Journal of Generalized Lie Theory and Applications

Abramova and Liivapuu, J Generalized Lie Theory Appl 2015, S1
http://dx.doi.org/10.4172/1736-4337.S1-006

Research Article — Open Access

N-Complex, Graded q-Differential Algebra and N-Connection on Modules

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Abstract

It is well known that given a differential module E with a differential d we can measure the non-exactness of this differential module by its homologies which are based on the key relation d^2=0. This relation is a basis for several important structures in modern mathematics and theoretical physics to point out only two of them which are the theory of de Rham cohomologies on smooth manifolds and the apparatus of BRST-quantization in gauge field theories.

Keywords: N-differential module; N-cochain complex; Cohomologies of N-cochain complex; Graded q-differential algebra; Algebra of connection form; N-connection form; Covariant N-differential; N-curvature Form; N-connection on module; Curvature of N-connection; N-connection consistent with Hermitian structure of module

Introduction

An idea to generalize the concept of a differential module and to elaborate the corresponding structures by giving the mentioned above key relation $d^2 = 0$ a more general form $d^N = 0,\ N \geq 2$ seems to be very natural. This idea has been proposed and studied in the series of papers [1-4] giving rise to the structures such as differential N-complex, N-cochain complex, generalized cohomologies of N-cochain complex and graded q-differential algebra, where q is a primitive Nth root of unity. The graded q-differential algebra can be viewed as a generalization of a graded differential algebra and can be used to develop the applications of structures based on relation $d^N = 0$ in non-commutative geometry. It is well known that the concepts of connection and its curvature are basic elements of the theory of fiber bundles and play an important role not only in a modern differential geometry but also in a modern theoretical physics namely in a gauge field theory. The development of a theory of connections has been closely related to the development of a theoretical physics. The advent of supersymmetric field theories in the 70's of the previous century gave rise to interest towards $\mathbb{Z}_2$-graded structures which became known in theoretical physics under the name of super structures. This direction of development has led to a concept of super connection which appeared in [5]. The emergence of non-commutative geometry in the 80's of the previous century was a powerful spur to the development of a theory of connections on modules [6-9]. A basic concept used in a theory of connections on modules is a concept of a graded differential algebra. Consequently using a concept of a graded q-differential algebra we can develop a more general theory of connections. The aim of this paper is to give a survey of several algebraic structures based on the relation $d^N = 0$ and to show the possible applications of these structures in non-commutative geometry.

In the Section 2 we give a short overview of N-structures such as N-differential module, generalized homologies of the N-differential module, N-cochain complex. At the end of this section we give an example of a positive N-complex proposed in [3]. In the Section 3 making use of the notion of N-complex we give a generalization of a concept of a graded differential algebra, which will be referred to as a graded q-differential algebra and was introduced and studied [2,4]. As it was investigated [10,11] it is possible to construct a realization of N-differential calculus of exterior forms on a smooth finite dimensional manifold. A construction of an analog of exterior calculus with N-differential d on a non-commutative space was proposed [12]. The space we consider is a reduced quantum plane. Our approach is based on a generalized Clifford algebra with four generators equipped with the N-differential d. We study the structure of the algebra of q-differential forms on a reduce quantum plane and show that the first order calculus induced by the differential d is a first order calculus. In the Section 4 we introduce a generalization of a concept of a connection form by means of a notion of graded q-differential algebra and covariant N-differential which can be viewed as analogs of a connection form in a graded differential algebra described [13]. We begin this section with an algebra of polynomials in the variables $\mathfrak{d}, a_1, a_2, \ldots$ and prove the power expansion formula for an n th power of the operator $\hat{\mathfrak{d}}_a = \hat{\mathfrak{d}} + a_1$. Applying this formula we show that the N th power of the covariant N-differential is the operator of multiplication by an element $F_A^{(N)}$, which we then define as the N-curvature form of a N-connection form A [14]. In the Section 5 we make use of a notion of a graded q-differential algebra to construct a N-connection on module introduced in [14-17], which may be viewed as a generalization of a classical connection. We define the notions such as dual N-connection, N-connection consistent with the Hermitian structure of a module. Assuming module to be a finitely generated free module we study the local structure of N-connection, define a curvature of N-connection.

N-complex

Let K be a commutative ring with a unit and E be a left K -module.

