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*Research Article*

# On Derivations of Some Classes of Leibniz Algebras

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**Abstract** In this paper, we describe the derivations of complex $n$-dimensional naturally graded filiform Leibniz algebras $\mathrm{NGF}_1$, $\mathrm{NGF}_2$, and $\mathrm{NGF}_3$. We show that the dimension of the derivation algebras of $\mathrm{NGF}_1$ and $\mathrm{NGF}_2$ equals $n+1$ and $n+2$, respectively, while the dimension of the derivation algebra of $\mathrm{NGF}_3$ is equal to $2n-1$. The second part of the paper deals with the description of the derivations of complex $n$-dimensional filiform non Lie Leibniz algebras, obtained from naturally graded non Lie filiform Leibniz algebras. It is well known that this class is split into two classes denoted by $\mathrm{FLb}_n$ and $\mathrm{SLb}_n$. Here we found that for $L \in \mathrm{FLb}_n$, we have $n-1 \leq \dim \mathrm{Der}(L) \leq n+1$ and for algebras $L$ from $\mathrm{SLb}_n$, the inequality $n-1 \leq \dim \mathrm{Der}(L) \leq n+2$ holds true.

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## 1 Introduction

A graded algebra is an algebra endowed with a gradation which is compatible with the algebra bracket. A choice of Cartan decomposition endows any semisimple Lie algebra with the structure of a graded Lie algebra. Any parabolic Lie algebra is also a graded Lie algebra. Lie algebra $\mathrm{sl}_2$ of trace-free $2 \times 2$ matrices is graded by the generators:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These satisfy the relations $[X,Y] = H$, $[H,X] = 2X$, and $[H,Y] = -2Y$. Hence, with

$$g_{-1} = \mathrm{span}(X), \quad g_0 = \mathrm{span}(H), \quad g_1 = \mathrm{span}(Y),$$

the decomposition $\mathrm{sl}_2 = g_{-1} \oplus g_0 \oplus g_1$ presents $\mathrm{sl}_2$ as a graded Lie algebra.

It is well-known that the natural gradation of nilpotent Lie and Leibniz algebras is very helpful in investigation of their structural properties. This technique is more effective when the length of the natural gradation is sufficiently large. In the case when it is maximal the algebra is called *filiform*. For applications of this technique, for instance, see [12] and Goze et al. [4] (for Lie algebras) and [1,2,7] (for Leibniz algebras) cases. In [12] Vergne introduced the concept of naturally graded filiform Lie algebras as those admitting a gradation associated with the lower central series. In that paper, she also classified them, up to isomorphism. Apart from that, several authors have studied algebras which admit a connected gradation of maximal length (i.e., the length is exactly the dimension of the algebra). So, Khakimdjanov started this study in [6], Reyes, in [3], continued this research by giving an induction classification method, and finally, Millionschikov in [9] gave the full list of these algebras (over an arbitrary field of zero characteristic).

Recall that an algebra $L$ over a field $K$ is called Leibniz algebra if it satisfies the following Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

where $[\cdot,\cdot]$ denotes the multiplication in $L$ (first the Leibniz algebras have been introduced in [8]). It is not difficult to see that the class of Leibniz algebras is "non-antisymmetric" generalization of the class of Lie algebras. In this paper, we are dealing with the derivations of some classes of complex Leibniz algebras.

The outline of the paper is as follows. Section 2 contains preliminary results on Leibniz algebras which we will use in the paper. The main results of the paper are in Section 3. The first part of this section deals with the description of derivations of naturally graded Leibniz algebras. In the second part (Section 3.2) we study derivations of filiform Leibniz algebras arising from naturally graded non Lie filiform Leibniz algebras. It is known that the last is split into two disjoint subclasses [2]. In this paper, we denote these classes by $\mathrm{FLb}_n$ and $\mathrm{SLb}_n$. We show that according to dimensions of the derivation algebras each class is split into subclasses as follows:

$$\mathrm{FLb}_n = F_{n-1} \cup F_n \cup F_{n+1}, \quad \mathrm{SLb}_n = S_{n-1} \cup S_n \cup S_{n+1} \cup S_{n+2},$$

where $F_i$ and $S_j$ are subclasses of $\mathrm{FLb}_n$ and $\mathrm{SLb}_n$, respectively, with the derivation algebras' dimensions $i$ and $j$.

Further all algebras considered are over the field of complex numbers $\mathbb{C}$ and omitted products of basis vectors are supposed to be zero.

## 2 Preliminaries

This section contains definitions and results which will be needed throughout the paper.

Let $L$ be a Leibniz algebra. We put

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$

**Definition 1** A Leibniz algebra $L$ is said to be nilpotent if there exists $s \in \mathbb{N}$ such that

$$L^1 \supset L^2 \supset \cdots \supset L^s = 0.$$

**Definition 2** An $n$-dimensional Leibniz algebra $L$ is said to be filiform if $\dim L^i = n - i$, where $2 \leq i \leq n$.

Obviously, a filiform Leibniz algebra is nilpotent.

**Definition 3** A linear transformation $d$ of a Leibniz algebra $L$ is called a derivation if

$$d([x,y]) = [d(x),y] + [x,d(y)], \quad \forall x,y \in L.$$

The set of all derivations of an algebra $L$ is denoted by $\mathrm{Der}(L)$. By $Lb_n$ we denote the set of all $n$-dimensional filiform Leibniz algebras, appearing from naturally graded non Lie filiform Leibniz algebras. For Lie algebras the study of derivations has been initiated in [5]. The derivations of naturally graded filiform Leibniz algebras were first considered by Omirov in [10]. In the following theorem, we declare the results of the papers [2, 12].

**Theorem 1.** *Any complex $n$-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non isomorphic algebras:*

$$\mathrm{NGF}_1 = \begin{cases} [e_1,e_1] = e_3, \\ [e_i,e_1] = e_{i+1}, \quad 2 \leq i \leq n-1, \end{cases}$$

$$\mathrm{NGF}_2 = \begin{cases} [e_1,e_1] = e_3, \\ [e_i,e_1] = e_{i+1}, \quad 3 \leq i \leq n-1, \end{cases}$$

$$\mathrm{NGF}_3 = \begin{cases} [e_i,e_1] = -[e_1,e_i] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_i,e_{n+1-i}] = -[e_{n+1-i},e_i] = \alpha(-1)^{i+1}e_n, & 2 \leq i \leq n-1. \end{cases}$$

*where $\alpha \in \{0,1\}$ for even $n$ and $\alpha = 0$ for odd $n$.*

Here is a result of the papers [2, 11] on decomposition of $Lb_n$ into two disjoint classes.

