

# Nonholonomic Ricci Flows of Riemannian Metrics and Lagrange-Finsler Geometry

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## Abstract

In this paper, the theory of the Ricci flows for manifolds is elaborated with nonintegrable (nonholonomic) distributions defining nonlinear connection structures. Such manifolds provide a unified geometrical arena for nonholonomic Riemannian spaces, Lagrange mechanics, Finsler geometry, and various models of gravity (the Einstein theory and string, or gauge, generalizations). Nonholonomic frames are considered with associated nonlinear connection structure and certain defined classes of nonholonomic constraints on Riemann manifolds for which various types of generalized Finsler geometries can be modelled by Ricci flows. We speculate upon possible applications of the nonholonomic flows in modern geometrical mechanics and physics.

**Keywords:** Nonlinear connections; Nonholonomic Riemann manifolds; Lagrange and Finsler geometry; Geometric flows

## Introduction

A series of the most remarkable results in mathematics are related to Grisha Perelman's proof of the Poincaré Conjecture [1-3] built on geometrization (Thurston) conjecture [4,5] for three dimensional Riemannian manifolds, and R. Hamilton's Ricci flow theory [6,7] see reviews and basic references explained by Kleiner [8-11]. Much of the works on Ricci flows has been performed and validated by experts in the area of geometrical analysis and Riemannian geometry. Recently, a number of applications in physics of the Ricci flow theory were proposed, by Vacaru [12-16]. Some geometrical approaches in modern gravity and string theory are connected to the method of moving frames and distributions of geometric objects on (semi) Riemannian manifolds and their generalizations to spaces provided with nontrivial torsion, nonmetricity and/or nonlinear connection structures [17,18]. The geometry of nonholonomic manifolds and non-Riemannian spaces is largely applied in modern mechanics, gravity, cosmology and classical/quantum field theory explained by Stavrinou [19-35]. Such spaces are characterized by three fundamental geometric objects: nonlinear connection (N-connection), linear connection and metric. There is an important geometrical problem to prove the existence of the "best possible" metric and linear connection adapted to a N-connection structure. From the point of view of Riemannian geometry, the Thurston conjecture only asserts the existence of a best possible metric on an arbitrary closed three dimensional (3D) manifold. It is a very difficult task to define Ricci flows of mutually compatible fundamental geometric structures on non-Riemannian manifolds (for instance, on a Finsler manifold). For such purposes, we can also apply the Hamilton's approach but correspondingly generalized in order to describe nonholonomic (constrained) configurations. The first attempts to construct exact solutions of the Ricci flow equations on nonholonomic Einstein and Riemann-Cartan (with nontrivial torsion) manifolds, generalizing well known classes of exact solutions in Einstein and string gravity, were performed and explained by Vacaru [13-16].

We take a unified point of view towards Riemannian and generalized Finsler-Lagrange spaces following the geometry of nonholonomic manifolds and exploit the similarities and emphasize

differences between locally isotropic and anisotropic Ricci flows. In our works, it will be shown when the remarkable Perelman-Hamilton results hold true for more general non-Riemannian configurations. It should be noted that this is not only a straightforward technical extension of the Ricci flow theory to certain manifolds with additional geometric structures. The problem of constructing the Finsler-Ricci flow theory contains a number of new conceptual and fundamental issues on compatibility of geometrical and physical objects and their optimal configurations. There are at least three important arguments supporting the investigation of nonholonomic Ricci flows: 1) The Ricci flows of a Riemannian metric may result in a Finsler-like metric if the flows are subjected to certain nonintegrable constraints and modelled with respect to nonholonomic frames (we shall prove it in this work). 2) Generalized Finsler-like metrics appear naturally as exact solutions in Einstein, string, gauge and noncommutative gravity, parametrized by generic off-diagonal metrics, nonholonomic frames and generalized connections and methods explained by Vacaru S [33-35]. It is an important physical task to analyze Ricci flows of such solutions as well of other physically important solutions (for instance, black holes, solitonic and pp-waves solutions, Taub NUT configurations [13-15] resulting in nonholonomic geometric configurations. 3) Finally, the fact that a 3D manifold establishes an appropriate Riemannian metric, which implies certain fundamental consequences (for instance) for our spacetime topology, allows us to consider other types of "also not bad" metrics with possible local anisotropy and nonholonomic gravitational interactions. What are the natural evolution equations for such configurations and how can we relate them to the topology of nonholonomic manifolds? We shall address such questions here (for regular Lagrange systems)

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and in further works. The notion of nonholonomic manifold was introduced independently by G. Vranceanu [36] and Horak [37] as there was a need for geometric interpretation of nonholonomic mechanical systems modern approaches, criticism and historical remarks explained by Vacaru [34,38,39]. A pair  $(M, \mathcal{D})$ , where  $M$  is a manifold and  $\mathcal{D}$  is a nonintegrable distribution on  $M$ , is called a nonholonomic manifold. Three well known classes of nonholonomic manifolds, where the nonholonomic distribution defines a nonlinear connection (N-connection) structure, are defined by the Finsler spaces [40-42] and their generalizations as Lagrange and Hamilton spaces [34,43] (usually such geometries are modelled on the tangent bundle  $TM$ ) More recent examples, related to exact off-diagonal solutions and nonholonomic frames in Einstein/string/gauge/noncommutative gravity and nonholonomic Fedosov manifolds [33,34,44] also emphasize nonholonomic geometric structures. Let us now sketch the Ricci flow program for nonholonomic manifolds and Lagrange-Finsler geometries. Different models of "locally anisotropic" spaces can be elaborated for different types of fundamental geometric structures (metric, nonlinear and linear connections). In general, such spaces contain nontrivial torsion and nonmetricity fields. It would be a very difficult technical task to generalize and elaborate new proofs for all types of non-Riemannian geometries. Our strategy will be different: We shall formulate the criteria to determine when certain types of Finsler like geometries can be "extracted" (by imposing the corresponding nonholonomic constraints) from "well defined" Ricci flows of Riemannian metrics. This is possible because such geometries can be equivalently described in terms of the Levi Civita connections or by metric configurations with nontrivial torsion induced by nonholonomic frames. By nonholonomic transforms of geometric structures, we shall be able to generate certain classes of nonmetric geometries and/or generalized torsion configurations. The aim of this paper (the first one in a series of works) is to formulate the Ricci flow equations on nonholonomic manifolds and prove the conditions under which such configurations (of Finsler-Lagrange type and in modern gravity) can be extracted from well defined flows of Riemannian metrics and evolution of preferred frame structures. Further works will be dedicated to explicit generalizations of Perelman results [1-3] for nonholonomic manifolds and spaces provided with almost complex structure generated by nonlinear connections. We shall also construct new classes of exact solutions of nonholonomic Ricci flow equations, with noncommutative and/or Lie algebroid symmetry, defining locally anisotropic flows of black hole, wormhole and cosmological configurations and developing the results from work of Vacaru [13-15,33-35]. The paper is organized as follows: We start with preliminaries on geometry of nonholonomic manifolds provided with nonlinear connection (N-connection) structure in Section 2. We show how nonholonomic configurations can be naturally defined in modern gravity and the geometry of Riemann-Finsler and Lagrange spaces in Section 3. Section 4 is devoted to the theory of anholonomic Ricci flows: we analyze the evolution of distinguished geometric objects and speculate on nonholonomic Ricci flows of symmetric and nonsymmetric metrics. In Section 5, we prove that the Finsler-Ricci flows can be extracted from usual Ricci flows by imposing certain classes of nonholonomic constraints and deformations of connections. We also study regular Lagrange systems and consider generalized Lagrange-Ricci flows. The Appendix outlines some necessary results from the local geometry of N-anholonomic manifolds.

### Notation remarks

We shall use both the free coordinate and local coordinate formulas which are both convenient to introduce compact denotations and

sketch some proofs. The left up/lower indices will be considered as labels of geometrical objects, for instance, on a nonholonomic Riemannian of Finsler space. The boldfaced letters will be used to denote that the objects (spaces) are adapted (provided) to (with) nonlinear connection structure.

### Preliminaries: Nonholonomic Manifolds

We recall some basic facts in the geometry of nonholonomic manifolds provided with nonlinear connection (N-connection) structure. The reader can refer to the concepts explained by Etayo [33,34,38,44] for details and proofs (for some important results we shall sketch the key points for such proofs). On nonholonomic vectors and (co-) tangent bundles and related Riemannian-Finsler and Lagrange-Hamilton geometries [34,41,42].

### N-connections

Consider a  $(n+m)$ -dimensional manifold  $\mathbf{V}$ , with  $n \geq 2$  and  $m \geq 1$  (for a number of physical applications, it is equivalently called to be a physical and/or geometric space). In a particular case,  $\mathbf{V} = TM$ , with  $n=m$  (i.e. a tangent bundle), or  $\mathbf{V} = E = (E, M)$ ,  $\dim M = n$ , is a vector bundle on  $M$ , with total space  $E$ . In a general case, we can consider a manifold  $\mathbf{V}$  provided with a local fibred structure into conventional "horizontal" and "vertical" directions. The local coordinates on  $\mathbf{V}$  are denoted in the form  $u = (x, y)$ , or  $u^\alpha = (x^i, y^a)$ , where the "horizontal" indices run the values  $i, j, k, \dots = 1, 2, \dots, n$  and the "vertical" indices run the values  $a, b, c, \dots = n+1, n+2, \dots, n+m$ . We denote by  $\pi^T : TV \rightarrow TM$  the differential of a map  $\pi : \mathbf{V} \rightarrow V$  defined by fiber preserving morphisms of the tangent bundles  $TV$  and  $TM$ . The kernel of  $\pi^T$  is only the vertical subspace  $v\mathbf{V}$  with a related inclusion mapping  $i : v\mathbf{V} \rightarrow TV$ .

**Definition 2.1:** A nonlinear connection (N-connection)  $\mathbf{N}$  on a manifold  $\mathbf{V}$  is defined by the splitting on the left of an exact sequence

$$0 \rightarrow v\mathbf{V} \xrightarrow{i} TV \rightarrow TV / v\mathbf{V} \rightarrow 0,$$

i. e. by a morphism of submanifolds  $\mathbf{N} : TV \rightarrow v\mathbf{V}$  such that  $\mathbf{N} \circ i$  is the unity in  $v\mathbf{V}$

Locally, a N-connection is defined by its coefficients  $N_i^a(u)$ ,

$$\mathbf{N} = N_i^a(u) dx^i \otimes \frac{\partial}{\partial y^a}. \tag{1}$$

Globalizing the local splitting, one proves:

**Proposition 2.1:** Any N-connection is defined by a Whitney sum of conventional horizontal ( $h$ ) subspace, ( $h\mathbf{V}$ ), and vertical ( $v$ ) subspace, ( $v\mathbf{V}$ ),

$$TV = h\mathbf{V} \oplus v\mathbf{V}. \tag{2}$$

The sum (2) states on  $TV$  a nonholonomic (equivalently, anholonomic, or nonintegrable) distribution of horizontal and vertical subspaces. The well known class of linear connections consists of a particular subclass with the coefficients being linear on  $y^a$ , i.e.  $N_i^a(u) = \Gamma_{ij}^a(x) y^b$ .

The geometric objects on  $\mathbf{V}$  can be defined in a form adapted to a N-connection structure, following certain decompositions being invariant under parallel transports preserving the splitting (2). In this case, we call them to be distinguished (by the N-connection structure), i.e. d-objects. For instance, a vector field  $\mathbf{X} \in TV$  is expressed

$$\mathbf{X} = (hX, vX), \text{ or } \mathbf{X} = X^\alpha e_\alpha = X^i e_i + X^a e_a,$$

where  $hX = X^i e_i$  and  $vX = X^a e_a$  state, respectively, the adapted to the N-connection structure horizontal ( $h$ ) and vertical ( $v$ ) components

of the vector. In brief,  $\mathbf{X}$  is called a distinguished vector, in brief,  $d$ -vector). In a similar fashion, the geometric objects on TV like tensors, spinors, connections,... are called respectively  $d$ -tensors,  $d$ -spinors,  $d$ -connections if they are adapted to the  $N$ -connection splitting (2).

**Definition 2.2:** The  $N$ -connection curvature is defined as the Neijenhuis tensor,

$$\mathbb{W}(\mathbf{X}, \mathbf{Y}) \doteq vX, vY + v[\mathbf{X}, \mathbf{Y}] - v[vX, \mathbf{Y}] - v[\mathbf{X}, vY]. \quad (3)$$

In local form, we have for (3)

$$\mathbb{W} = \frac{1}{2} \Omega_{ij}^a d^i \wedge d^j \otimes \partial_a,$$

with coefficients

$$\Omega_{ij}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \quad (4)$$

Any  $N$ -connection  $\mathbf{N}$  may be characterized by an associated frame (vierbein) structure  $\mathbf{e}_\nu = (\mathbf{e}_i, \mathbf{e}_a)$ , where

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a} \text{ and } \mathbf{e}_a = \frac{\partial}{\partial y^a}, \quad (5)$$

and the dual frame (coframe) structure  $\mathbf{e}^\mu = (e^i, e^a)$ , where

$$e^i = dx^i \text{ and } e^a = dy^a + N_i^a(u) dx^i. \quad (6)$$

These vielbeins are called respectively  $N$ -adapted frames and coframes. In order to preserve a relation with the previous denotations [33,34] we emphasize that  $\mathbf{e}_\nu = (\mathbf{e}_i, \mathbf{e}_a)$  and  $\mathbf{e}^\mu = (e^i, e^a)$  are correspondingly the former "N-elongated" partial derivatives  $\delta_\nu = \delta / \partial u^\nu = (\delta_i, \delta_a)$  and  $N$ -elongated differentials  $\delta^\mu = \delta u^\mu = (d^i, \delta^a)$ . This emphasizes that the operators (5) and (6) define certain "N-elongated" partial derivatives and differentials which are more convenient for tensor and integral calculations on such nonholonomic manifolds. The vielbeins (6) satisfy the nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma \quad (7)$$

with (antisymmetric) nontrivial anholonomy coefficients  $W_{\alpha\beta}^\gamma = \partial_\alpha N_\beta^\gamma - \partial_\beta N_\alpha^\gamma$  and  $W_{ji}^a = \Omega_{ij}^a$ . The above presented formulas present the proof of

**Proposition 2.2:** A  $N$ -connection on  $\mathbf{V}$  defines a preferred nonholonomic  $N$ -adapted frame (vierbein) structure  $\mathbf{e} = (e_i, e_a)$  and its dual  $\tilde{\mathbf{e}} = (\tilde{e}^i, \tilde{e}^a)$  with  $\mathbf{e}$  and  $\tilde{\mathbf{e}}$  linearly depending on  $N$ -connection coefficients.