Definition 2.1: *A module E endowed with an endomorphism d: E→ E is said to be a differential module with differential d if endomorphism d satisfies $d^2 = 0$. In the case when K is a field a differential module E will be referred to as a differential vector space.*

It is well known that the property d^2=0 of differential d: E→E implies $\mathrm{Im}\,d \subset \mathrm{Ker}\,d \subset E$, and this relation allows to measure the non-exactness of the sequence $E \xrightarrow{d} E \xrightarrow{d} E$ considering the homology $H(E)$=Ker d/Im d of a differential module E which has the structure of a quotient module.

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Received July 21, 2015; **Accepted** August 03, 2015; **Published** August 31, 2015

Citation: Abramov V, Liivapuu O (2015) N-Complex, Graded q-Differential Algebra and N-Connection on Modules. J Generalized Lie Theory Appl S1: 006. doi:10.4172/1736-4337.S1-006

Citation: Abramov V, Liivapuu O (2015) *N*-Complex, Graded *q*-Differential Algebra and *N*-Connection on Modules. J Generalized Lie Theory Appl S1: 006. doi:10.4172/1736-4337.S1-006

Let *E, F* be differential modules respectively with differentials $d: E \to E, d': F \to F$.

Definition 2.2: *A homomorphism of modules* $\varphi: E \to F$ *is said to be a homomorphism of differential modules E, F if it satisfies* $\phi \circ d = d' \circ \phi$.

It is easy to show that if φ is a homomorphism of differential modules then $\phi(\mathrm{Im} d) \subset \mathrm{Im} d'$, $\phi(Ker\ d) \subset Ker\ d'$ which means that a homomorphism of differential modules φ induces the homomorphism of homologies $\phi_*: H(E) \to H(F)$ of differential modules *E, F*. It can be proved [4]

Proposition 2.3: For an exact sequence of differential modules

$$0 \to E \xrightarrow{\phi} F \xrightarrow{\psi} G \to 0,$$

i.e. $Im f = Ker \psi, Ker f = 0, Im \psi = G$, *there exists a homomorphism of homologies* $\partial: H(G) \to H(E)$ *such that the triangle*

$$\begin{array}{ccc} & H(F) & \\ \varphi^* \nearrow & & \searrow \psi_* \\ H(E) & \xrightarrow[\partial]{} & H(G) \end{array} \qquad (1)$$

is exact.

Let us remind that a $\mathbb{Z}$-graded module *E* is a direct sum of sub modules $E^i \subset E$ labeled by integers $i \in \mathbb{Z}$, i.e. $E = \oplus_{i\in\mathbb{Z}} E^i$, where an element $u \in E^i$ is said to be a homogeneous element of $\mathbb{Z}$-graded module of degree *i*.

Definition 2.4: *A differential module E with differential d is said to be a cochain complex if E is a* $\mathbb{Z}$*-graded module* $E = \oplus_{i\in\mathbb{Z}} E^i$ *and its differential d has degree 1 with respect to a* $\mathbb{Z}$*-graded structure of E, i.e.* $d: E^i \to E^{i+1}$.

A $\mathbb{Z}$-graded structure of a cochain complex $E = \oplus_{i\in\mathbb{Z}} E^i$ induces the $\mathbb{Z}$-graded structure of its homology *H(E)*, i.e. $H(E) = \oplus_{i\in\mathbb{Z}} H^i(E)$, where $H^i(E) = \mathrm{Ker} d \cap E^i / \mathrm{Im} d \cap E^i$. A homology *H(E)* is usually referred to as a cohomology of a cochain complex *E*. It can be proved

Proposition 2.5: *For an exact sequence of cochain complexes*

$$0 \to E \xrightarrow{\phi} F \xrightarrow{\psi} G \to 0$$

there exists a homomorphism of homologies $\partial: H^i(G) \to H^{i+1}(E)$ *such that the sequence*

$$\ldots \xrightarrow{\partial} H^n(E) \xrightarrow{\phi_*} H^n(F) \xrightarrow{\psi_*} H^n(G) \xrightarrow{\partial} H^{n+1}(E) \to \ldots$$

is exact.

Let us remind that a positive cochain complex is $E = \oplus_{i\in\mathbb{Z}} E^i$ with $E^i = 0$ for $i < 0$ or equivalently $E = \oplus_{i\in\mathbb{N}} E^i$. The well-known way of constructing this kind of complexes is based on a notion of a pre-cosimplicial module.