**Theorem 2.** *Any complex $n$-dimensional filiform Leibniz algebra $L$, obtained from naturally graded non Lie filiform Leibniz algebra, admits a basis $e_1, e_2, \ldots, e_n$ such that the table of $L$ has one of the following forms:*

$$\mathrm{FLb}_n = \begin{cases} [e_1,e_1] = e_3, \\ [e_i,e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_1,e_2] = \alpha_4 e_4 + \alpha_5 e_5 + + \alpha_{n-1}e_{n-1} + \theta e_n, \\ [e_j,e_2] = \alpha_4 e_{j+2} + \alpha_5 e_{j+3} + \cdots + \alpha_{n+2-j}e_n, & 2 \leq j \leq n-2; \end{cases}$$

$$\mathrm{SLb}_n = \begin{cases} [e_1,e_1] = e_3, \\ [e_i,e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [e_1,e_2] = \beta_3 e_4 + \beta_4 e_5 + \cdots + \beta_{n-1}e_n, \\ [e_2,e_2] = \gamma e_n, \\ [e_j,e_2] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + \cdots + \beta_{n+1-j}e_n, & 3 \leq j \leq n-2. \end{cases}$$

We denote algebras from $\mathrm{FLb}_n$ and $\mathrm{SLb}_n$ by $L(\alpha_4, \alpha_5, \ldots, \alpha_{n-1}, \theta)$ and $L(\beta_3, \beta_4, \ldots, \beta_{n-1}, \gamma)$, respectively.

## 3 Main results

### 3.1 Derivations of graded Leibniz algebras

In this section, we study the derivations of $\mathrm{NGF}_i$, $i = 1,2,3$. In each case, we give a basis of the derivation algebra. Let $d$ be represented by a matrix $D = (d_k^l)$, $k,l = 1,2,3,\dots,n$, on the basis $\{e_1, e_2, \dots, e_n\}$. We describe the matrix $D$.

**Theorem 3.** *The dimension of the derivation algebras of* $\mathrm{NGF}_1$, $\mathrm{NGF}_2$, *and* $\mathrm{NGF}_3$ *are equal to* $n+1$, $n+2$, *and* $2n-1$, *respectively.*

*Proof.* Let us start from $\mathrm{NGF}_1$. We take $d(e_j) = \sum_{i=j}^{n} d_i^j e_i$, where $j = 1,2$. Since, $[e_1, e_1] = e_3$, we have

$$\begin{aligned} d(e_3) &= [d(e_1), e_1] + [e_1, d(e_1)] = \left[\sum_{i=1}^{n} d_i^1 e_i, e_1\right] + \left[e_1, \sum_{i=1}^{n} d_i^1 e_i\right] \\ &= d_1^1[e_1,e_1] + d_2^1[e_2,e_1] + \left[\sum_{i=3}^{n} d_i^1 e_i, e_1\right] + d_1^1[e_1,e_1] = (2d_1^1 + d_2^1)e_3 + \sum_{i=3}^{n-1} d_i^1 e_{i+1}. \end{aligned}$$

Therefore,

$$d(e_3) = (2d_1^1 + d_2^1)e_3 + \sum_{i=3}^{n-1} d_i^1 e_{i+1}. \tag{3.1}$$

From $[e_2, e_1] = e_3$, we find

$$\begin{aligned} d(e_3) &= [d(e_2), e_1] - [e_2, d(e_1)] = \left[\sum_{i=2}^{n} d_i^2 e_i, e_1\right] + \left[e_2, \sum_{i=1}^{n} d_i^1 e_i\right] \\ &= d_2^2[e_2,e_1] + \left[\sum_{i=3}^{n} d_i^2 e_i, e_1\right] + d_1^1[e_2,e_1] = (d_1^1 + d_2^2)e_3 + \sum_{i=3}^{n-1} d_i^2 e_{i+1}. \end{aligned}$$

Hence,

$$d(e_3) = (d_1^1 + d_2^2)e_3 + \sum_{i=3}^{n-1} d_i^2 e_{i+1}. \tag{3.2}$$

Comparing (3.1) and (3.2), we obtain

$$d_2^2 = d_1^1 + d_2^1, \quad d_i^2 = d_i^1, \quad \text{for } 3 \le i \le n-1.$$

According to the table of multiplication of $\mathrm{NGF}_1$, one has $[e_3, e_1] = e_4$. Thus

$$\begin{aligned} d(e_4) &= [d(e_3), e_1] - [e_3, d(e_1)] = \left[(2d_1^1 + d_2^1)e_3 + \sum_{i=3}^{n-1} d_i^2 e_{i+1}, e_1\right] + \left[e_3, \sum_{i=1}^{n} d_i^1 e_i\right] \\ &= (2d_1^1 + d_2^1)e_4 + \sum_{i=3}^{n-2} d_i^2 e_{i+2} + d_1^1[e_3,e_1] = (2d_1^1 + d_2^1)e_4 + \sum_{i=3}^{n-2} d_i^2 e_{i+2} + d_1^1 e_4 \\ &= (3d_1^1 + d_2^1)e_4 + \sum_{i=3}^{n-2} d_i^2 e_{i+2}. \end{aligned}$$

Therefore,

$$d(e_4) = (3d_1^1 + d_2^1)e_4 + \sum_{i=5}^{n} d_{i-2}^2 e_i.$$

For $k \geq 5$ one can find

$$d(e_k) = ((k-1)d_1^1 + d_2^1)e_k + \sum_{i=k+1}^{n} d_{i-k+2}^2 e_i. \tag{3.3}$$