For simplicity, we shall work with a particular class of nonholonomic manifolds:

**Definition 2.3:** A manifold  $\mathbf{V}$  is  $N$ -anholonomic if its tangent space  $\mathbf{TV}$  is enabled with a  $N$ -connection structure (2).

There are two important examples of  $N$ -anholonomic manifolds, when  $V=E$ , or  $TM$ :

**Example 2.1:** A vector bundle  $\mathbf{E} = (E, \pi, M, \mathbf{N})$ , defined by a surjective projection  $\pi : E \rightarrow M$ , with  $M$  being the base manifold,  $\dim M = n$ , and  $E$  being the total space,  $\dim E = n + m$ , and provided with a  $N$ -connection splitting (2) is called  $N$ -anholonomic vector bundle. A particular case is that of  $N$ -anholonomic tangent bundle  $\mathbf{TM} = (TM, \pi, M, \mathbf{N})$ , with dimensions  $n=m$

In a similar manner, we can consider different types of (super) spaces, Riemann or Riemann-Cartan manifolds, noncommutative bundles, or superbundles, provided with nonholonomic distributions (2) and preferred systems [33,34].

## Torsions and curvatures of $d$ -connections and $d$ -metrics

One can be defined  $N$ -adapted linear connection and metric structures:

**Definition 2.4:** A distinguished connection ( $d$ -connection)  $\mathbf{D}$  on a  $N$ -anholonomic manifold  $\mathbf{V}$  is a linear connection conserving under parallelism the Whitney sum (2).

For any  $d$ -vector  $\mathbf{X}$ , there is a decomposition of  $\mathbf{D}$  into  $h$ - and  $v$ -covariant derivatives,

$$\mathbf{D}_X \doteq \mathbf{X} \lrcorner \mathbf{D} = hX \lrcorner \mathbf{D} + vX \lrcorner \mathbf{D} = Dh_X + D_{vX} = hD_X + vD_X. \quad (8)$$

The symbol " $\lrcorner$ " in (8) denotes the interior product. We shall write conventionally that  $\mathbf{D} = (hD, vD)$ , or  $\mathbf{D}_\alpha = (D_i, D_a)$ . For convenience, in the Appendix, we present some local formulas for  $d$ -connections  $\mathbf{D} = \{G_{\alpha\beta}^\gamma = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a)\}$ , with  $hD = (L_{jk}^i, L_{bk}^a)$  and  $vD = (C_{jc}^i, C_{bc}^a)$ , see (6).

**Definition 2.5:** The torsion of a  $d$ -connection  $\mathbf{D} = (hD, vD)$ , for any  $d$ -vectors  $\mathbf{X}, \mathbf{Y}$  is defined by  $d$ -tensor field

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D}_X \mathbf{Y} - \mathbf{D}_Y \mathbf{X} - [\mathbf{X}, \mathbf{Y}]. \quad (9)$$

One has a  $N$ -adapted decomposition

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) = \mathbf{T}(hX, hY) + \mathbf{T}(hX, vY) + \mathbf{T}(vX, hY) + \mathbf{T}(vX, vY). \quad (10)$$

Considering  $h$ - and  $v$ -projections of (10) and taking into account that  $h[vX, vY] = 0$ , one proves

**Theorem 2.1:** The torsion  $\mathbf{T}$  of a  $d$ -connection  $\mathbf{D}$  is defined by five nontrivial  $d$ -tensor fields adapted to the  $h$ - and  $v$ -splitting by the  $N$ -connection structure

$$hT(hX, hY) \doteq D_{hX} hY - D_{hY} hX - h[\mathbf{X}, \mathbf{Y}],$$

$$vT(hX, hY) \doteq v[hY, hX],$$

$$vT(hX, vY) \doteq -vD_{vY} hX - h[hX, vY],$$

$$vT(hX, vY) \doteq vD_{hX} vY - v[hX, vY],$$

$$vT(vX, vY) \doteq vD_{vX} vY - vD_{vY} vX - v[vX, vY].$$

The  $d$ -torsions  $hT(hX, hY), vT(vX, vY), \dots$  are called respectively the  $h$  ( $hh$ )-torsion,  $v$  ( $vv$ )-torsion and so on. The local formulas (9) for torsion  $\mathbf{T}$  are given in the Appendix.

**Definition 2.6:** The curvature of a  $d$ -connection  $\mathbf{D}$  is defined

$$\mathbf{R}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D}_X \mathbf{D}_Y - \mathbf{D}_Y \mathbf{D}_X - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]} \quad (11)$$

for any  $d$ -vectors  $\mathbf{X}, \mathbf{Y}$

By straightforward calculations, one check the properties

$$hR(\mathbf{X}, \mathbf{Y}) vZ = 0, vR(\mathbf{X}, \mathbf{Y}) hZ = 0,$$

$$\mathbf{R}(\mathbf{X}, \mathbf{Y}) Z = hR(\mathbf{X}, \mathbf{Y}) hZ + vR(\mathbf{X}, \mathbf{Y}) vZ,$$

for any for any  $d$ -vectors  $\mathbf{X}, \mathbf{Y}, Z$ .

**Theorem 2.2:** The curvature  $\mathbf{R}$  of a  $d$ -connection  $\mathbf{D}$  is completely defined by six  $d$ -curvatures

$$\mathbf{R}(hX, hY) hZ = (D_{hX} D_{hY} - D_{hY} D_{hX} - D_{[hX, hY]} - vD_{[hX, hY]}) hZ,$$

$$\mathbf{R}(hX, hY) vZ = (D_{hX} D_{hY} - D_{hY} D_{hX} - D_{[hX, hY]} - vD_{[hX, hY]}) vZ,$$

$$\mathbf{R}(vX, hY) hZ = (D_{hX} D_{hY} - D_{hY} D_{vX} - D_{[vX, hY]} - vD_{[vX, hY]}) hZ,$$

$$\mathbf{R}(vX, hY) vZ = (D_{vX} D_{hY} - D_{hY} D_{vX} - D_{[vX, hY]} - D_{[vX, hY]}) vZ,$$

$$\mathbf{R}(vX, vY)hZ = (D_{vX}D_{vY} - D_{vY}D_{vX} - D_{v[vX, vY]})hZ,$$

$$\mathbf{R}(vX, vY)vZ = (D_{vX}D_{vY} - D_{vY}D_{vX} - D_{v[vX, vY]})vZ.$$

The formulas for local coefficients of d-curvatures  $\mathbf{R} = \{R^{\alpha}_{\beta\gamma\delta}\}$  are given in the Appendix, see (11).

**Definition 2.7:** A metric structure  $\tilde{g}$  on a N-anholonomic manifold  $\mathbf{V}$  is a symmetric covariant second rank tensor field which is non degenerated and of constant signature in any point  $\mathbf{u} \in \mathbf{V}$ .

In general, a metric structure is not adapted to a N-connection structure.

**Definition 2.8:** A d-metric  $\mathbf{g} = hg \oplus_N vg$  is a usual metric tensor which contracted to a d-vector results in a dual d-vector, d-covector (the duality being defined by the inverse of this metric tensor).

The relation between arbitrary metric structures and d-metrics is established by

**Theorem 2.3:** Any metric  $\tilde{g}$  can be equivalently transformed into a d-metric

$$\mathbf{g} = hg(hX, hY) + vg(vX, vY) \tag{12}$$

adapted to a given N-connection structure.

**Proof:** e introduce  $hg(hX, hY) = hg(hX, hY)$  and  $vg(vX, vY) = vg(vX, vY)$  and try to find a N-connection when

$$\tilde{g}(hX, vY) = 0 \tag{13}$$

for any d-vectors  $\mathbf{X}, \mathbf{Y}$ . In local form, the equation (13) is an algebraic equation for the N-connection coefficients  $N_i^a$ , see formulas (1) and (2) in the Appendix.

A distinguished metric (in brief, d-metric) on a N-anholonomic manifold  $\mathbf{V}$  is a usual second rank metric tensor  $\mathbf{g}$  which with respect to a N-adapted basis (6) can be written in the form

$$\mathbf{g} = g_j(x, y) e^j \otimes e^j + h_{ab}(x, y) e^a \otimes e^b \tag{14}$$

defining a N-adapted decomposition  $\mathbf{g} = hg \oplus_N vg = [hg, vg]$ .

From the class of arbitrary d-connections  $\mathbf{D}$  on  $\mathbf{V}$ , one distinguishes those which are metric compatible (metrical d-connections) satisfying the condition

$$\mathbf{D}\mathbf{g} = \mathbf{0} \tag{15}$$

including all h- and v-projections

$$D_j g_{kl} = 0, D_a g_{kl} = 0, D_j h_{ab} = 0, D_a h_{bc} = 0.$$

Different approaches to Finsler-Lagrange geometry modelled on  $\mathbf{TM}$  (or on the dual tangent bundle  $\mathbf{T}^*\mathbf{M}$ , in the case of Cartan-Hamilton geometry) were elaborated for different d-metric structures which are metric compatible [34,40] or not metric compatible [34,42].

### (Non) adapted linear connections

For any metric structure  $\mathbf{g}$  on a manifold  $\mathbf{V}$ , there is the unique metric compatible and torsionless Levi Civita connection  $\nabla$  for which  $\nabla^a T^a = 0$  and  $\nabla \mathbf{g} = \mathbf{0}$ . This is not a d-connection because it does not preserve under parallelism the N-connection splitting (2) (it is not adapted to the N-connection structure).

**Theorem 2.4** For any d-metric  $\mathbf{g} = [hg, vg]$  on a N-anholonomic manifold  $\mathbf{V}$ , there is a unique metric canonical d-connection  $\hat{\mathbf{D}}$  satisfying the conditions  $\hat{\mathbf{D}}\mathbf{g} = \mathbf{0}$  and with vanishing  $h(hh)$ -torsion,

$$v(vv)\text{-torsion, i. e. } h\hat{T}(hX, hY) = 0 \text{ and } v\hat{T}(vX, vY) = 0.$$

**Proof:** y straightforward calculations, we can verify that the d-connection with coefficients  $\hat{G}^{\gamma}_{\alpha\beta} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^j_{jc}, \hat{C}^a_{bc})$ , see (15) in the Appendix, satisfies the condition of Theorem.

**Definition 2.9:** A N-anholonomic Riemann-Cartan manifold  ${}^{RC}\mathbf{V}$  is defined by a d-metric  $\mathbf{g}$  and a metric d-connection  $\mathbf{D}$  structures. For a particular case, we can consider that a space  ${}^R\hat{\mathbf{V}}$  is a N-anholonomic Riemann manifold if its d-connection structure is canonical, i.e.,  $\mathbf{D} = \hat{\mathbf{D}}$ .

The d-metric structure  $\mathbf{g}$  on  ${}^{RC}\mathbf{V}$  is of type (14) and satisfies the metricity conditions (15). With respect to a local coordinate basis, the metric  $\mathbf{g}$  is parametrized by a generic off-diagonal metric ansatz (2). For a particular case, we can take  $\mathbf{D} = \hat{\mathbf{D}}$  and treat the torsion  $\hat{\mathbf{T}}$  as a nonholonomic frame effect induced by a nonintegrable N-splitting. We conclude that a N-anholonomic Riemann manifold is with nontrivial torsion structure (9) (defined by the coefficients of N-connection (1), and d-metric (14) and canonical d-connection (15)). Nevertheless, such manifolds can be described alternatively, equivalently, as a usual (holonomic) Riemann manifold with the usual Levi Civita for the metric (1) with coefficients (2). We do not distinguish the existing nonholonomic structure for such geometric constructions. For more general applications, we have to consider additional torsion components, for instance, by the so-called H-field in string gravity [45].

**Theorem 2.5:** The geometry of a (semi) Riemannian manifold  $V$  with prescribed  $(n+m)$ -splitting (nonholonomic h- and v-decomposition) is equivalent to the geometry of a canonical  ${}^R\hat{\mathbf{V}}$ .

**Proof:** et  $g_{\alpha\beta}$  be the metric coefficients, with respect to a local coordinate frame, on  $V$ . The  $(n+m)$ -splitting states for a parametrization of type (2) which allows us to define the N-connection coefficients  $N_i^a$  by solving the algebraic equations (3) (roughly speaking, the N-connection coefficients are defined by the "off-diagonal" N-coefficients, considered with respect to those from the blocks  $n \times n$  and  $m \times m$ ). Having defined  $\mathbf{N} = \{N_i^a\}$ , we can compute the N-adapted frames  $e_a$  (5) and  $e^b$  (6) by using frame transforms (4) and (5) for any fixed values  $e^i(u)$  and  $e_a^a(u)$ ; for instance, for coordinate frames  $e^i = \delta^i$  and  $e_a^a = \delta_a^a$ . As a result, the metric structure is transformed into a d-metric of type (14). We can say that  $V$  is equivalently re-defined as a N-anholonomic manifold  $\mathbf{V}$ .