Definition 2.6: *A sequence of modules* $(E^n)_n \in_{\mathbb{N}} = (E^0, E^1, \ldots, E^n, \ldots)$ *together with homomorphisms* $f_0, f_1, \ldots, f_{n+1}$ *such that each* f_i *determines a sequence*

$$E^0 \xrightarrow{f_i} E^1 \xrightarrow{f_i} E^2 \ldots \xrightarrow{f_i} E^n \xrightarrow{f_i} E^{n+1} \xrightarrow{f_i} \ldots,$$

where $f_i: E^n \to E^{n+1}$ *is a homomorphism of modules called a coface homomorphism, is said to be a pre-cosimplicial module if*

$$f_j \circ f_i = f_i \circ f_{j-1}, \qquad (2)$$

where $i, j \in \{0, 1, \ldots, n+1\}$ *and* $i < j$.

Given a pre-cosimplicial module $((E^n)_{n\in\mathbb{N}}; f_0, f_1, \ldots, f_{n+1})$ we can construct a positive cochain complex *E* with differential *d* by setting $E = \oplus_n \in_{\mathbb{N}} E^n$ and $d = \sum_{i=0}^{n+1} (-1)^i f_i$. It is easy to check that $d^2 = 0$ follows immediately from (2). We shall call this positive cochain complex the pre-cosimplicial complex and its differential *d* the simplicial differential.

It is clear that the basic relation which determines the structure of a differential module and allows to define the important characteristics of a non-exactness of a differential module such as homologies is $d^2 = 0$. This basic relation can be given a more general form $d^N = 0$, where *N* is an integer greater or equal to two. This generalization was proposed and considered in the framework of non-commutative geometry almost at the same time in the papers [1-3]. Now we turn to a theory which can be developed if one replaces the basic relation $d^2 = 0$ of a differential module by a more general one $d^N = 0$.

Let $N \geq 2$ be a positive integer and *E* be a left *K*-module.

Definition 2.7: *A left K-module E is said to be a N-differential module with N-differential d if* $d: E \to E$ *is an endomorphism of E satisfying* $d^N = 0$. *In the case when K is a field a N-differential module E will be referred to as a N-differential vector space.*

Obviously *N*-differential module can be viewed as a generalization of a concept of differential module to any integer $N \geq 2$. Now for each integer $1 \leq m \leq N - 1$ we can define the sub modules $Z_m(E) = \mathrm{Ker}(d^m) \subset E$ and $B_m(E) = \mathrm{Im}(d^{N-m}) \subset E$. From the relation $d^N = 0$ it follows that $B_m(E) \subset Z_m(E)$.

Definition 2.8: *The quotient modules* $H_m(E) := Z_m(E) / B_m(E)$ *are called the generalized homology of the N-differential module E.*

It can be proved [4] that in the case of a *N*-differential module *E* we have the statement analogous to Proposition 2.3 which is

Proposition 2.9: *If* $0 \to E \xrightarrow{\phi} F \xrightarrow{\psi} G \to 0$ *is an exact sequence of N-differential modules then for every m* $\in \{1,2,\ldots,N-1\}$ *there are homomorphisms* $\partial: H_m(G) \to H_{N-m}(E)$ *such that for every n* $\in \{1,2,\ldots,N-1\}$ *the following hexagons of homomorphisms are exact.*

$$\begin{array}{ccccc} & H_{(n)}(F) & \xrightarrow{\psi_*} & H_{(n)}(G) & \\ \varphi^* \nearrow & & & & \searrow \partial \\ H_{(n)}(E) & & & & H_{(N-n)}(E) \\ \partial \nwarrow & & & & \swarrow \varphi^* \\ & H_{(N-n)}(G) & \xleftarrow{\psi_*} & H_{(N-n)}(F) & \end{array}$$

Definition 2.10: *A N-differential module E with N-differential d is said to be a N-cochain complex or simply a N-complex if E is a* $\mathbb{Z}$*-graded module* $E = \oplus_{k\in\mathbb{Z}} E^k$ *and its N-differential d has degree 1, i.e.* $d: E^k \to E^{k+1}$.