Indeed, it is true for $k = 4$. Suppose that it is true for $k$ and show that it is the case for $k+1$. Considering $e_{k+1} = [e_k, e_1]$ we have

$$\begin{aligned}
d(e_{k+1}) &= [d(e_k), e_1] + [e_k, d(e_1)] = \left[((k-1)d_1^1 + d_2^1)e_k + \sum_{i=k+1}^{n} d_{i-k+2}^2 e_i, e_1\right] + \left[e_k, \sum_{i=1}^{n} d_i^1 e_i\right] \\
&= ((k-1)d_1^1 + d_2^1)e_{k+1} + \sum_{i=k+1}^{n-1} d_{i-k+2}^2 e_{i+1} + d_1^1[e_k, e_1] \\
&= ((k-1)d_1^1 + d_2^1)e_{k+1} + \sum_{i=k+2}^{n} d_{i-k+1}^2 e_i + d_1^1 e_{k+1} \\
&= (kd_1^1 + d_2^1)e_{k+1} + \sum_{i=k+2}^{n} d_{i-k+1}^2 e_i.
\end{aligned}$$

Hence, we get

$$d(e_{k+1}) = (kd_1^1 + d_2^1)e_{k+1} + \sum_{i=k+2}^{n} d_{i-k+1}^2 e_i.$$

In fact, $e_n = [e_{n-1}, e_1]$, therefore

$$d(e_n) = [d(e_{n-1}), e_1] + [e_{n-1}, d(e_1)].$$

We substitute $k$ by $n-1$ in (3.3) and obtain $d(e_{n-1}) = ((n-2)d_1^1 + d_2^1)e_{n-1} + d_3^2 e_n$. Therefore,

$$d(e_n) = [((n-2)d_1^1 + d_2^1)e_{n-1} + d_3^2 e_n, e_1] + \left[e_{n-1}, \sum_{i=1}^{n} d_i^1 e_i\right] = ((n-2)d_1^1 + d_2^1)e_n + d_1^1[e_{n-1}, e_1].$$

That is,

$$d(e_n) = ((n-1)d_1^1 + d_2^1)e_n.$$

The matrix of $d$ on the basis $\{e_1, e_2, e_3, \ldots, e_n\}$ has the following form:

$$\begin{bmatrix}
d_1^1 & 0 & 0 & \cdots & 0 & 0 \\
d_2^1 & d_1^1 + d_2^1 & 0 & \cdots & 0 & 0 \\
d_3^1 & d_3^1 & 2d_1^1 + d_2^1 & \cdots & 0 & 0 \\
d_4^1 & d_4^1 & d_3^1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d_{n-2}^1 & d_{n-2}^1 & d_{n-3}^1 & \cdots & 0 & 0 \\
d_{n-1}^1 & d_{n-1}^1 & d_{n-2}^1 & \cdots & (n-2)d_1^1 + d_2^1 & 0 \\
d_n^1 & d_n^2 & d_{n-1}^1 & \cdots & d_3^1 & (n-1)d_1^1 + d_2^1
\end{bmatrix}.$$

Consider the following system of vectors:

$$v_1 = E_{11} + \sum_{i=2}^{n}(i-1)E_{ii}, \quad v_k = E_{k1} + \sum_{i=2}^{n} E_{i,i-k+2},\ 2 \leq k \leq n-1, \quad v_n = E_{n1}, \quad v_{n+1} = E_{n2},$$

where $E_{ij}$ is the matrix with zero entries except for the element $a_{ij} = 1$. It is easy to see that the set $\{v_1, v_2, v_3, \ldots, v_{n+1}\}$ presents a basis of $\mathrm{Der}(\mathrm{NGF}_1)$, therefore, $\dim \mathrm{Der}(\mathrm{NGF}_1) = n+1$.

Next, we describe the derivation algebra of $\mathrm{NGF}_2$. Let $d(e_j) = \sum_{i=j}^{n} d_i^j e_i$, where $j = 1, 2$. Since $[e_1, e_1] = e_3$, then

$$d(e_3) = [d(e_1), e_1] + [e_1, d(e_1)] = \left[\sum_{i=1}^{n} d_i^1 e_i, e_1\right] + \left[e_1, \sum_{i=1}^{n} d_i^1 e_i\right]$$
$$= d_1^1[e_1, e_1] + d_2^1[e_2, e_1] + \left[\sum_{i=3}^{n} d_i^1 e_i, e_1\right] + d_1^1[e_1, e_1] = 2d_1^1 e_3 + \sum_{i=3}^{n-1} d_i^1 e_{i+1}.$$

If one uses $[e_2, e_1] = 0$, then

$$0 = [d(e_2), e_1] + [e_2, d(e_1)] = \left[\sum_{i=2}^{n} d_i^2 e_i, e_1\right] + \left[e_2, \sum_{i=1}^{n} d_i^1 e_i\right]$$
$$= d_2^2[e_2, e_1] + \left[\sum_{i=3}^{n} d_i^2 e_i, e_1\right] + d_1^1[e_2, e_1] = \sum_{i=3}^{n-1} d_i^2 e_{i+1}.$$

Therefore,

$$d_i^2 = 0, \quad \text{for } 3 \le i \le n-1.$$

Because of $[e_3, e_1] = e_4$, we find that

$$d(e_4) = [d(e_3), e_1] + [e_3, d(e_1)] = \left[2d_1^1 e_3 + \sum_{i=4}^{n} d_{i-1}^1 e_i, e_1\right] + \left[e_3, \sum_{i=1}^{n} d_i^1 e_i\right]$$
$$= 2d_1^1 e_4 + \sum_{i=4}^{n-1} d_{i-1}^1 e_{i+1} + d_1^1 e_4 = 3d_1^1 e_4 + \sum_{i=5}^{n} d_{i-2}^1 e_i.$$

Similarly,

$$d(e_k) = (k-1)d_1^1 e_k + \sum_{i=k+1}^{n-1} d_{i-k+2}^1 e_i, \;\; 4 \le k \le n-1.$$

Then the matrix of $d$ has the form

$$D = (d_k^l)_{k,l=1,2,3,\ldots,n},$$

where

$$d_i^1 \neq 0 \text{ for } 1 \le i \le n, \quad d_1^2 = 0, \text{ and } d_i^2 = 0 \text{ for } 3 \le i \le n-1, \quad d_n^2 \neq 0 \text{ and } d_2^2 \neq 0,$$
$$d_1^3 = d_2^3 = 0 \text{ and } d_3^3 = 2d_1^1, \quad d_i^3 = d_{i-1}^1 \text{ for } 4 \le i \le n-1.$$

From the view of $D$ it is easy to conclude that $\dim \mathrm{Der}(\mathrm{NGF}_2) = n + 2$.