It is also possible to compute the coefficients of canonical d-connection  $\hat{\mathbf{D}}$  following formulas (15). We conclude that the geometry of a (semi) Riemannian manifold  $V$  with prescribed  $(n+m)$ -splitting can be described equivalently by geometric objects on a canonical N-anholonomic manifold  ${}^R\hat{\mathbf{V}}$  with induced torsion  $\hat{\mathbf{T}}$  with the coefficients computed by introducing (15) into (9). The inverse construction also holds true: A d-metric (14) on  ${}^R\hat{\mathbf{V}}$  is also a metric on  $V$  but with respect to certain N-elongated basis (6). It can be also rewritten with respect to a coordinate basis having the parametrization (2). From this Theorem, by straightforward computations with respect to N-adapted bases (6) and (5), one follows

**Corollary 2.1:** The metric of a (semi) Riemannian manifold provided with a preferred N-adapted frame structure defines canonically two equivalent linear connection structures: the Levi Civita connection and the canonical d-connection.

**Proof.** n a manifold  ${}^R\hat{\mathbf{V}}$ , we can work with two equivalent linear connections. If we follow only the methods of Riemannian geometry, we have to choose the Levi Civita connection. In some cases, it may



be optimal to elaborate a N-adapted tensor and differential calculus for nonholonomic structures, i.e. to choose the canonical d-connection. With respect to N-adapted frames, the coefficients of one connection can be expressed via coefficients of the second one, see formulas (16) and (15). Both such linear connections are defined by the same off-diagonal metric structure. For diagonal metrics with respect to local coordinate frames, the constructions are trivial.

Having prescribed a nonholonomic  $n+m$  splitting on a manifold  $V$ , we can define two canonical linear connections  $\nabla$  and  $\hat{D}$ . Correspondingly, these connections are characterized by two curvature tensors,  ${}^i R^\alpha_{\beta\gamma\delta}(\nabla)$  (computed by introducing  $\Gamma^\alpha_{\beta\gamma}$  into (7) and (10)) and  $\mathbf{R}^\alpha_{\beta\gamma\delta}(\hat{D})$  (with the N-adapted coefficients computed following formulas (11)). Contracting indices, we can compute the Ricci tensor  $Ric(\nabla)$  and the Ricci d-tensor  $Ric(\hat{D})$  following formulas (12), correspondingly written for  $\nabla$  and  $\hat{D}$ . Finally, using the inverse d-tensor  $\mathbf{g}^{\alpha\beta}$  for both cases, we compute the corresponding scalar curvatures  ${}^s R(\nabla)$  and  ${}^s R(\hat{D})$ , see formulas (13) by contracting, respectively, with the Ricci tensor and Ricci d-tensor.

### Metrization procedure and preferred linear connections

On a N-anholonomic manifold  $V$ , with prescribed fundamental geometric structures  $\mathbf{g}$  and  $\mathbf{N}$ , we can consider various classes of d-connections  $\mathbf{D}$ , which, in general, are not metric compatible, i.e.  $\mathbf{D}\mathbf{g} \neq 0$ . The canonical d-connection  $\hat{D}$  is the "simplest" metrical one, with respect to which other classes of d-connections  $\mathbf{D} = \hat{D} + \mathbf{Z}$  can be distinguished by their deformation (equivalently, distorsion, or deflection) d-tensors  $\mathbf{Z}$ . Every geometric construction performed for a d-connection  $\mathbf{D}$  can be redefined for  $\hat{D}$ , and inversely, if  $\mathbf{Z}$  is well defined.

Let us consider the set of all possible nonmetrical and metrical d-connections constructed only from the coefficients of a d-metric and N-connection structure,  $g_{ij}, h_{ab}$  and  $N_i^a$ , and their partial derivatives. Such d-connections can be generated by two procedures of deformation,

$$\hat{G}^\gamma_{\alpha\beta} \rightarrow {}^{[K]}G^\gamma_{\alpha\beta} = G^\gamma_{\alpha\beta} + {}^{[K]}Z^\gamma_{\alpha\beta} \quad (\text{Kawaguchi's metrization kaw1, kaw2}),$$

$$\text{or } \rightarrow {}^{[M]}G^\gamma_{\alpha\beta} = \hat{G}^\gamma_{\alpha\beta} + {}^{[M]}Z^\gamma_{\alpha\beta},$$

where  ${}^{[K]}Z^\gamma_{\alpha\beta}$  and  ${}^{[M]}Z^\gamma_{\alpha\beta}$  are deformation d-tensors.

**Theorem 2.6:** For given d-metric  $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$  and N-connection  $\mathbf{N} = \{N_i^a\}$  structures, the deformation d-tensor

$${}^{[K]}Z^\gamma_{\alpha\beta} = \{ {}^{[K]}Z^i_{jk} = \frac{1}{2} g^{im} D_j g_{mk}, {}^{[K]}Z^a_{bc} = \frac{1}{2} h^{ac} D_c h_{ab},$$

$${}^{[K]}Z^i_{ja} = \frac{1}{2} g^{im} D_a g_{mj}, {}^{[K]}Z^a_{bc} = \frac{1}{2} h^{ad} D_c h_{db} \}$$

transforms a d-connection  $G^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bc}, C^i_{jc}, C^a_{bc})$  into a metric d-connection

$${}^{[K]}G^\gamma_{\alpha\beta} = (L^i_{jk} + {}^{[K]}Z^i_{jk}, L^a_{bc} + {}^{[K]}Z^a_{bc}, C^i_{jc} + {}^{[K]}Z^i_{ja}, C^a_{bc} + {}^{[K]}Z^a_{bc}).$$

**Proof:** t comes from a straightforward verification that the metricity conditions  ${}^{[K]}\mathbf{D}\mathbf{g} = 0$  are satisfied (similarly to Chapter 1 in for generalized Finsler-affine spaces).

**Theorem 2.7:** For fixed d-metric,  $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ , and N-connection,  $\mathbf{N} = \{N_i^a\}$ , structures the set of metric d-connections  ${}^{[M]}G^\gamma_{\alpha\beta} = \hat{G}^\gamma_{\alpha\beta} + {}^{[M]}Z^\gamma_{\alpha\beta}$  is defined by the deformation d-tensors

$${}^{[M]}Z^\gamma_{\alpha\beta} = \{ {}^{[M]}Z^i_{jk} = {}^{[-]}O^i_{km} Y^m_{lj}, {}^{[M]}Z^a_{bc} = {}^{[-]}O^{ea}{}_{bd} Y^m_{ej},$$

$${}^{[M]}Z^i_{ja} = {}^{[+]}O^m{}_{jk} Y^k_{mc}, {}^{[M]}Z^a_{bc} = {}^{[+]}O^{ea}{}_{bd} Y^d_{ec} \}$$

where the so-called Obata operators are defined

$${}^{[\pm]}O^i_{km} = \frac{1}{2} (\delta^i_k \delta^l_m \pm g_{km} g^{li}) \quad \text{and} \quad {}^{[\pm]}O^{ea}{}_{bd} = \frac{1}{2} (\delta^e_b \delta^a_d \pm h_{bd} h^{ea})$$

and  $Y^m_{lj}, Y^m_{ej}, Y^k_{mc}, Y^d_{ec}$  are arbitrary d-tensor fields.

**Proof:** t also comes from a straightforward verification. Here we note, that  ${}^{[M]}G^\gamma_{\alpha\beta}$  are generated with prescribed nontrivial torsion coefficients. If  ${}^{[M]}Z^\gamma_{\alpha\beta} = 0$ , the canonical d-connection  $\hat{G}^\gamma_{\alpha\beta}$  contains a nonholonomically induced torsion.

We can generalize the concept of N-anholonomic Riemann-Cartan manifold  ${}^{RC}V$  (see Definition 2.9):

**Definition 2.10:** A N-anholonomic metric-affine manifold  ${}^{ma}V$  is defined by three fundamental geometric objects: 1) a d-metric  $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ , 2) a N-connection  $\mathbf{N} = \{N_i^a\}$  and 3) a general d-connection  $\mathbf{D}$ , with nontrivial nonmetricity d-tensor field  $\mathbf{Q} = \mathbf{D}\mathbf{g}$ .

The geometry and classification of metric-affine manifolds and related generalized Finsler-affine spaces is considered in Part I of monograph explained by Vacaru [34]. From Theorems 2.6, 2.7 and 2.5, follows

**Conclusion 2.1:** The geometry of any manifold  ${}^{ma}V$  can be equivalently modelled by deformation tensors on Riemann manifolds provided with preferred frame structure. The constructions are elaborated in N-adapted form if we work with the canonical d-connection, or not adapted to the N-connection structure if we apply the Levi Civita connection.

Finally, in this section, we note that if the torsion and nonmetricity fields of  ${}^{ma}V$  are defined by the d-metric and N-connection coefficients (for instance, in Finsler geometry with Chern or Berwald connection, see below section 5.1) we can equivalently (nonholonomically) transform  ${}^{ma}V$  into a Riemann manifold with metric structure of type (1) and (2).

### Gravity and Lagrange-Finsler Geometry

We study N-anholonomic structures in Riemann-Finsler and Lagrange geometry modelled on nonholonomic Riemann-Cartan manifolds.

#### Generalized lagrange spaces

If a N-anholonomic manifold is stated to be a tangent bundle,  $V = \mathbf{TM}$  the dimension of the base and fiber space coincide,  $n = m$ , and we obtain a special case of N-connection geometry. For such geometric models, a N-connection is defined by Whitney sum

$$\mathbf{T}\mathbf{T}\mathbf{M} = h\mathbf{T}\mathbf{M} \oplus v\mathbf{T}\mathbf{M}, \quad (16)$$

with local coefficients  $\mathbf{N} = \{N_i^a(x^j, y^\alpha)\}$ , where it is convenient to distinguish h-indices  $i, j, k, \dots$  from v-indices  $a, b, c, \dots$ . On  $\mathbf{T}\mathbf{M}$ , there is an almost complex structure  $\mathbf{F} = \{\mathbf{F}^\beta_\alpha\}$  associated to  $\mathbf{N}$  defined by

$$\mathbf{F}(\mathbf{e}_i) = -\mathbf{e}_i \quad \text{and} \quad \mathbf{F}(\mathbf{e}_a) = \mathbf{e}_a, \quad (17)$$

where  $\mathbf{e}_i = \partial / \partial x^i - N_i^a \partial / \partial y^a$  and  $\mathbf{e}_a = \partial / \partial y^a$  and  $\mathbf{F}^\beta_\alpha \mathbf{F}^\alpha_\gamma = -\delta^\beta_\gamma$ . Similar constructions can be performed on N-anholonomic manifolds  $V^{n+n}$  where fibred structures of dimension  $n+n$  are modelled.

A general d-metric structure (14) on  $V^{n+n}$ , together with a

prescribed N-connection  $\mathbf{N}$ , defines a N-anholonomic Riemann-Cartan manifold of even dimension.

**Definition 3.1:** A generalized Lagrange space is modelled on  $\mathbf{V}^{n+n}$  (by a d-metric with  $g_{ij} = \delta_i^a \delta_j^b h_{ab}$ , i.e.

$${}^c g = h_{ij}(x, y) (e^i \otimes e^j + e^i \otimes e^j). \tag{18}$$

One calls  $\varepsilon = h_{ab}(x, y) y^a y^b$  to be the absolute energy associated to a  $h_{ab}$  of constant signature.

**Theorem 3.1:** For nondegenerated Hessians

$$\tilde{h}_{ab} = \frac{1}{2} \frac{\partial^2 \varepsilon}{\partial y^a \partial y^b}, \tag{19}$$

when  $\det |\tilde{h}| \neq 0$ , there is a canonical N-connection completely defined by  $h_{ij}$ ,

$${}^c N_i^a(x, y) = \frac{\partial G^a}{\partial y^i} \tag{20}$$

where

$$G^a = \frac{1}{2} \tilde{h}^{ab} \left( y^k \frac{\partial^2 \varepsilon}{\partial y^b \partial x^k} - \delta_b^k \frac{\partial \varepsilon}{\partial x^k} \right).$$

**Proof:** ne has to consider local coordinate transformation laws for some coefficients  $N_i^a$  preserving splitting (16). We can verify that  ${}^c N_i^a$  satisfy such conditions. The sketch of proof is given and explained by Vacaru [34] for **TM**. We can consider any nondegenerated quadratic form  $h_{ab}(x, y) = e_a^i e_b^j h_{ij}(x, y)$  on  $\mathbf{V}^{n+n}$  if we redefine the v-coordinates in the form  $y^i = y^i(x^j, y^a)$  and  $x^i = x^i x$

Finally, in this section, we state:

**Theorem 3.2:** For any generalized Lagrange space, there are canonical N-connection  ${}^c \mathbf{N}$ , almost complex  ${}^c \mathbf{F}$ , d-metric  ${}^c \mathbf{g}$  and d-connection  ${}^c \hat{\mathbf{D}}$  structures defined by an effective regular Lagrangian  ${}^c L(x, y) = \sqrt{|\varepsilon|}$  and its Hessian  $\tilde{h}_{ab}(x, y)$  (19).

**Proof:** t follows from formulas (19), (20), (17) and (19) and adapted d-connection (21) and d-metric structures (20) all induced by a  ${}^c L = \sqrt{|\varepsilon|}$ .