If *E* is an *N*-complex then its cohomologies $H_m(E)$ are $\mathbb{Z}$-graded modules, i.e. $H_m(E) = \oplus_{n\in\mathbb{Z}} H_m^n(E)$, where

$$H_m^n(E) = \mathrm{Ker}(d^m : E^n \to E^{n+m}) / d^{N-m}(E^{n+m-N}).$$

It should be noted that many notions related to *N*-complexes depend only on the underlying $\mathbb{Z}_N$-graduation, and for this purpose we define a $\mathbb{Z}_N$-complex to be a $\mathbb{Z}_N$-graded *N*-differential module with *N*-differential *d* of degree 1. We end this section by giving an example of a positive *N*-complex [3]. Let $q \in \mathbb{C}$ be a *Nth* root of unity and $((E^n)_{n\in\mathbb{N}}; f_0, f_1, \ldots, f_{n+1})$ be a pre-cosimplicial module, where $f_0, f_1, \ldots, f_{n+1}$ are coface homomorphisms. This pre-cosimplicial module induces the positive *N*-complexes if we construct then positively graded or $_{\mathbb{N}}$-graded module

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$E=\oplus_{n\in\mathbb{N}}E^n$ and equip it with the endomorphism $d_m: E\to E$, $i\le n+1$ of degree 1 defining it by

$$d_m=\delta_{m+1}+q^{n-m+1}\sum_{r=0}^{m}(-1)^r f_{n-m+r+1}, \tag{3}$$

where

$$\delta_m=\sum_{i=0}^{n-m+1}q^i f_i. \tag{4}$$

For $m=0,1$ we get

$$d_0=\sum_{i=0}^{n+1}q^i f_i,\quad d_1=\sum_{i=0}^{n}q^i f_i-q^n f_{n+1}. \tag{5}$$

It can be verified that $d_m: E^n\to E^{n+1}$, $d_m^N=0$ which means that d_m is the N-differential, and we obtain the N-complexes (E,d_0), (E,d_1),…, (E,d_{n+1}).

Graded q-differential algebra

It is well known that if we endow a cochain complex with an additional structure which is an associative unital law of multiplication in such a way that a differential of a cochain complex satisfies the graded Leibniz rule with respect to this multiplication then we obtain a concept of a graded differential algebra. Analogously if we equip a N-complex with an associative unital law of multiplication in such a way that a N-differential of a N-complex satisfies the q-graded Leibniz rule, where q is a primitive Nth root of unity, then we obtain a generalization of a graded differential algebra which is called graded q-differential algebra. The aims of this section are to remind a concept of a graded differential algebra, to give a definition of a graded q-differential algebra, to show in what a way an associative graded algebra can be endowed with a structure of a graded q-differential algebra and finally to show a relation of a graded q-differential algebra with noncommutative geometry by giving a construction based on a generalized Clifford algebra which leads to a realization of a graded q-differential algebra by analogs of differential forms on a reduced quantum plane.

Let $\Omega=\oplus_{n\in\mathbb{N}}\Omega^n$ be an associative unital graded $\mathbb{C}$-algebra. The subspace of elements of degree zero Ω^0 Ω is the subalgebra of Ω and we denote this subalgebra by $\mathfrak{A}$, i.e. $\mathfrak{A}=\Omega^0$. Any subspace $\Omega^k\subset\Omega$ of elements of degree $k\in\mathbb{Z}$ is the $(\mathfrak{A},\mathfrak{A})$ -bimodule.

Definition 3.1: *A graded differential algebra (GDA) is an associative unital graded $\mathbb{C}$-algebra equipped with a linear mapping d of degree 1 such that the sequence*

$$\ldots\xrightarrow{d}\Omega^{k-1}\xrightarrow{d}\Omega^k\xrightarrow{d}\Omega^{k+1}\xrightarrow{d}\ldots$$

is a cochain complex, and d is an antiderivation, i.e. it satisfies the graded Leibniz rule

$$d(\omega\cdot\theta)=d\omega\cdot\theta+(-1)^k\omega\cdot d\theta,$$

where $\omega\in\Omega^k$, $\theta\in\Omega$.

Let us mention that if Ω is a GDA then Ker d is the graded unital subalgebra of Ω whereas Im d is the graded two-sided ideal of Ker d. Hence the cohomology $H(\Omega)$ is the unital associative graded algebra.

Definition 3.2: *Let* A *be an associative unital algebra and $\mathfrak{M}$ be a $(\mathfrak{A},\mathfrak{A})$ - bimodule. The triple $(\mathfrak{A},\mathrm{d},\mathfrak{M})$ is said to be a first order differential calculus over an algebra $\mathfrak{A}$ if $d:\mathfrak{A}\to\mathfrak{M}$ is a homomorphism of $(\mathfrak{A},\mathfrak{A})$ -bimodules satisfying the Leibniz rule*

$d(ab)=d(a)b+ad(b)$,

where $a,b\in\mathfrak{A}$. A homomorphism d is referred to as a differential of a first order differential calculus.

