tag{A.2m}$$
$$E_2 \otimes E_2 \otimes E_2 : \hat{q}_{22}\hat{q}_{21}E_0 = \hat{q}_{22}\hat{q}_{12}E_0, \tag{A.2n}$$
$$E_2 \otimes E_2 \otimes E_3 : \hat{q}_{23}\hat{q}_{23}E_3 = \hat{q}_{22}\hat{q}_{13}E_3, \tag{A.2o}$$
$$E_2 \otimes E_3 \otimes E_2 : \hat{q}_{32}\hat{q}_{23}E_3 = \hat{q}_{23}\hat{q}_{32}E_3, \tag{A.2p}$$
$$E_3 \otimes E_2 \otimes E_2 : \hat{q}_{22}\hat{q}_{31}E_3 = \hat{q}_{32}\hat{q}_{32}E_3, \tag{A.2q}$$
$$E_2 \otimes E_3 \otimes E_3 : \hat{q}^1_{33}\hat{q}_{21}E_0 \oplus \hat{q}^2_{33}\hat{q}_{22}E_1 \oplus \hat{q}^0_{33}\hat{q}_{20}E_2 \oplus [\hat{q}^3_{33}]\hat{q}_{23}2E_3 = \hat{q}_{23}(\hat{q}^0_{33}E_0 \oplus \hat{q}^1_{33}E_1 \oplus \hat{q}^2_{33}E_2 \oplus [\hat{q}^3_{33}]2E_3), \tag{A.2r}$$
$$E_3 \otimes E_2 \otimes E_3 : \hat{q}_{23}(\hat{q}^0_{33}E_0 \oplus \hat{q}^1_{33}E_1 \oplus \hat{q}^2_{33}E_2 \oplus [\hat{q}^3_{33}]2E_3) = \hat{q}_{32}(\hat{q}^0_{33}E_0 \oplus \hat{q}^1_{33}E_1 \oplus \hat{q}^2_{33}E_2 \oplus [\hat{q}^3_{33}]2E_3), \tag{A.2s}$$
$$E_3 \otimes E_3 \otimes E_2 : \hat{q}_{32}(\hat{q}^0_{33}E_0 \oplus \hat{q}^1_{33}E_1 \oplus \hat{q}^2_{33}E_2 \oplus [\hat{q}^3_{33}]2E_3) = \hat{q}^1_{33}\hat{q}_{12}E_0 \oplus \hat{q}^2_{33}\hat{q}_{22}E_1 \oplus \hat{q}^0_{33}\hat{q}_{02}E_2 \oplus [\hat{q}^3_{33}]\hat{q}_{32}2E_3, \tag{A.2t}$$
$$E_3 \otimes E_3 \otimes E_3 : [\hat{q}^3_{33}](\hat{q}^0_{33}2E_0 \oplus \hat{q}^1_{33}2E_1 \oplus \hat{q}^2_{33}2E_2) = [\hat{q}^3_{33}](\hat{q}^0_{33}2E_0 \oplus \hat{q}^1_{33}2E_1 \oplus \hat{q}^2_{33}2E_2), \tag{A.2u}$$
$$\hat{q}^0_{33}\hat{q}_{30}E_3 \oplus \hat{q}^1_{33}\hat{q}_{31}E_3 \oplus \hat{q}^2_{33}\hat{q}_{32}E_3 \oplus [\hat{q}^3_{33}]^2 4E_3 = \hat{q}^0_{33}\hat{q}_{03}E_3 \oplus \hat{q}^1_{33}\hat{q}_{13}E_3 \oplus \hat{q}^2_{33}\hat{q}_{23}E_3 \oplus [\hat{q}^3_{33}]^2 4E_3, \tag{A.2v}$$

where $[\hat{q}^3_{33}]$ is a 2×2-matrix on the 6-dimensional $2E_3$.

Then by (A.2b) and (A.2c),

$$\hat{q}_{11}\hat{q}_{22} = \hat{q}_{12} = \hat{q}_{21},$$

by (A.2l)

$$\hat{q}_{13} = \hat{q}_{31},$$

by (A.2h)

$$(\hat{q}_{13})^2 = \hat{q}_{11}\hat{q}_{23},$$

which together with (A.2j) gives

$$\hat{q}_{23} = \hat{q}_{32},$$

further by (A.2o)

$$(\hat{q}_{23})^2 = \hat{q}_{22}\hat{q}_{13}$$

and the last conditions are given by (A.2k) and (A.2r) as follows:

$$\hat{q}^0_{33} = \hat{q}_{13}\hat{q}^1_{33} = \hat{q}_{23}\hat{q}^2_{33}, \quad \hat{q}^0_{33}\hat{q}_{23} = \hat{q}^1_{33}\hat{q}_{12}, \quad \hat{q}^0_{33}\hat{q}_{13} = \hat{q}^2_{33}\hat{q}_{12}, \quad \hat{q}^1_{33}\hat{q}_{11} = \hat{q}^2_{33}\hat{q}_{13}, \quad \hat{q}^1_{33}\hat{q}_{23} = \hat{q}^2_{33}\hat{q}_{22}.$$

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