**Lagrange-finsler spaces**

The class of Lagrange-Finsler geometries is usually defined on tangent bundles but it is possible to model such structures on general N-anholonomic manifolds, for instance, in (pseudo) Riemannian and Riemann-Cartan geometry, if nonholonomic frames are introduced into consideration [33,34]. Let us consider two such important examples when the N-anholonomic structures are modelled on **TM**. One denotes by  $\widetilde{TM} = TM \setminus \{0\}$  where  $\{0\}$  means the set of null sections of surjective map  $\pi : TM \rightarrow M$ .

**Example 3.1:** A Lagrange space is a pair  $L^a = [M, L(x, y)]$  with a differentiable fundamental Lagrange function  $L(x, y)$  defined by a map  $L : (x, y) \in TM \rightarrow L(x, y) \in \mathbb{R}$  of class  $C^\infty$  on  $\widetilde{TM}$  and continuous on the null section  $0 : M \rightarrow TM$  of  $\pi$ . The Hessian (19) is defined

$${}^L g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \tag{21}$$

when  $rank |g_{ij}| = n$  on  $\widetilde{TM}$  and the left up "L" is an abstract label pointing that certain values are defined by the Lagrangian L.

The notion of Lagrange space was introduced by Kern [43] and elaborated as a natural extension of Finsler geometry. In a more particular case, we have

**Example 3.2:** A Finsler space defined by a fundamental Finsler function  $F(x, y)$ , being homogeneous of type  $F(x, \lambda y) = \lambda F(x, y)$ , for nonzero  $\lambda \in \mathbb{R}$ , may be considered as a particular case of Lagrange geometry when  $L = F^2$ .

Our approach to the geometry of N-anholonomic spaces (in particular, to that of Lagrange, or Finsler, spaces) is based on canonical d-connections. It is more related to the existing standard models of gravity and field theory allowing to define Finsler generalizations of spinor fields, noncommutative and supersymmetric models, discussed in by Vacaru [33,34]. Nevertheless, a number of schools and authors on Finsler geometry prefer linear connections which are not metric compatible (for instance, the Berwald and Chern connections, see below Definition 5.1) which define new classes of geometric models and alternative physical theories with nonmetricity field, see details in [34,40-42]. From a geometrical point of view [46,47], all such approaches are equivalent. It can be considered as a particular realization, for nonholonomic manifolds, of the Poincaré's idea on duality of geometry and physical models stating that physical theories can be defined equivalently on different geometric spaces [48].

From the Theorem 3.2, one follows:

**Conclusion 3.1:** Any mechanical system with regular Lagrangian  $L(x, y)$  (or any Finsler geometry with fundamental function  $F(x, y)$ ) can be modelled as a nonholonomic Riemann geometry with canonical structures  ${}^L \mathbf{N}$ ,  ${}^L \mathbf{g}$  and  ${}^L \hat{\mathbf{D}}$  (or  ${}^F \mathbf{N}$ ,  ${}^F \mathbf{g}$  and  ${}^F \hat{\mathbf{D}}$ , for  $L = F^2$ ) defined on a N-anholonomic manifold  $\mathbf{V}^{n+n}$ . In equivalent form, such Lagrange-Finsler geometries can be described by the same metric and N-anholonomic distributions but with the corresponding not adapted Levi Civita connections

Let us denote by  $\mathbf{Ric}(\mathbf{D}) = C(1,4)\mathbf{R}(\mathbf{D})$ , where  $C(1,4)$  means the contraction on the first and fourth indices of the curvature  $\mathbf{R}(\mathbf{D})$ , and  $\mathbf{Sc}(\mathbf{D}) = C(1,2)\mathbf{Ric}(\mathbf{D}) = {}^s \mathbf{R}$ , where  $C(1,2)$  is defined by contracting  $\mathbf{Ric}(\mathbf{D})$  with the inverse d-metric, respectively, the Ricci tensor and the curvature scalar defined by any metric d-connection  $\mathbf{D}$  and d-metric  $\mathbf{g}$  on  ${}^{RC}\mathbf{V}$ , see also the component formulas (12), (13) and (14) in Appendix. The Einstein equations are

$$\mathbf{En}(\mathbf{D}) \doteq \mathbf{Ric}(\mathbf{D}) - \frac{1}{2} \mathbf{g} \mathbf{Sc}(\mathbf{D}) = \mathfrak{i}, \tag{22}$$

where the source  $\mathfrak{i}$  reflects any contributions of matter fields and corrections from, for instance, string/brane theories of gravity. In a physical model, the equations (22) have to be completed with equations for the matter fields and torsion (for instance, in the Einstein-Cartan theory one considers algebraic equations [49] for the torsion and its source). It should be noted here that because of nonholonomic structure of  ${}^{RC}\mathbf{V}$ , the tensor  $\mathbf{Ric}(\mathbf{D})$  is not symmetric and  $\mathbf{D}[\mathbf{En}(\mathbf{D})] \neq 0$ . This imposes a more sophisticated form of conservation laws on such spaces with generic "local anisotropy" [34], (a similar situation arises in Lagrange mechanics when nonholonomic constraints modify the definition of conservation laws). For  $\mathbf{D} = \hat{\mathbf{D}}$ , all constructions can be equivalently redefined for the Levi Civita connection  $\nabla$ , when  $\nabla[\mathbf{En}(\nabla)] = 0$ . A very important class of models can be elaborated when  $\mathfrak{i} = \text{diag}[\lambda^h(\mathbf{u}) \text{hg}, \lambda^v(\mathbf{u}) \text{vg}]$ , which defines the so-called N-anholonomic Einstein spaces with "nonhomogeneous" cosmological constant (various classes of exact solutions in gravity and nonholonomic Ricci flow theory were constructed and analyzed in [13-15,33,34]).

**Anholonomic Ricci Flows**

The Ricci flow theory was elaborated by Hamilton [6,7] and applied as a method approaching the Poincaré Conjecture and Thurston

Geometrization Conjecture [4,5] Perelman's works [1-3] and reviews of results [8,10].

### Holonomic Ricci flows

For a one parameter  $\tau$  family of Riemannian metrics  $\underline{g}(\tau) = \{g_{\alpha\beta}(\tau, u^\gamma)\}$  on a N-anholonomic manifold  $\mathbf{V}$ , one introduces the Ricci flow equation

$$\frac{\partial g_{\alpha\beta}}{\partial \tau} = -2 R_{\alpha\beta}, \tag{23}$$

where  $R_{\alpha\beta}$  is the Ricci tensor for the Levi Civita connection  $\nabla = \{\Gamma_{\beta\gamma}^\alpha\}$  with the coefficients defined with respect to a coordinate basis  $\partial_\alpha = \partial / \partial u^\alpha$ . The equation (23) is a tensor nonlinear generalization of the scalar heat equation  $\partial\phi / \partial\tau = \Delta\phi$ , where  $\Delta$  is the Laplace operator defined by  $\underline{g}$ . Usually, one considers normalized Ricci flows defined by

$$\frac{\partial}{\partial \tau} g_{\alpha\beta} = -2 R_{\alpha\beta} + \frac{2r}{5} g_{\alpha\beta}, \tag{24}$$

$$g_{\alpha\beta, \tau=0} = g_{\alpha\beta}^{(0)}(u), \tag{25}$$

where the normalizing factor  $r = \int RdV / dV$  is introduced in order to preserve the volume  $V$ , the boundary conditions are stated for  $\tau=0$  and the solutions are searched for  $\tau_0 > \tau \geq 0$ . For simplicity, we shall work with equations (23) if the constructions do not result in ambiguities. It is important to study the evolution of tensors in orthonormal frames and coframes on nonholonomic manifolds. Let  $(\mathbf{V}, g_{\alpha\beta}(\tau)), 0 \leq \tau < \tau_0$ , be a Ricci flow with  $R_{\alpha\beta} = R_{\alpha\beta}$  and consider the evolution of basis vector fields

$$e_\alpha(\tau) = e_\alpha^\alpha(\tau) \partial_\alpha \text{ and } e^\beta(\tau) = e_\beta^\beta(\tau) du^\beta$$

which are  $\underline{g}(0)$ -orthonormal on an open subset  $\mathbf{U} \subset \mathbf{V}$ . We evolve this local frame flows according to the formula

$$\frac{\partial}{\partial \tau} e_\alpha^\alpha = g^{\alpha\beta} R_{\beta\gamma} e_\alpha^\gamma. \tag{26}$$

There are unique solutions for such linear ordinary differential equations for all time  $\tau \in (0, \tau_0)$ .

Using the equations (24), (25) and (26), one can define the evolution equations under Ricci flow, for instance, for the Riemann tensor, Ricci tensor, Ricci scalar and volume form stated in coordinate frames (see, for example, the Theorem 3.13 in [10]). In this section, we shall consider such nonholonomic constraints on the evolution equation where the geometrical object will evolve in N-adapted form; we shall also model sets of N-anholonomic geometries, in particular, flows of geometric objects on nonholonomic Riemann manifolds and Finsler and Lagrange spaces.

### Ricci flows and N-anholonomic distributions

On manifold  $\mathbf{V}$ , the equations (24) and (25) describe flows not adapted to the N-connections  $N_i^\alpha(\tau, u)$ . For a prescribed family of such N-connections, we can construct from  $\underline{g}_{\alpha\beta}(\tau, u^\gamma)$  the corresponding set of d-metrics  $\mathbf{g}_{\alpha\beta}(\tau, u) = [g_{ij}(\tau, u), h_{ab}(\tau, u)]$  and the set of N-adapted frames on  $(\mathbf{V}, \mathbf{g}_{\alpha\beta}(\tau)), 0 \leq \tau < \tau_0$ . The evolution of such N-adapted frames is not defined by the equations (26) but satisfies the

**Proposition 4.1:** For a prescribed  $n+m$  splitting, the solutions of the system (24) and (25) define a natural flow of preferred N-adapted frame structures.

**Proof:** Following formulas (1), (2) and (3), the boundary conditions (25) state the values  $N_i^\alpha(\tau=0, u)$  and  $\mathbf{g}_{\alpha\beta}(\tau=0, u) = [g_{ij}(\tau=0, u), h_{ab}(\tau=0, u)]$ . Having a well defined solution  $\underline{g}_{\alpha\beta}(\tau, u)$ , we can construct the coefficients of N-connection  $N_i^\alpha(\tau, u)$  and d-metric  $\mathbf{g}(\tau, u) = [g(\tau, u), h(\tau, u)]$  for any  $\tau \in (0, \tau_0)$ : the associated set of frame (vielbein) structures  $\mathbf{e}_\nu(\tau) = (e_i(\tau), e_a)$ , where

$$e_i(\tau) = \frac{\partial}{\partial x^i} - N_i^\alpha(\tau, u) \frac{\partial}{\partial y^\alpha} \text{ and } e_a = \frac{\partial}{\partial y^a}, \tag{27}$$

and the set of dual frame (coframe) structures  $\mathbf{e}^\mu(\tau) = (e^i, e^a(\tau))$ , where

$$e^i = dx^i \text{ and } e^a(\tau) = dy^a + N_i^\alpha(\tau, u) dx^i. \tag{28}$$

We conclude that prescribing the existence of a nonintegrable  $(n+m)$ -decomposition on a manifold for any  $\tau \in (0, \tau_0)$ , from any solution of the Ricci flow equations (26), we can extract a set of preferred frame structures with associated N-connections, with respect to which we can perform the geometric constructions in N-adapted form.

We shall need a formula relating the connection Laplacian on contravariant one-tensors with Ricci curvature and the corresponding deformations under N-anholonomic maps. Let  $\mathbf{A}$  be a d-tensor of rank  $k$ . Then we define  $\nabla^2 \mathbf{A}$ , for  $\nabla$  being the Levi Civita connection, to be a contravariant tensor of rank  $k+2$  given by

$$\nabla^2 \mathbf{A}(\mathbf{X}, \mathbf{Y}) = (\nabla_{\mathbf{X}} \nabla_{\mathbf{Y}} \mathbf{A})(\cdot) - \nabla_{\nabla_{\mathbf{X}} \mathbf{Y}} \mathbf{A}(\cdot). \tag{29}$$

This defines the (Levi Civita) Laplacian connection

$$\Delta \mathbf{A} \doteq \mathbf{g}^{\alpha\beta} (\nabla^2 \mathbf{A})(e_\alpha, e_\beta), \tag{30}$$

for tensors, and

$$\Delta f \doteq \text{tr} \nabla^2 f = \mathbf{g}^{\alpha\beta} (\nabla^2 f)_{\alpha\beta},$$

for a scalar function on  $\mathbf{V}$ . In a similar manner, by substituting  $\nabla$  with  $\hat{\mathbf{D}}$ , we can introduce the canonical d-connection Laplacian, for instance,

$$\hat{\Delta} \mathbf{A} \doteq \mathbf{g}^{\alpha\beta} (\hat{\mathbf{D}}^2 \mathbf{A})(e_\alpha, e_\beta). \tag{31}$$

**Proposition 4.2** The Laplacians  $\hat{\Delta}$  and  $\Delta$  are related by formula

$$\Delta \mathbf{A} = \hat{\Delta} \mathbf{A} + \Delta \mathbf{A} \tag{32}$$

where the deformation d-tensor of the Laplacian,  $\Delta$ , is defined canonically by the N-connection and d-metric coefficients.