If Ω is a GDA with differential d then clearly the triple $\mathfrak{A}$, d, Ω_q is a first order differential calculus over the algebra $\mathfrak{A}$.

Definition 3.3: *If Ω is a GDA with differential d and $\mathfrak{A}$ is its subalgebra of elements of degree zero then the triple $(\mathfrak{A}, d, \Omega_q)$ will be referred to as the differential calculus over the algebra $\mathfrak{A}$.*

Now we are going to describe a generalization of a GDA introduced and studied [2,4] by giving the basic condition $d^2=0$ of a GDA a more general form $d^N=0$, $N\ge 2$. In the following of this section, K is the field of complex numbers $\mathbb{C}$ and q is a primitive N *th* root of unity, where $N\ge 2$.

Definition 3.4: *A graded q-differential algebra (q-GDA) is an associative unital -graded ($\mathbb{Z}_N$-graded) $\mathbb{C}$-algebra $\Omega_q=\oplus_{k\in\mathbb{Z}}\Omega_q^k$ endowed with a linear mapping d of degree one such that the sequence*

$$\ldots\xrightarrow{d}\Omega_q^{k-1}\xrightarrow{d}\Omega_q^k\xrightarrow{d}\Omega_q^{k+1}\xrightarrow{d}\ldots$$

is a N-complex with N-differential d satisfying the graded q-Leibniz rule

$$d(\omega\cdot\theta)=d\omega\cdot\theta+q^k\omega\cdot d\theta, \tag{6}$$

where $\omega\in\Omega_q^k$, $\theta\in\Omega_q$.

Obviously the subspace of elements of degree zero $\mathfrak{A}=\Omega_q^0\subset\Omega_q$ is the subalgebra of a q-GDA Ω_q. Clearly the triple $(\mathfrak{A},d,\Omega_q^1)$ is the first order differential calculus over the algebra A. The triple (A, d, Ω_q) will be referred to as a N-differential calculus over the algebra A.

Let us remind that a graded q-center of an associative unital graded $\mathbb{C}$-algebra $\mathcal{A}=\oplus_{k\in\mathbb{Z}}\mathcal{A}^k$ is the graded subspace $Z(\mathcal{A})=\oplus_{k\in\mathbb{Z}}Z^k(\mathcal{A})$ of A generated by the homogeneous elements $v\in\mathcal{A}^k$, where $k\in\mathbb{Z}$, satisfying

$vw=q^{km}wv$ for any $w\in\mathcal{A}^m$, $m\in\mathbb{Z}$. It is easy to verify that the graded q-center $Z(\mathcal{A})$ is the graded subalgebra of $\mathcal{A}$.

Definition 3.5: *A graded q-derivation of degree $k\in\mathbb{Z}$ of A is a homogeneous linear mapping $D:\mathcal{A}\to\mathcal{A}^{m+k}$ satisfying the graded q-Leibniz rule*

$$D(vw)=D(v)w+q^{km}vD(w),$$

where $v\in A^m$, $w\in A$.

It is well known that given a homogeneous element $v\in\mathcal{A}^k$ of $\mathcal{A}$ we can associate to it the graded q-derivation of degree k by means of a graded q-commutator $[\,,\,]_q$ as follows

$$v\mapsto \mathrm{ad}_q(v)w=[v,w]_q=vw-q^{km}wv, \tag{7}$$

where $w\in\mathcal{A}^m$. Clearly $\mathrm{ad}_q(v):\mathcal{A}\to\mathcal{A}^{m+k}$, and the graded q-derivation $\mathrm{ad}_q(v)$ associated to a homogeneous element v is referred to as an inner graded q-derivation of degree k of A. Let $v\in A^1$ be a homogeneous element of degree one and $d_v=\mathrm{ad}_q(v):\mathcal{A}\to\mathcal{A}^{m+k}$ be the inner graded q-derivation associated to v. The kth power of this inner graded q-derivation can be expanded as follows:

Lemma 3.6: *For any integer $k\ge 2$ it holds*

$$d_v^k w=\sum_{i=0}^{k}p_i^{(k)}v^{k-i}wv^i, \tag{8}$$

where w is a homogeneous element of $\mathcal{A}$ and

$$p_i^{(k)}=(-1)^i q^{|w|_i}\frac{[k]_q!}{[i]_q![k-i]_q!}=(-1)^i q^{|w|_i}\begin{bmatrix}k\\ i\end{bmatrix}. \tag{9}$$

$$|w|_i = i\,|w| + \frac{i(i-1)}{2} \tag{10}$$

Making use of this Lemma we can prove [18]

Theorem 3.7: *If $\mathcal{A}$ is an associative unital $\mathbb{Z}$-graded $\mathbb{C}$-algebra and $v \in A^1$ is an element of degree one satisfying $v^N \in Z(A)$, where $N \geq 2$, then the inner graded q-derivation $d_v = \mathrm{ad}_q(v) : \mathcal{A}^k \to \mathcal{A}^{k+1}$ associated to v is the N-differential of an algebra A. Hence an algebra A regarded with respect to the N-differential d_v is the q-GDA.*

We can apply this theorem to a generalized Clifford algebra in order to construct a *q*-GDA.

Definition 3.8: *A generalized Clifford algebra is an algebra over the complex numbers $\mathbb{C}$ generated by a set of canonical generators $\{\xi_1, \xi_2, \ldots, \xi_p\}$ which are subjected to the relations*

$$\xi_i \xi_j = q^{\mathrm{sg}(j-i)} \xi_j \xi_i, \quad \xi_i^N = 1, \quad i, j = 1, 2, \ldots, p \tag{11}$$

where sg(k) is the usual sign function, and 1 *is the identity element of an algebra.*

Since our aim in this section is to construct the analogs of differential forms on a reduced quantum plane we shall consider the generalized Clifford algebra with four generators, i.e. *p* = *4*. Let us denote the generalized Clifford algebra with four generators by $\mathfrak{C}_N$. We split the set of generators of this algebra into two pairs ξ_1, ξ_3 and ξ_2, ξ_4 denoting the generators of the first pair by *x, y*, i.e. $x = \xi_1$, $y = \xi_3$, and the generators of the second pair by *u, v*, i.e. $u = \xi_2$, $v = \xi_4$. From (11) it follows

$$xy = q\, yx, \qquad x^N = y^N = 1, \tag{12}$$

$$xu = q\, ux, \qquad xv = q\, vx, \tag{13}$$

$$yu = q^{-1} uy, \quad yv = q\, vy, \tag{14}$$

$$uv = q\, vu, \qquad u^N = v^N = 1. \tag{15}$$

Let $\mathfrak{P}_N$ be the subalgebra of the algebra $\mathfrak{C}_N$ generated by *x, y*. The relations (12) show that the generators *x, y* can be interpreted as coordinates of a reduced quantum plane [19], and correspondingly the subalgebra $\mathfrak{P}_N$ can be interpreted as the algebra of polynomial functions on a reduced quantum plane. Keeping in mind this interpretation we shall call an element $f \in \mathfrak{P}_N$ a polynomial function on a reduced quantum plane. Our next step is to construct a *N*-differential calculus on a reduced quantum plane. For this purpose we take the generalized Clifford algebra $\mathfrak{C}_N$ and applying the Theorem 3.7 equip it with the structure of a *q*-GDA in such a way that the subalgebra $\mathfrak{P}_N$ of polynomial functions will be the subalgebra of elements of degree zero of this *q*-GDA. We define the $\mathbb{Z}_N$-graded structure of $\mathfrak{C}_N$ as follows: we assign the degree zero to the generators x, y and the degree one to the generators *u, v*. Hence denoting the degree of an element *w* by |*w*| we can write

$$|x| = |y| = \bar{0}, \; |u| = |v| = \bar{1}, \tag{16}$$

where $\bar{0}, \bar{1}$ are the residue classes of 0, 1 modulo *N*. As usual the degree of any monomial composed of generators *x, y, u, v* equals to the sum of degrees of its components. Obviously $\mathfrak{C}_N = \oplus_{i \in \mathbb{Z}_N} \mathfrak{C}_N^i$, where $\mathfrak{C}_N^i$ is the subspace of homogeneous elements of degree *i*, and $\mathfrak{C}_N^{\bar{0}} = \mathfrak{P}_N$.