Let us now consider the derivation algebra of $\mathrm{NGF}_3$. We take $d(e_j) = \sum_{i=j}^{n} d_i^j e_i$, where $j = 1, 2$. Then due to $[e_2, e_1] = e_3$ one has

$$d(e_3) = [d(e_2), e_1] + [e_2, d(e_1)] = \left[\sum_{i=2}^{n} d_i^2 e_i, e_1\right] + \left[e_2, \sum_{i=1}^{n} d_i^1 e_i\right]$$
$$= d_2^2[e_2, e_1] + \left[\sum_{i=3}^{n} d_i^2 e_i, e_1\right] + d_1^1[e_2, e_1] + d_{n-1}^1[e_2, e_{n-1}]$$
$$= d_2^2 e_3 + \sum_{i=3}^{n_1} d_i^2 e_{i+1} + d_1^1 e_3 - \alpha d_{n-1}^1 e_n$$
$$= (d_1^1 + d_2^2)e_3 + \sum_{i=4}^{n-1} d_{i-1}^2 e_i + (d_{n-1}^2 - \alpha d_{n-1}^1)e_n.$$

Hence,

$$d(e_3) = (d_1^1 + d_2^2)e_3 + \sum_{i=4}^{n-1} d_{i-1}^2 e_i + (d_{n-1}^2 - \alpha d_{n-1}^1)e_n.$$

Consider $e_4 = [e_3, e_1]$, then

$$\begin{aligned}
d(e_4) &= [d(e_3), e_1] + [e_3, d(e_1)] \\
&= \left[(d_1^1 + d_2^2)e_3 + \sum_{i=4}^{n-1} d_{i-1}^2 e_i + (d_{n-1}^2 - \alpha d_{n-1}^1)e_n, e_1\right] + \left[e_3, \sum_{i=1}^{n} d_i^1 e_i\right] \\
&= (d_1^1 + d_2^2)e_4 + \sum_{i=4}^{n-1} d_{i-1}^2 e_{i+1} + d_1^1[e_3, e_1] + d_{n-2}^1[e_3, e_{n-2}] \\
&= (d_1^1 + d_2^2)e_4 + \sum_{i=4}^{n-1} d_{i-1}^2 e_{i+1} + d_1^1 e_4 + \alpha d_{n-2}^1 e_n \\
&= (2d_1^1 + d_2^2)e_4 + \sum_{i=5}^{n-1} d_{i-2}^2 e_i + (d_{n-2}^2 + \alpha d_{n-2}^1)e_n.
\end{aligned}$$

Therefore,

$$d(e_4) = (2d_1^1 + d_2^2)e_4 + \sum_{i=5}^{n-1} d_{i-2}^2 e_i + (d_{n-2}^2 + \alpha d_{n-2}^1)e_n.$$

Similarly,

$$d(e_k) = ((k-2)d_1^1 + d_2^2)e_k + \sum_{i=k+1}^{n-1} d_{i-k+2}^2 e_i + (d_{n-k+2}^2 + \alpha(-1)^{i+1} d_{n-k+2}^1)e_n, \quad 4 \le k \le n-1.$$

Then the matrix of derivations has the form

$$D = (d_k^l)_{k,l=1,2,3,\ldots,n},$$

where

$$\begin{aligned}
d_i^1 &\neq 0, \text{ for } 1 \le i \le n, \quad d_i^2 \neq 0, \text{ for } 2 \le i \le n, \\
d_{ii}^i &= (i-2)d_1^1 + d_2^2, \text{ for } 2 \le i \le n-1, \\
d_j^{i+1} &= d_{j-1}^i, \text{ for } 2 \le i \le n-1 \text{ and } 4 \le j \le n-i.
\end{aligned}$$

Thus, the dimension of $\mathrm{Der}(\mathrm{NGF}_3)$ is $2n-1$. □

3.2 Derivations of filiform Leibniz algebras

Now we study the derivations of classes from Theorem 2.

**Theorem 4.** *The dimensions of the derivation algebras of* $\mathrm{FLb}_n$ *are equal to* $n-1$, $n$ *or* $n+1$.

*Proof.* Depending on constraints for the structure constants $\alpha_4, \alpha_5, \dots, \alpha_{n-1}$ and $\theta$, we have the following distribution for dimensions of the derivation algebras of elements from $\mathrm{FLb}_n$:

$$\dim \mathrm{Der}(L) = \begin{cases} n+1, \text{ if } 1.\ \theta = 0 \text{ and } \alpha_i = 0,\ 4 \le i \le n-1. \\ \qquad 2.\ \theta \neq 0,\ \alpha_4 \neq 0,\ \alpha_5 \neq 0, \text{ and there exists } i \in \{6,7,\dots,n-1\} \\ \qquad\qquad \text{such that } \alpha_i \neq 0,\ \alpha_j = 0 \text{ for } j \neq i. \\ \qquad 3.\ \theta = 0,\ \alpha_4 \neq 0,\ \alpha_5 \neq 0, \text{ and there exists } i \in \{6,7,\dots,n-1\} \\ \qquad\qquad \text{such that } \alpha_i \neq 0,\ \alpha_j = 0 \text{ for } j \neq i. \\ \qquad 4.\ \theta \neq 0,\ \alpha_4 = 0,\ \alpha_5 = 0, \text{ and there exists } i \in \{6,7,\dots,n-1\} \\ \qquad\qquad \text{such that } \alpha_i \neq 0,\ \alpha_j = 0 \text{ for } j \neq i. \\ \qquad 5.\ \theta = 0,\ \alpha_4 = 0,\ \alpha_5 = 0, \text{ and there exists } i \in \{6,7,\dots,n-1\} \\ \qquad\qquad \text{such that } \alpha_i \neq 0,\ \alpha_j = 0 \text{ for } j \neq i. \\ \qquad 6.\ \theta \neq 0, \text{ and there exists } i \in \{4,5,6,\dots,n-1\} \\ \qquad\qquad \text{such that } \alpha_i = 0,\ \alpha_j \neq 0 \text{ for } j \neq i. \\ \qquad 7.\ \theta = 0, \text{ and there exists } i \in \{4,5,6,\dots,n-1\} \\ \qquad\qquad \text{such that } \alpha_i = 0,\ \alpha_j \neq 0 \text{ for } j \neq i. \\ \qquad 8.\ \theta \neq 0,\ \alpha_4 \neq 0,\ \alpha_5 \neq 0, \alpha_6 = 0, \text{ and there exists } \ell \in \{7,8,\dots,n-1\} \\ \qquad\qquad \text{such that } \alpha_i \neq 0,\ \forall i \ge \ell \text{and } \alpha_i = 0, \text{ if } i < \ell. \\ \qquad 9.\ \theta = 0,\ \alpha_4 \neq 0,\ \alpha_5 \neq 0, \alpha_6 = 0, \text{ and there exists } \ell \in \{7,8,\dots,n-1\} \\ \qquad\qquad \text{such that } \alpha_i \neq 0,\ \forall i \ge \ell \text{ and } \alpha_i = 0, \text{ if } i < \ell. \\ n, \text{ if } 1.\ \theta \neq 0, \text{ and } \alpha_i \neq 0,\ 4 \le i \le n-1. \\ \qquad 2.\ \theta \neq 0, \text{ and } \alpha_i = 0,\ 4 \le i \le n-1. \\ \qquad 3.\ \theta = 0,\ \alpha_4 \neq 0,\ \alpha_5 \neq 0, \text{ and } \alpha_i = 0,\ 6 \le i \le n-1. \\ \qquad 4.\ \theta \neq 0,\ \alpha_4 = 0 \text{ and there exists } \ell \in \{5,6,7,8,\dots,n-1\} \\ \qquad\qquad \text{such that } \alpha_i \neq 0,\ \forall i \ge \ell \text{ and } \alpha_i = 0, \text{ if } i < \ell. \\ n-1, \text{ if } 1.\ \theta \neq 0,\ \alpha_4 \neq 0,\ \alpha_5 \neq 0, \text{ and } \alpha_i = 0,\ 6 \le i \le n-1. \\ \qquad 2.\ \theta \neq 0,\ \alpha_4 = 0,\ \alpha_5 \neq 0, \text{ and } \alpha_i = 0,\ 6 \le i \le n-1. \end{cases}$$

We will treat only one case, where $\theta \neq 0, \alpha_i \neq 0$, for $4 \le i \le n-1$. The other cases are similar. Put $d(e_j) = \sum_{i=j}^{n} d_i^j e_i$, where $j = 1,2$. Then owing to $[e_1, e_1] = e_3$, one has

$$\begin{aligned} d(e_3) &= [d(e_1), e_1] + [e_1, d(e_1)] = \left[\sum_{i=1}^{n} d_i^1 e_i, e_1\right] + \left[e_1, \sum_{i=1}^{n} d_i^1 e_i\right] \\ &= d_1^1 [e_1, e_1] + d_2^1 [e_2, e_1] + \left[\sum_{i=3}^{n} d_i^1 e_i, e_1\right] + d_1^1 [e_1, e_1] + d_2^1 [e_1, e_2] \\ &= (2d_1^1 + d_2^1) e_3 + \sum_{i=3}^{n-1} d_i^1 e_{i+1} + d_2^1 \sum_{i=3}^{n-1} (\alpha_i e_i + \theta e_n) \\ &= (2d_1^1 + d_2^1) e_3 + \sum_{i=4}^{n-1} (d_{i-1}^1 + d_2^1 \alpha_i) e_i + (d_{n-1}^1 + d_2^1 \theta) e_n. \end{aligned}$$

Due to $[e_2, e_1] = e_3$, we have

$$d(e_3) = [d(e_2), e_1] + [e_2, d(e_1)] = \left[\sum_{i=1}^{n} d_i^2 e_i, e_1\right] + \left[e_2, \sum_{i=1}^{n} d_i^1 e_i\right]$$

$$= d_2^2[e_2,e_1] + \left[\sum_{i=3}^{n} d_i^2 e_i, e_1\right] + d_1^1[e_2,e_1] + d_2^1[e_2,e_2] = d_2^2 e_3 + \sum_{i=3}^{n-1} d_i^2 e_{i+1} + d_1^1 e_3 + d_2^1 \sum_{i=4}^{n} \alpha_i e_i$$

$$= (d_1^1 + d_2^2)e_3 + \sum_{i=4}^{n} d_{i-1}^2 e_i + d_2^1 \sum_{i=4}^{n} \alpha_i e_i = (d_1^1 + d_2^2)e_3 + \sum_{i=4}^{n-1} (d_{i-1}^2 + d_2^1\alpha_i)e_i + (d_{n-1}^2 + d_2^1\alpha_n)e_n.$$

Comparing the last two expressions for $d(e_3)$, we obtain

$$d_2^2 = d_1^1 + d_2^1,\ d_i^2 = d_i^1, \quad \text{for } 3 \le i \le n-1 \text{ and } d_{n-1}^2 = d_{n-1}^1 + d_2^1(\theta - \alpha_n).$$

From $[e_3,e_1] = e_4$, one has

$$d(e_4) = [d(e_3),e_1] + [e_3,d(e_1)] = \left[(d_1^1 + d_2^2)e_3 + \sum_{i=4}^{n-1} (d_{i-1}^2 + d_2^1\alpha_i)e_i + (d_{n-1}^2 + d_2^1\alpha_n)e_n, e_1\right] + \left[e_3, \sum_{i=1}^{n} d_i^1 e_i\right]$$

$$= (d_1^1 + d_2^2)e_4 + \sum_{i=4}^{n-1} (d_{i-1}^2 + d_2^1\alpha_i)e_{i+1} + d_1^1[e_3,e_1] + d_2^1[e_3,e_2]$$

$$= (d_1^1 + d_2^2)e_4 + \sum_{i=4}^{n-1} (d_{i-1}^2 + d_2^1\alpha_i)e_{i+1} + d_1^1 e_4 + d_2^1 \sum_{i=4}^{n-1} \alpha_i e_{i+1}$$

$$= (2d_1^1 + d_2^2)e_4 + \sum_{i=5}^{n} (d_{i-2}^2 + 2d_2^1\alpha_{i-1})e_i.$$

Let us consider $[e_4,e_1] = e_5$. Then,

$$d(e_5) = [d(e_4),e_1] + [e_4,d(e_1)] = \left[(2d_1^1 + d_2^2)e_4 + \sum_{i=5}^{n} (d_{i-2}^2 + 2d_2^1\alpha_{i-1})e_i, e_1\right] + \left[e_4, \sum_{i=1}^{n} d_i^1 e_i\right]$$