**Proof:** sketch the method of computation  $\Delta$ . Using the formula (17), we have

$$\nabla_{\mathbf{X}} = \hat{\mathbf{D}}_{\mathbf{X}} + Z_{\mathbf{X}} \tag{33}$$

where  $Z_{\mathbf{X}} = \mathbf{X}^\alpha Z_{\alpha\beta}^\gamma$  is defined for any  $\mathbf{X}^\alpha$  with  $Z_{\alpha\beta}^\gamma$  computed following formulas (17); all such coefficients depend on N-connection and d-metric coefficients and their derivatives, i.e. on generic off-diagonal metric coefficients (2) and their derivatives. Introducing (33) into (29) and (30), and separating the terms depending only on  $\hat{\mathbf{D}}_{\mathbf{X}}$  we get  $\hat{\Delta} \mathbf{A}$  (31). The rest of terms with linear or quadratic dependence on  $Z_{\alpha\beta}^\gamma$  and their derivatives define

$$\Delta \mathbf{A} \doteq \mathbf{g}^{\alpha\beta} (\Delta \mathbf{A}),$$

where

$$\Delta \mathbf{A} = \hat{\mathbf{D}}_{\mathbf{X}} (\mathbf{Z}_{\mathbf{Y}} \mathbf{A}) + Z_{\mathbf{X}} (\hat{\mathbf{D}}_{\mathbf{Y}} \mathbf{A}) + Z_{\mathbf{X}} (\mathbf{Z}_{\mathbf{Y}} \mathbf{A}) - \hat{\mathbf{D}}_{\mathbf{Z}_{\mathbf{Y}} \mathbf{A}} - Z_{\hat{\mathbf{D}}_{\mathbf{X}} \mathbf{Y}} \mathbf{A} - Z_{\mathbf{Z}_{\mathbf{X}} \mathbf{Y}} \mathbf{A}.$$

In a similar form as for Proposition 4.2, we prove

**Proposition 4.3:** The curvature, Ricci and scalar tensors of the Levi Civita connection  $\nabla$  and the canonical d-connection  $\hat{\mathbf{D}}$  are defined by formulas

$$R(\mathbf{X}, \mathbf{Y}) = \hat{\mathbf{R}}(\mathbf{X}, \mathbf{Y}) + \hat{\mathbf{Z}}(\mathbf{X}, \mathbf{Y}),$$

$$Ric(\nabla) = Ric(\hat{\mathbf{D}}) + Ric(\hat{\mathbf{Z}}),$$

$$Sc(\nabla) = Sc(\hat{\mathbf{D}}) + Sc(\hat{\mathbf{Z}}),$$

where

$$\hat{\mathbf{Z}}(\mathbf{X}, \mathbf{Y}) = \mathbf{D}_X Z_Y - Z_Y \mathbf{D}_X - Z_{[X, Y]}$$

$$Ric(\hat{\mathbf{Z}}) = C(1, 4) \hat{\mathbf{Z}}, Sc(\hat{\mathbf{Z}}) = C(1, 2) Ric(\hat{\mathbf{Z}})$$

for  $\hat{\mathbf{R}}$  computed following formula (11) and  $Sc(\hat{\mathbf{D}}) = {}^s \hat{\mathbf{R}}$ .

In the theory of Ricci flows, one considers tensors quadratic in the curvature tensors, for instance, for any given  $\mathbf{g}^{\beta\beta}$  and  $\mathbf{D}$

$$\mathbf{B}_{\alpha\gamma\alpha'\gamma'} = \mathbf{g}^{\beta\beta} \mathbf{g}^{\delta\delta} \mathbf{R}_{\alpha\beta\gamma\delta} \mathbf{R}_{\alpha'\beta'\gamma'\delta'}, \quad (34)$$

$$\underline{\mathbf{B}}_{\alpha\gamma\alpha'\gamma'} \doteq \mathbf{B}_{\alpha\gamma\alpha'\gamma'} - \mathbf{B}_{\alpha\gamma\alpha'\gamma'} - \mathbf{B}_{\alpha\gamma\alpha'\gamma'} + \mathbf{B}_{\alpha\gamma\alpha'\gamma'},$$

$$\underline{\mathbf{B}}_{\alpha\gamma'} \doteq \mathbf{D}_\alpha \mathbf{D}_{\gamma'} {}^s \mathbf{R} - \mathbf{g}^{\beta\beta} \left( \mathbf{D}_\beta \mathbf{D}_\alpha \mathbf{R}_{\gamma'\beta} + \mathbf{D}_\beta \mathbf{D}_{\gamma'} \mathbf{R}_{\alpha\beta} \right).$$

Using the connections  $\nabla$ , or  $\hat{\mathbf{D}}$ , we similarly define and compute the values  $\underline{\mathbf{B}}_{\alpha\gamma\alpha'\gamma'}$ ,  $\underline{\mathbf{B}}_{\alpha\gamma\alpha'\gamma'}$  and  $\underline{\mathbf{B}}_{\alpha\gamma'}$ , or  $\hat{\mathbf{B}}_{\alpha\gamma\alpha'\gamma'}$ ,  $\hat{\mathbf{B}}_{\alpha\gamma\alpha'\gamma'}$  and  $\hat{\mathbf{B}}_{\alpha\gamma'}$ .

### Evolution of distinguished geometrical objects

There are d-objects (d-tensors, d-connections) with N-adapted evolution completely defined by solutions of the Ricci flow equations (26).

**Definition 4.1:** A geometric structure/object is extracted from a (Riemannian) Ricci flow (for the Levi Civita connection) if the corresponding structure/object can be redefined equivalently, prescribing a  $(n+m)$ -splitting, as a N-adapted structure/d-object subjected to N-anholonomic flows.

Following the Propositions 4.2 and 4.3 and formulas (34), we prove

**Theorem 4.1:** The evolution equations for the Riemann and Ricci tensors and scalar curvature defined by the canonical d-connection are extracted respectively:

$$\frac{\partial}{\partial \tau} \hat{\mathbf{R}}_{\alpha\beta\gamma\delta} = \hat{\Delta} \hat{\mathbf{R}}_{\alpha\beta\gamma\delta} + 2 \hat{\underline{\mathbf{B}}}_{\alpha\beta\gamma\delta} + \hat{\mathbf{Q}}_{\alpha\beta\gamma\delta},$$

$$\frac{\partial}{\partial \tau} \hat{\mathbf{R}}_{\alpha\beta} = \hat{\Delta} \hat{\mathbf{R}}_{\alpha\beta} + \hat{\mathbf{Q}}_{\alpha\beta},$$

$$\frac{\partial}{\partial \tau} {}^s \hat{\mathbf{R}} = \hat{\Delta} {}^s \hat{\mathbf{R}} + 2 \hat{\underline{\mathbf{R}}}_{\alpha\beta} \hat{\mathbf{R}}^{\alpha\beta} + \hat{\mathbf{Q}}$$

where, for

$${}^s \mathbf{R}_{\alpha\beta\gamma\delta} = \hat{\mathbf{R}}_{\alpha\beta\gamma\delta} + Z_{\alpha\beta\gamma\delta}, \underline{\mathbf{B}}_{\alpha\beta\gamma\delta} = \hat{\underline{\mathbf{B}}}_{\alpha\beta\gamma\delta} + \hat{\underline{\mathbf{Z}}}_{\alpha\beta\gamma\delta}, Z = \mathbf{g}^{\alpha\beta} Z_{\alpha\beta},$$

$${}^s \mathbf{R}_{\alpha\beta} = \hat{\mathbf{R}}_{\alpha\beta} + Z_{\alpha\beta}, \underline{\mathbf{B}}_{\alpha\gamma'} = \hat{\underline{\mathbf{B}}}_{\alpha\gamma'} + \hat{\underline{\mathbf{Z}}}_{\alpha\gamma'}, {}^s \mathbf{R} = {}^s \hat{\mathbf{R}} + Z,$$

the  $\hat{\mathbf{Q}}$ -terms (defined by the coefficients of canonical d-connection,  $N_i^a$  and  $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$  and their derivatives) are

$$\hat{\mathbf{Q}}_{\alpha\beta\gamma\delta} = -\frac{\partial}{\partial \tau} Z_{\alpha\beta\gamma\delta} + \hat{\Delta} \hat{\mathbf{R}}_{\alpha\beta\gamma\delta} + 2 \hat{\underline{\mathbf{Z}}}_{\alpha\beta\gamma\delta},$$

$$\hat{\mathbf{Q}}_{\alpha\beta} = -\frac{\partial}{\partial \tau} Z_{\alpha\beta} + \hat{\Delta} \hat{\mathbf{R}}_{\alpha\beta} + \hat{\underline{\mathbf{Z}}}_{\alpha\beta},$$

$$\hat{\mathbf{Q}} = -\frac{\partial}{\partial \tau} Z + \hat{\Delta} Z + \hat{\Delta} {}^s \hat{\mathbf{R}} + 2 \hat{\underline{\mathbf{R}}}_{\alpha\beta} Z^{\alpha\beta} + 2 Z_{\alpha\beta} \hat{\mathbf{R}}^{\alpha\beta} + 2 Z_{\alpha\beta} Z^{\alpha\beta}.$$

In Ricci flow theory, it is important to have the formula for the evolution of the volume form:

**Remark 4.1:** The deformation of the volume form is stated by equation

$$\frac{\partial}{\partial \tau} dvol(\tau, u^a) = -({}^s \hat{\mathbf{R}} + Z) dvol(\tau, u^a)$$

which is just that for the Levi Civita connection and  $dvol(\tau, u^a) \doteq \sqrt{|\det \underline{\mathbf{g}}_{\alpha\beta}(\tau, u^a)|}$ , where  $\underline{\mathbf{g}}_{\alpha\beta}(\tau)$  are metrics of type (1).

The evolution equations from Theorem 4.1 and Remark 4.1 transform into similar ones from Theorem 3.13 [10].

For any solution of equations (24) and (25), on  $\mathbf{U} \subset \mathbf{V}$ , we can construct for any  $\tau \in (0, \tau_0)$  a parametrized set of canonical d-connections  $\hat{\mathbf{D}}(\tau) = \{\hat{G}_{\alpha\beta}^{\gamma}(\tau)\}$  (15) defining the corresponding canonical Riemann d-tensor (11), nonsymmetric Ricci d-tensor  $\hat{\mathbf{R}}_{\alpha\beta}$  (12) and scalar (13). The coefficients of d-objects are defined with respect to evolving N-adapted frames (27) and (28). One holds

**Conclusion 4.1:** The evolution of corresponding d-objects on N-anholonomic Riemann manifolds can be canonically extracted from the evolution under Ricci flows of geometric objects on Riemann manifolds.

In the sections 5.3 and 5.1, we shall consider how Finsler and Lagrange configurations can be extracted by more special parametrizations of metric and nonholonomic constraints.

### Nonholonomic Ricci flows of (non) symmetric metrics

The Ricci flow equations were introduced by Hamilton [6] in a heuristic form similarly to how A. Einstein proposed his equations by considering possible physically grounded equalities between the metric and its first and second derivatives and the second rank Ricci tensor. On (pseudo) Riemannian spaces the metric and Ricci tensors are both symmetric and it is possible to consider the parameter derivative of metric and/or correspondingly symmetrized energy-momentum of matter fields as sources for the Ricci tensor. On N-anholonomic manifolds there are two alternative possibilities: The first one is to postulate the Ricci flow equations in symmetric form, for the Levi Civita connection, and then to extract various N-anholonomic configurations by imposing corresponding nonholonomic constraints. The bulk of our former and present work is related to symmetric metric configurations.

In the second case, we can start from the very beginning with a nonsymmetric Ricci tensor for a non-Riemannian space. In this section, we briefly speculate on such geometric constructions: The nonholonomic Ricci flows even beginning with a symmetric metric tensor may result naturally in nonsymmetric metric tensors  $\hat{\mathbf{g}}_{\alpha\beta} = \underline{\mathbf{g}}_{\alpha\beta} + \bar{\mathbf{g}}_{\alpha\beta}$ , where  $\bar{\mathbf{g}}_{\alpha\beta} = -\bar{\mathbf{g}}_{\beta\alpha}$ . Nonsymmetric metrics in gravity were originally considered by Einstein [50] and Eisenhart [51], see modern approaches [52].

**Theorem 4.2:** With respect to N-adapted frames, the canonical nonholonomic Ricci flows with nonsymmetric metrics defined by equations

$$\frac{\partial}{\partial \tau} g_{ij} = -2 \hat{R}_{ij} + 2 \lambda g_{ij} - h_{cd} \frac{\partial}{\partial \tau} (N_i^c N_j^d), \quad (35)$$

$$\frac{\partial}{\partial \tau} h_{ab} = -2 \hat{R}_{ab} + 2 \lambda h_{ab}, \quad (36)$$



$$\frac{\partial}{\partial \tau} \hat{g}_{ia} = \hat{R}_{ia}, \frac{\partial}{\partial \tau} \hat{g}_{ai} = \hat{R}_{ai} \quad (37)$$

where  $\hat{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$  with respect to N-adapted basis (6),  $\lambda = r/5$ ,  $y^3 = v$  and  $\tau$  can be, for instance, the time like coordinate,  $\tau = t$ , or any parameter or extra dimension coordinate.

**Proof.** It follows from a redefinition of equations (24) with respect to N-adapted frames (by using the frame transform (4) and (5)), and considering respectively the canonical Ricci d-tensor (12) constructed from  $[g_{ij}, h_{ab}]$ . Here we note that normalizing factor  $r$  is considered for the symmetric part of metric.

One follows:

**Conclusion 4.2:** Nonholonomic Ricci flows (for the canonical d-connection) resulting in symmetric d-metrics are parametrized by the constraints

$$\hat{g}_{\alpha\beta} = 0 \text{ and } \hat{R}_{ia} = \hat{R}_{ai} = 0. \quad (38)$$

The system of equations (35), (36) and (38), for "symmetric" nonholonomic Ricci flows, was introduced and analyzed in [13,14].