Proposition 3.9: *For any λ,μ ∈ $\mathbb{C}$ an element $\omega = \lambda u + \mu\, v \in \mathfrak{C}_N^{\bar{1}}$ satisfies $\omega^N \in Z(\mathfrak{C}_N)$.*

Proof: For any $2 \leq k \leq N$ we have 1

$$\omega^k = \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix}_q \lambda^{k-l} \mu^l v^l u^{k-l}. \tag{17}$$

Since *q* is a primitive *N* th root of unity we have $\begin{bmatrix} N \\ l \end{bmatrix}_q = 0$ for $1 \leq l \leq N-1$.

Thus taking *k* = *N* in (17) we obtain

$$\omega^N = \lambda^N u^N + \mu^N v^N = (\lambda^N + \mu^N) 1 \in Z(\mathfrak{C}_N).$$

Now it follows from the Theorem 3.7 that the inner graded *q*-derivation $d_\omega = ad_q(\omega)$ associated to an element $\omega = \lambda u + \mu v \in \mathfrak{C}_N^{\bar{1}}$ is the *N* –differential of the $\mathbb{Z}_N$-graded algebra $\mathfrak{C}_N$. Hence the generalized Clifford algebra with four generators $\mathfrak{C}_N$ regarded with respect to the $\mathbb{Z}_N$-graded structure defined by (16) and to *N*-differential d_ω is the *q*-GDA, and the triple $(\mathfrak{P}_N, d_\omega, \mathfrak{C}_N)$ is the *N*-differential calculus over the algebra of polynomial functions $\mathfrak{P}_N$ of the reduced quantum plane. In what follows of this section we consider ω as a fixed element, and keeping this in mind we simplify the symbol for *N*- differential d_ω omitting ω and writing *d* instead of d_ω:

It should be mention that the structure of the *q*-GDA $\mathfrak{C}_N$ depends on a choice of element ω, and consequently the numbers *λ*, *μ* can be considered as the parameters of this structure.

The *N*-differential d induces the differentials of coordinates $x \xrightarrow{d} dx$, $y \xrightarrow{d} dy$ and later in this section we will show that any element of degree $k > \bar{0}$ of the *q*-GDA $\mathfrak{C}_N$ can be expressed in terms of the differentials of coordinates *dx, dy*. Since *N*-differential *d* satisfies $d^N = 0$ it can be viewed as an analog of exterior differentiation of higher order on the reduced quantum plane. Proceeding with this analogy we will call the *q*-GDA $\mathfrak{C}_N$ the algebra of q-differential forms on the reduced quantum plane and its elements of degree $k > \bar{0}$ expressed in terms of the differentials *dx,dy* the *q*-differential *k*-forms. Let us mention that in the particular case of *N* = 2 the above construction yields an analog of exterior calculus with exterior differential *d* satisfying $d^2 = 0$ on the reduced quantum plane with coordinates *x, y* which obey

$$x^2 = y^2 = 1, \quad xy = -yx.$$

In analogy with exterior calculus we shall call a *q*-differential form *θ* a *m*-closed *q*-differential form, where $1 \leq m \leq N-1$, if $d^m \theta = 0$, and we shall call a *q*-differential n-form *θ* a *q*-differential l-exact form if there exists a *q*-differential *(n – l)*-form *ρ* such that $d^l = \theta$. It follows from $d^N = 0$ that each *q*-differential *(N – m)*-exact form is *m*-closed.

Our next aim is to describe the structure of the algebra of *q*-differential forms in terms of the differentials of coordinate's *dx, dy*. We begin with the first order differential calculus which is the triple $(\mathfrak{P}_N, d, \mathfrak{C}_N^{\bar{1}})$, where $\mathfrak{P}_N$ is the algebra of polynomial functions, *d* is the *N*-differential and $\mathfrak{C}_N^{\bar{1}}$ is the $(\mathfrak{P}_N, \mathfrak{P}_N)$-bimodule of *q*-differential 1-forms. Evidently $dx, dy \in \mathfrak{C}_N^{\bar{1}}$. Let us express the differentials *dx, dy* in terms of the generators of $\mathfrak{C}_N$. It is worth mentioning that in what follows we shall use the structure of the right $\mathfrak{P}_N$-module of $\mathfrak{C}_N^{\bar{1}}$ have the relations to write *q*-differential 1-forms in terms of differentials. We

$$x\omega = q\, \omega x, \qquad y\omega = \omega_1 y, \tag{18}$$

where $\omega = \lambda u + \mu v$, $\omega_1 = q^{-1} \lambda u + q \mu v$. Using these relations we obtain

$$dx = [\omega, x]_q = (1-q)\, \omega x, \quad dy = [\omega, y]_q = (\omega - \omega_1) y, \tag{19}$$

where

$$\omega - \omega_1 = (1 - q^{-1}) \lambda u + (1-q) \mu v \tag{20}$$

Citation: Abramov V, Liivapuu O (2015) *N*-Complex, Graded *q*-Differential Algebra and *N*-Connection on Modules. J Generalized Lie Theory Appl S1: 006. doi:10.4172/1736-4337.S1-006