$$= \left[(2d_1^1 + d_2^2)e_4 + \sum_{i=5}^{n} (d_{i-2}^2 + 2d_2^1\alpha_{i-1})e_i, e_1\right] + d_1^1[e_4,e_1] + d_2^1[e_4,e_2]$$

$$= (2d_1^1 + d_2^2)e_5 + \sum_{i=5}^{n-1} (d_{i-2}^2 + 2d_2^1\alpha_{i-1})e_{i+1} + d_1^1 e_5 + d_2^1 \sum_{i=4}^{n-2} \alpha_i e_{i+2}$$

$$= (3d_1^1 + d_2^2)e_5 + \sum_{i=5}^{n-1} (d_{i-2}^2 + 2d_2^1\alpha_{i-1})e_{i+1} + d_2^1 \sum_{i=5}^{n-1} \alpha_{i-1} e_{i+1}$$

$$= (3d_1^1 + d_2^2)e_5 + \sum_{i=5}^{n-1} (d_{i-2}^2 + 3d_2^1\alpha_{i-1})e_{i+1}$$

$$= (3d_1^1 + d_2^2)e_5 + \sum_{i=6}^{n} (d_{i-3}^2 + 3d_2^1\alpha_{i-2})e_i.$$

Similarly,

$$d(e_k) = ((k-2)d_1^1 + d_2^2)e_k + \sum_{i=k+1}^{n} (d_{i-k+2}^2 + (k-2)d_2^1\alpha_{i-k+3})e_i. \tag{3.4}$$

From $e_n = [e_{n-1},e_1]$ we get

$$d(e_n) = [d(e_{n-1}),e_1] + [e_{n-1},d(e_1)].$$

The substitution $k$ by $n-1$ in (3.4) gives $d(e_{n-1}) = ((n-3)d_1^1 + d_2^2)e_{n-1} + (d_3^2 + (n-3)d_2^1\alpha_4)e_n$ and then

$$d(e_n) = \left[((n-3)d_1^1 + d_2^2)e_{n-1} + (d_3^2 + (n-3)d_2^1\alpha_4)e_n, e_1\right] + d_1^1[e_{n-1},e_1] = \left((n-3)d_1^1 + d_2^2\right)e_n + d_1^1 e_n.$$

As a result one has

$$d(e_n) = \left((n-2)d_1^1 + d_2^2\right) e_n. \tag{3.5}$$

On the other hand,

$$[e_{n-2}, e_2] = \alpha_4 e_n.$$

Notice that $\alpha_4 \neq 0$, therefore

$$d(e_n) = \frac{1}{\alpha_4} d([e_{n-2}, e_2]).$$

This implies that

$$d(e_n) = \frac{1}{\alpha_4}([d(e_{n-2}), e_2] + [e_{n-2}, d(e_2)]).$$

We substitute $k$ by $n-2$ in (3.4) to obtain

$$d(e_{n-2}) = ((n-4)d_1^1 + d_2^2)e_{n-2} + \sum_{i=n-1}^{n} (d_{i-n+4}^2 + (n-4)d_2^1\alpha_{i-n+5})e_i.$$

Then

$$\begin{aligned}
d(e_n) &= \frac{1}{\alpha_4}\left(\left[((n-4)d_1^1 + d_2^2)e_{n-2} + \sum_{i=n-1}^{n} (d_{i-n+4}^2 + (n-4)d_2^1\alpha_{i-n+5})e_i, e_2\right] + [e_{n-2}, d(e_2)]\right) \\
&= \frac{1}{\alpha_4}\left(\left[((n-4)d_1^1 + d_2^2)e_{n-2} + \sum_{i=n-1}^{n} (d_{i-n+4}^2 + (n-4)d_2^1\alpha_{i-n+5})e_i, e_2\right] + d_2^2[e_{n-2}, e_2]\right) \\
&= \frac{1}{\alpha_4}\left(((n-4)d_1^1 + d_2^2)\alpha_4 e_n + d_1^2 e_{n-1} + d_2^2\alpha_4 e_n\right) \\
&= \frac{1}{\alpha_4}\left(((n-4)d_1^1 + d_2^2)\alpha_4 e_n + + d_1^2 e_{n-1} + d_2^2\alpha_4 e_n\right) \\
&= \frac{d_1^2}{\alpha_4} e_{n-1} + \left((n-4)d_1^1 + 2d_2^2\right) e_n.
\end{aligned}$$

Thus,

$$d(e_n) = \frac{d_1^2}{\alpha_4} e_{n-1} + \left((n-4)d_1^1 + 2d_2^2\right) e_n. \tag{3.6}$$

Comparing (3.5) and (3.6), we obtain

$$d_1^2 = 0, \quad d_2^2 = 2d_1^1. \tag{3.7}$$

The matrix of $d$ has the form $D = (d_k^l)_{k,l=1,2,3,\dots,n}$, where

$$d_2^1 = d_1^1,\ d_2^2 = 2d_1^1,\ d_1^2 = 0,\ d_i^2 = d_i^1,\ 3 \le i \le n-2,\ d_{n-1}^2 = d_{n-1}^1 + d_2^1(\theta + \alpha_n).$$

Hence, in this case the dimension of $\mathrm{Der}(L)$ for $L \in \mathrm{FLb}_n$ is $n$. □

Now we describe the derivation algebra of elements from $\mathrm{SLb}_n$.

**Theorem 5.** *The dimensions of the derivation algebras for elements of* $\mathrm{SLb}_n$ *vary between* $n-1$ *and* $n+2$.