**Example 4.1:** The conditions (38) are satisfied by any ansatz of type (14) in 3D, 4D, or 5D, with coefficients of type

$$g_i = g_i(x^k), h_a = h_a(x^k, v), N_i^3 = w_i(x^k, v), N_i^4 = n_i(x^k, v), \quad (39)$$

for  $i, j, \dots = 1, 2, 3$  and  $a, b, \dots = 4, 5$  (the 3D and 4D being parametrized by eliminating the cases  $i=1$  and, respectively,  $i=1, 2$ );  $y^4 = v$  being the so-called "anisotropic" coordinate. Such metrics are off-diagonal with the coefficients depending on 2 and 3 coordinates but positively not depending on the coordinate  $y^5$ .

We constructed and investigated various types of exact solutions of the nonholonomic Einstein equations and Ricci flow equations [33-35] and [13-15]. They are parametrized by ansatz of type (39) which positively constrains the Ricci flows to be with symmetric metrics. Such solutions can be used as backgrounds for investigating flows of Eisenhart (generalized Finsler-Eisenhart geometries) if the constraints (38) are not completely imposed. We shall not analyze this type of N-anholonomic Ricci flows in this series of works.

## Generalized Finsler-Ricci Flows

The aim of this section is to provide some examples illustrating how different types of nonholonomic constraints on Ricci flows of Riemannian metrics model different classes of N-anholonomic spaces (defined by Finsler metrics and connections, geometric models of Lagrange mechanics and generalized Lagrange geometries).

### Finsler-Ricci flows

Let us consider a  $\tau$ -parametrized family (set) of fundamental Finsler functions  $F(\tau) = F(\tau, x^i, y^a)$ , see Example 3.2. For a family of nondegenerated Hessians

$${}^F h_{ij}(\tau, x, y) = \frac{1}{2} \frac{\partial^2 F^2(\tau, x, y)}{\partial y^i \partial y^j}, \quad (40)$$

see formula (21) for effective  $\varepsilon(\tau) = L(\tau) = F^2(\tau)$ , we can model Finsler metrics on  $\mathbf{V}^{n+n}$  (or on  $\mathbf{TM}$ ) and the corresponding family of canonical N-connections, see (20),

$${}^c N_i^a(\tau) = \frac{\partial G^a(\tau)}{\partial y^i}, \quad (41)$$

where  $G^a(\tau) = \frac{1}{2} {}^F h^{ab}(\tau) \left( y^k \frac{\partial^2 F^2(\tau)}{\partial y^b \partial x^k} - \delta_b^k \frac{\partial F^2(\tau)}{\partial x^k} \right)$  and  ${}^F h^{ab}(\tau)$  are inverse to  ${}^F h_{ij}(\tau)$ .

**Proposition 5.1:** Any family of fundamental Finsler functions  $F(\tau)$  with nondegenerated  ${}^F h_{ij}(\tau)$  defines a corresponding family of Sasaki type metrics

$${}^c g(\tau) = {}^F h_{ij}(\tau, x, y) \left( e^i \otimes e^j + {}^c e^i(\tau) \otimes {}^c e^j(\tau) \right), \quad (42)$$

with  ${}^F g_{ij}(\tau) = {}^F h_{ij}(\tau, x, y)$ , where  ${}^c e^a(\tau) = dy^a + {}^c N_i^a(\tau, u) dx^i$  are defined by the N-connection (41).

**Proof.** It follows from the explicit construction (42).

For  $\mathbf{V}^{n+n} = \mathbf{TM} = (TM, \pi, M, {}^c N_i^a)$  with injective  $\pi: TM \rightarrow M$ , we can model by  $F(\tau)$  various classes of Finsler geometries. In explicit form, we work on  $\widetilde{TM} \doteq TM \setminus \{0\}$  and consider the pull-back bundle  $\pi^* TM$ . One generates sets of geometric objects on pull-back cotangent bundle  $\pi^* T^* M$  and its tensor products:

on  $\pi^* T^* M \otimes \pi^* T^* M \otimes \pi^* T^* M$ , a corresponding family of Cartan tensors

$$A(\tau) = A_{ijk}(\tau) dx^i \otimes dx^j \otimes dx^k,$$

$$A_{ijk}(\tau) \doteq \frac{F(\tau)}{2} \frac{\partial g_{ij}(\tau)}{\partial y^k};$$

on  $\pi^* T^* M$ , a family of Hilbert forms  $\omega(\tau) \doteq \frac{\partial F(\tau)}{\partial y^i} dx^i$  and the d-connection 1-form

$$\omega_j^i(\tau) = L_{jk}^i(\tau) dx^k \quad (43)$$

$$L_{jk}^i(\tau) = \frac{1}{2} {}^F g^{ih} \left( {}^c e_k^F g_{jh} + {}^c e_j^F g_{kh} - {}^c e_h^F g_{jk} \right).$$

**Theorem 5.1:** The set of fundamental Finsler functions  $F(\tau)$  defines on  $\pi^* TM$  a unique set of linear connections, called the Chern connections, characterized by the structure equations:

$$d(dx^i) - dx^j \wedge \omega_j^i(\tau) = -dx^i \wedge \omega_j^i(\tau) = 0,$$

i.e. the torsion free condition;

$$dg_{ij}(\tau) - {}^F g_{ij}(\tau) \omega_k^i(\tau) - {}^F g_{ik}(\tau) \omega_j^k(\tau) = 2 \frac{A_{ija}(\tau)}{F(\tau)} {}^c e^a(\tau),$$

i.e. the almost metric compatibility condition.

**Proof:** It follows from straightforward computations. For any fixed value  $\tau = \tau_0$ , it is just the Chern's Theorem 2.4.1. from, In order to elaborate a complete geometric model on  $TM$ , which also allows us to perform the constructions for N-anholonomic manifolds, we have to extend the above considered forms with nontrivial coefficients with respect to  ${}^c e^a(\tau)$ .

**Definition 5.1:** A family of fundamental Finsler metrics  $F(\tau)$  defines models of Finsler geometry (equivalently, space) with d-connections  $\Gamma_{\beta\gamma}^\alpha(\tau) = (L_{jk}^i(\tau), C_{jk}^i(\tau))$  on a corresponding N-anholonomic manifold  $\mathbf{V}$ :

• of Cartan type if  $L_{jk}^i(\tau)$  is that from (43) and

$$C_{jk}^i(\tau) = \frac{1}{2} {}^F g^{ih} \left( \frac{\partial}{\partial x^k} {}^F g_{jh}(\tau) + \frac{\partial}{\partial x^j} {}^F g_{kh}(\tau) - \frac{\partial}{\partial x^h} {}^F g_{jk}(\tau) \right), \quad (44)$$

which is similar to formulas (21) but for  $L = F^2(\tau)$ ;

• of Chern type if  $L_{jk}^i(\tau)$  is given by (43) and  $C_{jk}^i(\tau) = 0$ ;

• of Berwald type if  $L_{jk}^i(\tau) = \partial^c N_j^i / \partial y^k$  and  $C_{jk}^i(\tau) = 0$ ;

• of Hashiguchi type if  $L_{jk}^i(\tau) = \partial^c N_j^i / \partial y^k$  and  $C_{jk}^i(\tau)$  is given by (44).

Various classes of remarkable Finsler connections have been investigated by Bejancu [41,42]. On modelling Finsler like structures in

Einstein and string gravity and in noncommutative gravity. It should be emphasized that the models of Finsler geometry with Chern, Berwald or Hashiguchi type d-connections are with nontrivial nonmetricity field [33,34]. So, in general, a family of Finsler fundamental metric functions  $F(\tau)$  may generate various types of N-anholonomic metric-affine geometric configurations, see Definition 2.10, but all components of such induced nonmetricity and/or torsion fields are defined by the coefficients of corresponding families of generic off-diagonal metrics of type (1), when the ansatz (2) is parametrized for  $g_{ij} = h_{ij} = F h_{ij}(\tau)$  and  $N_i^a = {}^c N_i^a(\tau)$ . Applying the results of Theorem 2.7, we can transform the families of “nonmetric” Finsler geometries into corresponding metric ones and model the Finsler configurations on N-anholonomic Riemannian spaces, see Conclusion 2.1. In the “simplest” geometric and physical manner (convenient both for applying the former Hamilton-Perelman results on Ricci flows for Riemannian metrics, as well for further generalizations to noncommutative Finsler geometry, supersymmetric models and so on...), we restrict our analysis to Finsler-Ricci flows with canonical d-connection of Cartan type when  ${}^F \hat{G}_{\beta\gamma}(\tau) = (L^i_{jk}(\tau), C^i_{jk}(\tau))$  is with  $L^i_{jk}(\tau)$  from (43) and  $C^i_{jk}(\tau)$  from (44). This provides a proof for

**Lemma 5.1:** A family of Finsler geometries defined by  $F(\tau)$  can be characterized equivalently by the corresponding canonical d-connections (in N-adapted form) and Levi Civita connections (in not N-adapted form) related by formulas

$${}^F \Gamma_{\alpha\beta}^\gamma = {}^F \hat{G}_{\alpha\beta}^\gamma + Z_{\alpha\beta}^\gamma \tag{45}$$

where  $Z_{\alpha\beta}^\gamma$  is computed following formulas (18) for  $g_{ij} = h_{ij} = F h_{ij}(\tau)$  and  $N_i^a = {}^c N_i^a(\tau)$ .

Following the Lemma 5.1 and section 4.1, we obtain the proof of

**Theorem 5.2:** The Finsler-Ricci flows for fundamental metric functions

$F(\tau)$  can be extracted from usual Ricci flows of Riemannian metrics parametrized in the form

$${}^F g_{\alpha\beta}(\tau) = \begin{bmatrix} {}^F g_{ij} + {}^c N_i^a {}^c N_j^b {}^F g_{ab} & {}^c N_j^e {}^F g_{ae} \\ {}^c N_i^e {}^F g_{be} & {}^F g_{ab} \end{bmatrix} \tag{46}$$

and satisfying the equations (for instance, for normalized flows)

$$\frac{\partial}{\partial \tau} {}^F g_{\alpha\beta} = -2 {}^F R_{\alpha\beta} + \frac{2r}{5} {}^F g_{\alpha\beta},$$

$${}^F g_{\alpha\beta}(\tau=0) = {}^F g_{\alpha\beta}^{[0]}(u).$$

The Finsler-Ricci flows are distinguished from the usual (unconstrained) flows of Riemannian metrics by existence of additional evolutions of preferred N-adapted frames (see Proposition 2.2):

**Corollary 5.1** The evolution, for all “time”  $\tau \in (0, \tau_0)$ , of preferred frames on a Finsler space

$${}^F e_a(\tau) = {}^F e_a^\alpha(\tau, u) \partial_\alpha$$

is defined by the coefficients

$${}^F e_a^\alpha(\tau, u) = \begin{bmatrix} {}^F e_i^i(\tau, u) & {}^c N_i^b(\tau, u) {}^F e_b^a(\tau, u) \\ 0 & {}^F e_a^a(\tau, u) \end{bmatrix}, \tag{47}$$

with  ${}^F g_{ij}(\tau) = {}^F e_i^i(\tau, u) {}^F e_j^j(\tau, u) \eta_{ij}$ , where  $\eta_{ij} = \text{diag}[\pm 1, \dots, \pm 1]$  establish the signature of  ${}^F g_{\alpha\beta}^{[0]}(u)$ , is given by equations

$$\frac{\partial}{\partial \tau} {}^F e_a^\alpha = {}^F g^{\alpha\beta} {}^F R_{\beta\gamma} {}^F e_a^\gamma, \tag{48}$$

where  ${}^F g^{\alpha\beta}$  is inverse to (46) and  ${}^F R_{\beta\gamma}$  is the Ricci tensor constructed from the Levi Civita coefficients of (46).

**Proof.** We have to introduce the metric and N-connection coefficients (42) and (41), defined by  ${}^F(\tau)$ , into (4). The equations (48) are similar to (26), but in our case for the N-adapted frames (47). We note that the evolution of the Riemann and Ricci tensors and scalar curvature defined by the Cartan d-connection, i.e. the canonical d-connection,  ${}^F \hat{G}_{\alpha\beta}^\gamma$ , can be extracted as in Theorem 4.1 when the values are redefined for the metric (46) and (45). Finally, in this section, we conclude that the Ricci flows of Finsler metrics can be extracted from Ricci flows of Riemannian metrics by corresponding metric ansatz, nonholonomic constraints and deformations of linear connections, all derived canonically from fundamental Finsler functions.

### Ricci flows of regular lagrange systems

There were elaborated different approaches to geometric mechanics. We follow those related to formulations in terms of almost symplectic geometry [27] and generalized Finsler and Lagrange geometry [43]. We note that Lagrange-Finsler spaces can be equivalently modelled as almost Kähler geometries (see formulas (17) defining the almost complex structure) and, which is important for applications of the theory of anholonomic Ricci flows, modelled as nonholonomic Riemann manifolds, see Conclusion 3.1.

For regular mechanical systems, we can formulate the problem: Which fundamental Lagrange function  $L(\tau) = L(\tau, x^i, y^j)$  from a class of Lagrangians parametrized by  $\tau \in (0, \tau_0)$  will define the evolution of Lagrange geometry, from a theory of Ricci flows? The aim of this section is to present the key results solving this problem.