*Proof.* Similarly to the case of $\mathrm{FLb}_n$ for the class $\mathrm{SLb}_n$, we have the distribution for dimension of derivation algebra as follows:

$$\dim \mathrm{Der}(L) = \begin{cases} n+2, \text{ if } 1.\ \gamma = 0 \text{ and } \beta_i = 0,\ 3 \le i \le n-1. \\ \qquad 2.\ \gamma = 0 \text{ and there exists } i \in \{3,4,5,\dots,n-1\} \text{ such that } \beta_i \neq 0,\ \beta_j = 0 \text{ for } j \neq i. \\ n+1, \text{ if } 1.\ \gamma \neq 0 \text{ and } \beta_i = 0,\ 3 \le i \le n-1. \\ \qquad 2.\ \gamma = 0 \text{ and } \beta_i \neq 0,\ 3 \le i \le n-1. \\ \qquad 3.\ \gamma \neq 0 \text{ and there exists } \ell \in \{3,4,5,\dots,n-1\} \text{ such that } \beta_i \neq 0,\ \forall i \ge \ell \\ \qquad\quad \text{and } \beta_i = 0, \text{ if } i < \ell, \text{ where } n = 2\ell - 1. \\ \qquad 4.\ \gamma \neq 0 \text{ and } \beta_i \neq 0,\ 3 \le i \le n-1. \\ \qquad 5.\ \gamma \neq 0 \text{ and there exists } i \in \{3,4,5,\dots,n-1\} \text{ such that } \beta_i \neq 0,\ \beta_j = 0 \text{ for } j \neq i. \\ \qquad 6.\ \gamma \neq 0 \text{ and there exists } i \in \{3,4,5,\dots,n-1\} \text{ such that } \beta_i = 0,\ \beta_j \neq 0 \text{ for } j \neq i. \\ \qquad 7.\ \gamma = 0 \text{ and there exists } i \in \{3,4,5,\dots,n-1\} \text{ such that } \beta_i = 0,\ \beta_j \neq 0 \text{ for } j \neq i. \\ n, \text{ if } 1.\ \gamma \neq 0 \text{ and there exists } \ell \in \{3,4,5,\dots,n-1\} \text{ such that } \beta_i \neq 0,\ \forall i \ge \ell \\ \qquad\quad \text{and } \beta_i = 0, \text{ if } i < \ell, \text{ where } n \neq 2\ell - 1. \\ \qquad 2.\ \gamma \neq 0,\ \beta_{n-1} \neq 0, \text{ and } \beta_i = 0,\ 3 \le i \le n-2. \\ \qquad 3.\ \gamma \neq 0,\ \beta_{n-1} = 0,\ \beta_{n-2} \neq 0, \text{ and } \beta_i = 0,\ 3 \le i \le n-3. \\ \qquad 4.\ \gamma = 0,\ \beta_{n-1} = 0,\ \beta_{n-2} \neq 0, \text{ and } \beta_i = 0,\ 3 \le i \le n-3. \\ \qquad 5.\ \gamma \neq 0,\ \beta_{n-1} \neq 0,\ \beta_{n-2} \neq 0, \text{ and } \beta_i = 0,\ 3 \le i \le n-3. \\ n-1, \text{ if } 1.\ \gamma = 0,\ \beta_{n-1} \neq 0,\ \beta_{n-2} \neq 0, \text{ and } \beta_i = 0,\ 3 \le i \le n-3. \\ \qquad 2.\ \gamma = 0,\ \beta_3 \neq 0, \text{ and } \beta_i = 0,\ 4 \le i \le n-1. \end{cases}$$

Let $\gamma = 0$, $\beta_{n-1} \neq 0$, $\beta_{n-2} \neq 0$, and $\beta_i = 0$, for $i = 3,4,5,\dots,n-3$. Put $d(e_j) = \sum_{i=j}^{n} d_i^j e_i$, where $j = 1,2$. Since $[e_1,e_1] = e_3$, we have

$$\begin{aligned} d(e_3) &= [d(e_1),e_1] + [e_1,d(e_1)] = \left[\sum_{i=1}^{n} d_i^1 e_i, e_1\right] + \left[e_1, \sum_{i=1}^{n} d_i^1 e_i\right] \\ &= d_1^1[e_1,e_1] + d_2^1[e_2,e_1] + \left[\sum_{i=3}^{n} d_i^1 e_i, e_1\right] + d_1^1[e_1,e_1] + d_2^1[e_1,e_2] \\ &= d_1^1 e_3 + \sum_{i=3}^{n-1} d_i^1 e_{i+1} + d_1^1 e_3 + d_2^1 \beta_{n-2} e_{n-1} + d_2^1 \beta_{n-1} e_n \\ &= (2d_1^1)e_3 + \sum_{i=3}^{n-1} d_i^1 e_{i+1} + d_2^1 \beta_{n-2} e_{n-1} + d_2^1 \beta_{n-1} e_n \\ &= (2d_1^1)e_3 + \sum_{i=4}^{n-2} d_{i-1}^1 e_i + (d_{n-2}^1 + d_2^1 \beta_{n-2})e_{n-1} + (d_{n-1}^1 + d_2^1 \beta_{n-1})e_n. \end{aligned}$$

Thus,

$$d(e_3) = (2d_1^1)e_3 + \sum_{i=4}^{n-2} d_{i-1}^1 e_i + (d_{n-2}^1 + d_2^1 \beta_{n-2})e_{n-1} + (d_n^1 + d_2^1 \beta_{n-1})e_n.$$

From $[e_2,e_1] = 0$ we get

$$\begin{aligned} 0 &= [d(e_2),e_1] + [e_2,d(e_1)] = \left[\sum_{i=2}^{n} d_i^2 e_i, e_1\right] + \left[e_2, \sum_{i=1}^{n} d_i^1 e_i\right] \\ &= d_2^2[e_2,e_1] + \left[\sum_{i=3}^{n} d_i^2 e_i, e_1\right] + d_1^1[e_2,e_1] + d_2^1[e_2,e_2] = \left[\sum_{i=3}^{n} d_i^2 e_i, e_1\right] = \sum_{i=4}^{n-1} d_{i-1}^2 e_i. \end{aligned}$$