Following the formulas from Result 6.1 and the methods elaborated in previous section 5.1, when  $F^2(\tau) \rightarrow L(\tau)$ ;  ${}^F h_{ij}(\tau) \rightarrow {}^L g_{ij}(\tau)$ , see (40) and (21);  ${}^c N_i^a(\tau) \rightarrow {}^L N_i^a(\tau)$ , see (41) and (19);  ${}^c g(\tau) \rightarrow {}^L g(\tau)$ , see (42) and (20);  ${}^F \hat{G}_{\beta\gamma}^\alpha(\tau) \rightarrow {}^L \hat{G}_{\beta\gamma}^\alpha(\tau)$ , see (45) and (21), where all values labeled by up-left “L” are canonically defined by  $L(\tau)$ , we prove (generalizations of Theorem 5.2 and Corollary 5.1):

**Theorem 5.3:** The Lagrange-Ricci flows for regular Lagrangians  $L(\tau)$  can be extracted from usual Ricci flows of Riemannian metrics parametrized as

$${}^L g_{\alpha\beta}(\tau) = \begin{bmatrix} {}^L g_{ij} + {}^L N_i^a {}^L N_j^b {}^L g_{ab} & {}^L N_j^e {}^L g_{ae} \\ {}^L N_i^e {}^L g_{be} & {}^L g_{ab} \end{bmatrix}$$

and satisfying the equations (for instance, normalized)

$$\frac{\partial}{\partial \tau} {}^L g_{\alpha\beta} = -2 {}^L R_{\alpha\beta} + \frac{2r}{5} {}^L g_{\alpha\beta},$$

$${}^L g_{\alpha\beta}(\tau=0) = {}^L g_{\alpha\beta}^{[0]}(u),$$

where  ${}^L R_{\alpha\beta}(\tau)$  are the Ricci tensors constructed from the Levi Civita connections of metrics  ${}^L g_{\alpha\beta}(\tau)$ .

The Lagrange-Ricci flows are characterized by the evolutions of preferred N-adapted frames (see Proposition 2.2):

**Corollary 5.2:** The evolution, for all time  $\tau \in (0, \tau_0)$ , of preferred frames on a Lagrange space

$${}^L e_a(\tau) = {}^L e_a^\alpha(\tau, u) \partial_\alpha$$

is defined by the coefficients

$${}^L e_\alpha^a(\tau, u) = \begin{bmatrix} {}^L e_i^i(\tau, u) & {}^L N_i^b(\tau, u) {}^L e_b^a(\tau, u) \\ 0 & {}^L e_a^a(\tau, u) \end{bmatrix},$$

with  ${}^L g_{ij}(\tau) = {}^L e_i^i(\tau, u) {}^L e_j^j(\tau, u) \eta_{ij}$ , where  $\eta_{ij} = \text{diag}[\pm 1, \dots, \pm 1]$  establish the signature of  ${}^L g_{\alpha\beta}^{[0]}(u)$ , is given by equations

$$\frac{\partial}{\partial \tau} {}^L e_\alpha^a = {}^L g^{\alpha\beta} {}^L R_{\beta\gamma} {}^L e_\alpha^\gamma.$$

We conclude that the Ricci flows of Lagrange metrics can be extracted from Ricci flows of Riemannian metrics by corresponding metric ansatz, nonholonomic constraints and deformations of linear connections, all derived canonically for regular Lagrange functions.

### Generalized Lagrange–Ricci flows

We have the result that any mechanical system with a regular Lagrangian  $L(x, y)$  can be geometrized canonically in terms of nonholonomic Riemann geometry, see Conclusion 3.1, and for certain conditions such configurations generate exact solutions of the gravitational field equations in the Einstein gravity and/or its string/gauge generalizations, see Result 6.2 and Theorem 6.1. In other words, for any symmetric tensor  $g_{ij} = \delta_i^a \delta_j^b h_{ab}(x, y)$  on a manifold  $V^{n+n}$  we can generate a Lagrange space model, see section 3.1. The aim of this section is to show how we can construct nonholonomic Ricci flows with effective Lagrangians starting from an arbitrary family  $g_{ij}(\tau) = \delta_i^a \delta_j^b h_{ab}(\tau, x, y)$ .

The values  $h_{ab}(\tau)$  of constant signature defines a family of absolute energies  $\varepsilon(\tau) = h_{ab}(\tau, x, y) y^a y^b$  and d–metrics of type (18),

$${}^e g(\tau) = h_{ij}(\tau, x, y) (e^i \otimes e^j + {}^e e^i(\tau) \otimes {}^e e^j(\tau)),$$

$${}^e e^i(\tau) = dy^i + {}^e N_i^a(\tau, x, y) dx^a, \tag{49}$$

where the  $\tau$ –parametrized N–connection coefficients

$${}^e N_i^a(\tau, x, y) = \frac{\partial {}^e G^a(\tau)}{\partial y^i} \tag{50}$$

with  ${}^e G^a(\tau) = \frac{1}{2} {}^e \tilde{h}^{ab}(\tau) \left( y^k \frac{\partial^2 \varepsilon(\tau)}{\partial y^b \partial x^k} - \delta_b^k \frac{\partial \varepsilon(\tau)}{\partial x^k} \right)$  are defined for nondegenerated Hessians

$${}^e \tilde{h}_{ab}(\tau) = \frac{1}{2} \frac{\partial^2 \varepsilon(\tau)}{\partial y^a \partial y^b}, \tag{51}$$

when  $\det |\tilde{h}| \neq 0$ .

For any fixed value of  $\tau$ , the existence of fundamental geometric objects (49), (50) and (51) follows from Theorem 3.1. Similarly, the Theorem 3.2 states a modelling by  $h_{ab}(\tau)$  of families of Lagrange spaces enabled with canonical N–connections  ${}^e N(\tau)$ , almost complex structure  ${}^e F(\tau)$ , d–metrics  ${}^e g(\tau)$  and d–connections  ${}^e \hat{D}(\tau)$  structures defined respectively by effective regular Lagrangians  ${}^e L(\tau, x, y) = \sqrt{|\varepsilon(\tau, x, y)|}$  and their Hessians  ${}^e \tilde{h}_{ab}(\tau, x, y)$  (51). The results of previous section 5.3 can be reformulated in the form (with proofs being similar for those for Theorem 5.2 and Corollary 5.1, but with  ${}^e L$  instead of  $F^2$  and  ${}^e N_i^a$  instead of  ${}^e N_i^a, \dots$ ):

**Theorem 5.4:** *The generalized Lagrange–Ricci flows for regular effective Lagrangians  ${}^e L(\tau)$  derived from a family of symmetric tensors  $h_{ab}(\tau, x, y)$  can be extracted from usual Ricci flows of Riemannian metrics parametrized in the form*

$${}^e g_{\alpha\beta}(\tau) = \begin{bmatrix} {}^e \tilde{h}_{ij} + {}^e N_i^a {}^e N_j^b {}^e \tilde{h}_{ab} & {}^e N_j^e {}^e \tilde{h}_{ae} \\ {}^e N_i^e {}^e \tilde{h}_{be} & {}^e \tilde{h}_{ab} \end{bmatrix}$$

and satisfying the equations (for instance, normalized)

$$\frac{\partial}{\partial \tau} {}^e g_{\alpha\beta} = -2 {}^e R_{\alpha\beta} + \frac{2r}{5} {}^e g_{\alpha\beta},$$

$${}^e g_{\alpha\beta}{}_{\tau=0} = {}^e g_{\alpha\beta}^{[0]}(u),$$

where  ${}^e R_{\alpha\beta}(\tau)$  are the Ricci tensors constructed from the Levi Civita connections of metrics  ${}^e g_{\alpha\beta}(\tau)$ .

The evolutions of preferred N–adapted frames (see Proposition 2.2) defined by generalized Lagrange–Ricci flows is stated by

**Corollary 5.3:** *The evolution, for all time  $\tau \in (0, \tau_0)$ , of preferred frames on an effective Lagrange space*

$${}^e e_\alpha(\tau) = {}^e e_\alpha^a(\tau, u) \partial_a$$

is defined by the coefficients

$${}^e e_\alpha^a(\tau, u) = \begin{bmatrix} {}^e e_i^i(\tau, u) & {}^e N_i^b(\tau, u) {}^e e_b^a(\tau, u) \\ 0 & {}^e e_a^a(\tau, u) \end{bmatrix},$$

with  ${}^e \tilde{h}_{ij}(\tau) = {}^e e_i^i(\tau, u) {}^e e_j^j(\tau, u) \eta_{ij}$ , where  $\eta_{ij} = \text{diag}[\pm 1, \dots, \pm 1]$  establish the signature of  ${}^e g_{\alpha\beta}^{[0]}(u)$ , is given by equations

$$\frac{\partial}{\partial \tau} {}^e e_\alpha^a = {}^e g^{\alpha\beta} {}^e R_{\beta\gamma} {}^e e_\alpha^\gamma.$$

In Introduction and Part I of the monograph [34], it was proven that certain types of gravitational interactions can be modelled as generalized Lagrange–Finsler geometries and inversely, certain classes of generalized Finsler geometries can be modelled on N–anholonomic manifolds, even as exact solutions of gravitational field equations. The approach elaborated by Romanian geometers and physicists [33–35] originates from Vranceanu G and Horac Z works [36,37] on nonholonomic manifolds and mechanical systems, see a review of results and recent developments explained by Bejancu [38]. Recently, there were proposed various models of “analogous gravity”, a review [53], which do not apply the methods of Finsler geometry and the formalism of nonlinear connections.

### Local Geometry Of N–Anholonomic Manifolds

Let us consider a metric structure on N–anholonomic manifold  $V$ ,

$$\tilde{g} = \underline{g}_{\alpha\beta}(u) du^\alpha \otimes du^\beta \tag{1}$$

defined with respect to a local coordinate basis  $du^\alpha = (dx^i, dy^a)$  by coefficients

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \tag{2}$$

Such a metric (2) is generic off–diagonal, i.e. it can not be diagonalized by coordinate transforms if  $N_i^a(u)$  are any general functions. The condition (13), for  $hX \rightarrow e_i$  and  $vY \rightarrow e_a$ , transforms into

$$\tilde{g}(e_i, e_a) = 0, \text{ equivalently } \underline{g}_{ia} - N_i^b h_{ab} = 0, \tag{3}$$

where  $\underline{g}_{ia} \doteq g(\partial/\partial x^i, \partial/\partial y^a)$ , which allows us to define in a unique form the coefficients  $N_i^b = h^{ab} \underline{g}_{ia}$  where  $h^{ab}$  is inverse to  $h_{ab}$ . We can write the metric  $\tilde{g}$  with ansatz (2) in equivalent form, as a d–metric (14) adapted to a N–connection structure, see Definition 2.8, if we

define  $g_{ij} \doteq g(e_i, e_j)$  and  $h_{ab} \doteq g(e_a, e_b)$  and consider the vielbeins  $e_\alpha$  and  $e^a$  to be respectively of type (5) and (6).

We can say that the metric  $\tilde{g}$  (1) is equivalently transformed into (14) by performing a frame (vielbein) transform

$$e_\alpha = e_\alpha^a \partial_a \text{ and } e^\beta = e_\beta^b du^b.$$

with coefficients

$$e_\alpha^a(u) = \begin{bmatrix} e_i^a(u) & N_i^b(u)e_b^a(u) \\ 0 & e_a^a(u) \end{bmatrix}, \quad (4)$$

$$e_\beta^b(u) = \begin{bmatrix} e_i^b(u) & -N_i^b(u)e_i^k(u) \\ 0 & e_a^b(u) \end{bmatrix}, \quad (5)$$

being linear on  $N_i^a$ . We can consider that a N-anholonomic manifold  $V$  provided with metric structure  $\tilde{g}$  (1) (equivalently, with d-metric (14)) is a special type of a manifold provided with a global splitting into conventional "horizontal" and "vertical" subspaces (2) induced by the "off-diagonal" terms  $N_i^b(u)$  and a prescribed type of nonholonomic frame structure (7).

The N-adapted components  $G_{\alpha\beta}^a$  of a d-connection  $D_\alpha = (e_\alpha \lrcorner D)$ , where " $\lrcorner$ " denotes the interior product, are defined by the equations

$$D_\alpha e_\beta = G_{\alpha\beta}^\gamma e_\gamma, \text{ or } G_{\alpha\beta}^\gamma(u) = (D_\alpha e_\beta) \lrcorner e^\gamma. \quad (6)$$

The N-adapted splitting into h- and v-covariant derivatives is stated by

$$hD = \{D_k = (L_{jk}^i, L_{bk}^a)\}, \text{ and } vD = \{D_c = (C_{jc}^i, C_{bc}^a)\},$$

where, by definition,

$$L_{jk}^i = (D_k e_j) \lrcorner e^i, \quad L_{bk}^a = (D_k e_b) \lrcorner e^a, \quad C_{jc}^i = (D_c e_j) \lrcorner e^i, \quad C_{bc}^a = (D_c e_b) \lrcorner e^a.$$

The components  $G_{\alpha\beta}^\gamma = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a)$  completely define a d-connection  $D$  on a N-anholonomic manifold  $V$ .