Therefore, we obtain

$$d_i^2 = 0, \quad 3 \le i \le n-1.$$

Consider $[e_3, e_1] = e_4$. Then

$$\begin{aligned}
d(e_4) &= [d(e_3), e_1] + [e_3, d(e_1)] \\
&= \left[(2d_1^1)e_3 + \sum_{i=4}^{n-2} d_{i-1}^1 e_i + d_2^1(d_{n-1}^1 + \beta_{n-2})e_{n-1} + (d_n^1 + d_2^1\beta_{n-1})e_n, e_1\right] + \left[e_3, \sum_{i=1}^{n} d_i^1 e_i\right] \\
&= (2d_1^1)e_4 + \sum_{i=4}^{n-2} d_{i-1}^1 e_{i+1} + (d_{n-1}^1 + d_2^1\beta_{n-2})e_n + \left[e_3, \sum_{i=1}^{n} d_i^1 e_i\right] \\
&= (2d_1^1)e_4 + \sum_{i=4}^{n-2} d_{i-1}^1 e_{i+1} + (d_{n-1}^1 + d_2^1\beta_{n-2})e_n + d_1^1 e_4 \\
&= (3d_1^1)e_4 + \sum_{i=4}^{n-2} d_{i-1}^1 e_{i+1} + (d_{n-1}^1 + d_2^1\beta_{n-2})e_n.
\end{aligned}$$

Thus,

$$d(e_4) = (3d_1^1)e_4 + \sum_{i=4}^{n-2} d_{i-1}^1 e_{i+1} + (d_{n-1}^1 + d_2^1\beta_{n-2})e_n.$$

Take $[e_4, e_1] = e_5$. Then

$$\begin{aligned}
d(e_5) = [d(e_4), e_1] + [e_4, d(e_1)] &= \left[(3d_1^1)e_4 + \sum_{i=4}^{n-2} d_{i-1}^1 e_{i+1} + d_2^1(d_{n-1}^1 + \beta_{n-2})e_n, e_1\right] + \left[e_4, \sum_{i=1}^{n} d_i^1 e_i\right] \\
&= (3d_1^1)e_5 + \sum_{i=4}^{n-2} d_{i-1}^1 e_{i+2} + d_i^1 e_5 = (4d_1^1)e_5 + \sum_{i=6}^{n} d_{i-3}^1 e_i.
\end{aligned}$$

Hence,

$$d(e_5) = (4d_1^1)e_5 + \sum_{i=6}^{n} d_{i-3}^1 e_i.$$

Similarly,

$$d(e_k) = (k-1)d_1^1 e_k + \sum_{i=k+1}^{n} d_{i-k+2}^1 e_i. \tag{3.8}$$

It is clear that this relation is true for $k \ge 5$. Consider $e_n = [e_{n-1}, e_1]$, then $d(e_n) = d([e_{n-1}, e_1])$. So,

$$d(e_n) = [d(e_{n-1}), e_1] + [e_{n-1}, d(e_1)].$$

We substitute $k$ by $n-1$ in (3.8), and obtain

$$d(e_n) = [(n-2)d_1^1 e_{n-1} + d_3^1 e_n, e_1] + \left[e_{n-1}, \sum_{i=1}^{n} d_i^1 e_i\right] = ((n-2)d_1^1)e_n + d_1^1 e_n = ((n-1)d_1^1)e_n.$$

Thus,

$$d(e_n) = ((n-1)d_1^1)e_n. \tag{3.9}$$

On the other hand,

$$[e_1, e_2] = \beta_{n-2}e_{n-1} + \beta_{n-1}e_n.$$

Then

$$d(e_n) = \frac{1}{\beta_{n-1}} \left( d([e_1, e_2]) - \beta_{n-2} d(e_{n-1}) \right).$$

And then

$$\begin{aligned}
d(e_n) &= \frac{1}{\beta_{n-1}} \left( \left[ \sum_{i=1}^{n} d_i^1 e_i, e_2 \right] + \left[ e_1, \sum_{i=2}^{n} d_i^2 e_i \right] - \beta_{n-2} d(e_{n-1}) \right) \\
&= \frac{1}{\beta_{n-1}} \left( d_1^1 [e_1, e_2] + d_2^1 [e_2, e_2] + d_3^1 [e_3, e_2] + d_2^2 [e_1, e_2] - \beta_{n-2} d(e_{n-1}) \right) \\
&= \frac{1}{\beta_{n-1}} \left( d_1^1 (\beta_{n-2} e_{n-1} + \beta_{n-1} e_n) + d_3^1 \beta_{n-2} e_n + d_2^2 (\beta_{n-2} e_{n-1} + \beta_{n-1} e_n) - \beta_{n-2} d(e_{n-1}) \right) \\
&= \frac{1}{\beta_{n-1}} \left( (d_1^1 + d_2^2)(\beta_{n-2} e_{n-1} + \beta_{n-1} e_n) + d_3^1 \beta_{n-2} e_n - \beta_{n-2}((n-2) d_1^1 e_{n-1} + d_3^1 e_n) \right) \\
&= \frac{1}{\beta_{n-1}} \left( ((3-n) d_1^1 + d_2^2) \beta_{n-2} e_{n-1} + ((d_1^1 + d_2^2) \beta_{n-1} + d_3^1 \beta_{n-2} e_n - \beta_{n-2} d_3^1) e_n \right).
\end{aligned}$$

Thus,

$$d(e_n) = \frac{1}{\beta_{n-1}} \left( ((3-n) d_1^1 + d_2^2) \beta_{n-2} e_{n-1} \right) + ((d_1^1 + d_2^2)) e_n. \tag{3.10}$$

Comparing (3.9) and (3.10), we obtain

$$(3-n) d_1^1 + d_2^2 = 0 \text{ that is } d_2^2 = (n-3) d_1^1 \text{ and } (d_1^1 + d_2^2) = (n-1) d_1^1 \text{ which implies } d_2^2 = (n-2) d_1^1. \tag{3.11}$$

From (3.11) we get

$$d_1^1 = 0, \quad d_2^2 = 0.$$

The matrix of $d$ has the form $D = (d_k^l)_{k,l} = 1, 2, 3, \ldots, n$, where

$$d_2^2 = d_1^1 = 0, \quad d_2^1 = 0, \quad d_1^2 = 0, \quad d_i^2 = 0, \quad 3 \le i \le n-1.$$

The dimension of the derivation algebra of $L \in \mathrm{SLb}_n$ is $n-1$.

The other cases are treated similarly.

And at last, if $L \in \mathrm{SLb}_n$ with $\gamma = 0$ and $\alpha_i = 0$, for $i = 4, 5, \ldots, n-1$, then the dimension of the derivation algebra of $L$ is $n+2$, which is immediate from the case of $\mathrm{NGF}_2$. □

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