The simplest way to perform computations with d-connections is to use N-adapted differential forms like

$$G_\beta^\alpha = G_{\beta\gamma}^\alpha e^\gamma \quad (7)$$

with the coefficients defined with respect to (6) and (5). For instance, torsion can be computed in the form

$$T^\alpha \doteq D e^\alpha = d e^\alpha + \Gamma_{\beta\gamma}^\alpha e^\beta \wedge e^\gamma. \quad (8)$$

Locally it is characterized by (N-adapted) d-torsion coefficients

$$T_{jk}^i = L_{jk}^i - L_{kj}^i, T_{ja}^i = -T_{aj}^i = C_{ja}^i, T_{ji}^a = \Omega_{ji}^a, \quad (9)$$

$$T_{bi}^a = -T_{ib}^a = \frac{\partial N_i^a}{\partial y^b} - L_{bi}^a, T_{bc}^a = C_{bc}^a - C_{cb}^a.$$

By a straightforward d-form calculus, we can find the N-adapted components of the curvature

$$R_\beta^\alpha \doteq D G_\beta^\alpha = G_{\beta\gamma}^\alpha - G_{\gamma\beta}^\alpha \wedge G_\gamma^\alpha = R_{\beta\gamma\delta}^\alpha e^\gamma \wedge e^\delta, \quad (10)$$

of a d-connection  $D$ , i.e. the d-curvatures from Theorem 2.2:

$$R_{hjk}^i = e_k L_{hj}^i - e_j L_{hk}^i + L_{hj}^m L_{mk}^i - L_{hk}^m L_{mj}^i - C_{ha}^i \Omega_{ij}^a, \quad (11)$$

$$R_{bjk}^a = e_k L_{bj}^a - e_j L_{bk}^a + L_{bj}^c L_{ck}^a - L_{bk}^c L_{cj}^a - C_{bc}^a \Omega_{kj}^c,$$

$$R_{jka}^i = e_a L_{jk}^i - D_k C_{ja}^i + C_{jb}^i T_{ka}^b,$$

$$R_{bka}^c = e_a L_{bk}^c - D_k C_{ba}^c + C_{bd}^c T_{ka}^d,$$

$$R_{jbc}^i = e_c C_{jb}^i - e_b C_{jc}^i + C_{jb}^h C_{hc}^i - C_{jc}^h C_{hb}^i,$$

$$R_{bcd}^a = e_d C_{bc}^a - e_c C_{bd}^a + C_{bc}^e C_{ed}^a - C_{bd}^e C_{ec}^a.$$

Contracting respectively the components of (11), one proves that the Ricci tensor  $R_{\alpha\beta} \doteq R_{\alpha\beta}^\gamma$  is characterized by h- v-components, i.e. d-tensors,

$$R_j^i \doteq R_{jk}^k, R_a^i \doteq -R_{ka}^k, R_{ai} \doteq R_{ab}^b, R_{ab} \doteq R_{abc}^c. \quad (12)$$

It should be noted that this tensor is not symmetric for arbitrary d-connections  $D$ . The scalar curvature of a d-connection is

$${}^s R \doteq g^{\alpha\beta} R_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab}, \quad (13)$$

defined by a sum the h- and v-components of (12) and d-metric (14).

The Einstein tensor is defined and computed in standard form

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} {}^s R \quad (14)$$

There is a minimal extension of the Levi Civita connection  $\nabla$  to a canonical d-connection  $\tilde{D}$  which is defined only by a metric  $\tilde{g}$  is metric compatible, with  $\tilde{T}^i_{jk} = 0$  and  $\tilde{T}^a_{bc} = 0$  but  $\tilde{T}^i_{ja}, \tilde{T}^a_{ji}$  and  $\tilde{T}^a_{bi}$  are not zero, see (9). The coefficient  $\tilde{G}_{\alpha\beta}^\gamma = (\tilde{L}_{jk}^i, \tilde{L}_{bk}^a, \tilde{C}_{jc}^i, \tilde{C}_{bc}^a)$  of this connection, with respect to the N-adapted frames, are defined :

$$\tilde{L}_{jk}^i = \frac{1}{2} g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \quad (15)$$

$$\tilde{L}_{bk}^a = e_b (N_k^a) + \frac{1}{2} h^{ac} (e_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d),$$

$$\tilde{C}_{jc}^i = \frac{1}{2} g^{ik} e_c g_{jk}, \quad \tilde{C}_{bc}^a = \frac{1}{2} h^{ad} (e_c h_{bd} + e_d h_{cd} - e_d h_{bc}).$$

The Levi Civita linear connection  $\nabla = \{\Gamma_{\beta\gamma}^\alpha\}$ , uniquely defined by the conditions  $\nabla T = 0$  and  $\nabla g = 0$ , is not adapted to the distribution (2). Let us parametrize the coefficients in the form

$${}^i \Gamma_{\beta\gamma}^\alpha = ({}^i L_{jk}^\alpha, {}^i L_{jk}^\alpha, {}^i L_{bk}^\alpha, {}^i L_{bk}^\alpha, {}^i C_{jc}^\alpha, {}^i C_{jc}^\alpha, {}^i C_{bc}^\alpha, {}^i C_{bc}^\alpha),$$

where

$$\nabla_{e_k} (e_j) = {}^i L_{jk}^\alpha e_i + {}^i L_{jk}^\alpha e_a, \nabla_{e_k} (e_b) = {}^i L_{bk}^\alpha e_i + {}^i L_{bk}^\alpha e_a,$$

$$\nabla_{e_b} (e_j) = {}^i C_{jb}^\alpha e_i + {}^i C_{jb}^\alpha e_a, \nabla_{e_c} (e_b) = {}^i C_{bc}^\alpha e_i + {}^i C_{bc}^\alpha e_a.$$

A straightforward calculus<sup>1</sup> shows that the coefficients of the Levi-Civita connection can be expressed in the form

$${}^i L_{jk}^\alpha = L_{jk}^\alpha, {}^i L_{jk}^\alpha = -C_{jb}^\alpha g_{ik} h^{ab} - \frac{1}{2} \Omega_{jk}^\alpha, \quad (16)$$

$${}^i L_{bk}^\alpha = \frac{1}{2} \Omega_{jk}^\alpha h_{cb} g^{ji} - \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C_{hb}^\alpha,$$

$${}^i L_{bk}^\alpha = L_{bk}^\alpha + \frac{1}{2} (\delta_c^a \delta_d^b + h_{cd} h^{ab}) [L_{bk}^c - e_b (N_k^c)],$$

$${}^i C_{kb}^\alpha = C_{kb}^\alpha + \frac{1}{2} \Omega_{jk}^\alpha h_{cb} g^{ji} + \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C_{hb}^\alpha,$$

$${}^i C_{jb}^\alpha = -\frac{1}{2} (\delta_c^a \delta_b^d - h_{cb} h^{ad}) [L_{aj}^c - e_d (N_j^c)], C_{bc}^a = C_{bc}^a,$$

$${}^i C_{ab}^\alpha = -\frac{g^{ij}}{2} \{ [L_{aj}^c - e_a (N_j^c)] h_{cb} + [L_{bj}^c - e_b (N_j^c)] h_{ca} \},$$

where  $\Omega_{jk}^\alpha$  are computed as in formula (4). For certain considerations, it is convenient to express

<sup>1</sup>Such results were originally considered by R. Miron and M. Anastasiei for vector bundles provided with N-connection and metric structures, see Ref. [?]. Similar proofs hold true for any nonholonomic manifold provided with a prescribed N-connection structures.



$$\Gamma_{\alpha\beta}^\gamma = \hat{G}_{\alpha\beta}^\gamma + Z_{\alpha\beta}^\gamma \quad (17)$$

where the explicit components of distortion tensor  $Z_{\alpha\beta}^\gamma$  can be defined by comparing the formulas (16) and (15):

$$\begin{aligned} Z_{jk}^i &= 0, Z_{jk}^a = -C_{jb}^i g_{ik} h^{ab} - \frac{1}{2} \Omega_{jk}^a, \\ Z_{bk}^i &= \frac{1}{2} \Omega_{jk}^i h_{cb} g^{ji} - \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C_{hb}^i, \\ Z_{bk}^a &= \frac{1}{2} (\delta_c^a \delta_d^b + h_{cd} h^{ab}) [L_{bk}^c - e_b(N_k^c)], \\ Z_{kb}^i &= \frac{1}{2} \Omega_{jk}^i h_{cb} g^{ji} + \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C_{hb}^i, \\ Z_{jb}^a &= -\frac{1}{2} (\delta_c^a \delta_b^d - h_{cb} h^{ad}) [L_{dj}^c - e_d(N_j^c)], Z_{bc}^a = 0, \\ Z_{ab}^i &= -\frac{g^{ij}}{2} \{ [L_{aj}^c - e_a(N_j^c)] h_{cb} + [L_{bj}^c - e_b(N_j^c)] h_{ca} \}. \end{aligned} \quad (18)$$

It should be emphasized that all components of  $\Gamma_{\alpha\beta}^\gamma, \hat{G}_{\alpha\beta}^\gamma$  and  $Z_{\alpha\beta}^\gamma$  are defined by the coefficients of d-metric (14) and N-connection (1), or equivalently by the coefficients of the corresponding generic off-diagonal metric (2).

For a differentiable Lagrangian  $L(x, y)$ , i.e. a fundamental Lagrange function, is defined by a map  $L : (x, y) \in TM \rightarrow L(x, y) \in \mathbb{R}$  of class  $C^\infty$  on  $\widetilde{TM} = TM \setminus \{0\}$  and continuous on the null section  $0 : M \rightarrow TM$  of  $\pi$  [34] the following results are derived:

## Result

1. The Euler-Lagrange equations

$$\frac{d}{d\tau} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0$$

where  $y^i = \frac{dx^i}{d\zeta}$  for  $(\ )$  depending on parameter  $\zeta$ , are equivalent to the “nonlinear” geodesic equations

$$\frac{d^2 x^i}{d\tau^2} + 2G^i(x^k, \frac{dx^k}{d\zeta}) = 0$$

defining paths of a canonical semispray

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$$

where

$$2G^i(x, y) = \frac{1}{2} L_{g^{ij}} \left( \frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right)$$

with  $L_{g^{ij}}$  being inverse to (21).

2. There exists on  $V \simeq \widetilde{TM}$  a canonical N-connection

$${}^L N_j^i = \frac{\partial G^i(x, y)}{\partial y^j} \quad (19)$$

defined by the fundamental Lagrange function  $L(x, y)$ , which prescribes nonholonomic frame structures of type (5) and (6),  ${}^L e_\nu = (e_i, e_a)$  and  ${}^L e^\mu = (e^i, e^a)$ .

3. There is a canonical metric structure

$${}^L g = g_{ij}(x, y) e^i \otimes e^j + g_{ij}(x, y) e^i \otimes e^j \quad (20)$$

constructed as a Sasaki type lift from  $M$  for  $g_{ij}(x, y) = {}^L g_{ij}(x, y)$ , see (21).

4. There is a unique metrical and, in this case, torsionless canonical

d-connection  ${}^L D = (hD, vD)$  with the nontrivial coefficients with respect to  ${}^L e_\nu$  and  ${}^L e^\mu$  parametrized respectively  ${}^L \hat{G}_{\beta\gamma}^\alpha = (\hat{L}_{jk}^i, \hat{C}_{bc}^a)$ , for  $\hat{L}_{jk}^i = \frac{1}{2} g^{ih} (e_k g_{jh} + e_j g_{kh} - e_h g_{jk})$ ,  $\hat{C}_{jk}^i = \frac{1}{2} g^{ih} (e_k g_{jh} + e_j g_{kh} - e_h g_{jk})$  (21)

defining the generalized Christoffel symbols, where (for simplicity, we omitted the left up labels  $(L)$  for N-adapted bases).

We conclude that any regular Lagrange mechanics can be geometrized as a nonholonomic Riemann manifold  $V$  equipped with canonical N-connection (19) and adapted d-connection (21) and d-metric structures (20) all induced by a  $L(x, y)$ .

Let us show how N-anholonomic configurations can be defined in gravity theories explained by Vacaru [33,34]. In this case, it is convenient to work on a general manifold  $V, \dim V = n + m$  enabled with a global N-connection structure, instead of the tangent bundle  $\widetilde{TM}$ .

**Result 6.2:** Various classes of vacuum and nonvacuum exact solutions of (22) parametrized by generic off-diagonal metrics, nonholonomic vielbeins and Levi Civita or non-Riemannian connections in Einstein and extra dimension gravity models define explicit examples of N-anholonomic Einstein-Cartan (in particular, Einstein) spaces.

It should be noted that a subclass of N-anholonomic Einstein spaces was related to generic off-diagonal solutions in general relativity by such nonholonomic constraints when  $\text{Ric}(\hat{D}) = \text{Ric}(\nabla)$  even  $\hat{D} \neq \nabla$  where  $\hat{D}$  is the canonical d-connection and  $\nabla$  is the Levi-Civita connection.

A direction in modern gravity is connected to analogous gravity models when certain gravitational effects and, for instance, black hole configurations are modelled by optical and acoustic media. Following our approach on geometric unification of gravity and Lagrange regular mechanics in terms of N-anholonomic spaces, one holds

**Theorem 6.1:** A Lagrange (Finsler) space can be canonically modelled as an exact solution of the Einstein equations (22) on a N-anholonomic Riemann-Cartan space if and only if the canonical N-connection  ${}^L N$  ( ${}^L N$ ), d-metric  ${}^L g$  ( ${}^L g$ ) and d-connection  ${}^L \hat{D}$  ( ${}^L \hat{D}$ ) structures defined by the corresponding fundamental Lagrange function  $L(x, y)$  (Finsler function  $F(x, y)$ ) satisfy the gravitational field equations for certain physically reasonable sources

**Proof.** It can be performed in local form by considering the Einstein tensor (14) defined by the  ${}^L N$  ( ${}^L N$ ) in the form (19) and  ${}^L g$  ( ${}^L g$ ) in the form (20) inducing the canonical d-connection  ${}^L \hat{D}$  ( ${}^L \hat{D}$ ). For certain zero or nonzero  $\tau$ , such N-anholonomic configurations may be defined by exact solutions of the Einstein equations for a d-connection structure [53].

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