It is evident that the right $\mathfrak{P}_N$ -module $\mathfrak{C}_N^{\bar{1}}$ is a free right module and $\{u, v\}$ is the basis for this right module.

Proposition 3.10: *For any integer $N \geq 3$ the right $\mathfrak{P}_N$ -module of q-differential 1-forms $\mathfrak{C}_N^{\bar{1}}$ is freely generated by the differentials of coordinates dx, dy.*

Proof: Let $f, h \in \mathfrak{P}_N$ be polynomial functions on the reduced quantum plane. Using (19),(20) and the fact that $\{u, v\}$ is the basis for the right $\mathfrak{P}_N$ –module $\mathfrak{C}_N^{\bar{1}}$ we can show that the equality $dx\,f + dy\,h = 0$ is equivalent to the system of equations

$$(1-q)xf + (1-q^{-1})yh = 0,$$

$$xf + yh = 0.$$

Multiplying the second equation by $q - 1$ and adding it to the first equation we obtain $(q-q^{-1})h = 0$. As $q-q^{-1} \neq 0$ for $N \geq 3$ we conclude that $h = 0$. In the same way we show that $f = 0$, and this proves that the differentials dx, dy are linearly independent q-differential 1-forms.

In order to prove that any q-differential 1-form is a linear combination of differentials we find the transition matrix from the basis $\{u, v\}$ to the basis $\{dx, dy\}$. Let us denote the algebra of square matrices of order 2 whose entries are the elements of the algebra $\mathfrak{P}_N$ by $\mathfrak{M}_2(\mathfrak{P}_N)$. Then $(dx\ dy) = (u\ v) \bullet A$, where $A \in \mathfrak{M}_2(\mathfrak{P}_N)$, and from (19) we find

$$A = \begin{pmatrix} (1-q)\lambda x & (1-q^{-1})\lambda y \\ (1-q)\mu x & (1-q)\mu y \end{pmatrix},$$

It should be noted that the transition matrix depends on the coordinates of a point of a reduced quantum plane. As the coordinates of a reduced quantum plane obey the relations $x^N = 1$, $y^N = 1$ they are invertible elements of the algebra $\mathfrak{P}_N$ and $x^{-1} = x^{N-1}$, $y^{-1} = y^{N-1}$. If $N \geq 3$ then the matrix A is an invertible matrix and

$$A^{-1} = \frac{1}{(1-q^2)} \begin{pmatrix} \lambda^{-1} q x^{-1} & \mu^{-1} x^{-1} \\ -\lambda^{-1} q y^{-1} & \mu^{-1} q y^{-1} \end{pmatrix}.$$

Consequently we have

$$u = \frac{q}{(1-q^2)\lambda}(dx x^{-1} - dy y^{-1}), \tag{21}$$

$$v = \frac{1}{(1-q^2)\mu}(dx x^{-1} + q dy y^{-1}). \tag{22}$$

Now any q-differential 1-form $\theta = u\,f + v\,h \in \mathfrak{C}_N^{\bar{1}}$, where $f, h \in \mathfrak{P}_N$, can be expressed in terms of the differentials, and this ends the proof.

From the Proposition 3.10 it follows that the first order differential calculus $\left(\mathfrak{P}_N, d, \mathfrak{C}_N^{\bar{1}}\right)$ is the coordinate calculus with coordinate differential d [20]. If we have a coordinate calculus then a coordinate differential of this calculus induces the partial derivatives which satisfy the twisted Leibniz rule. The second term in the right hand side of the twisted Leibniz rule for a partial derivative depends on the homomorphism from the algebra $\mathfrak{P}_N$ to the algebra of (2×2)-matrices with entries from $\mathfrak{P}_N$, which we denote by $\mathfrak{M}_2(\mathfrak{P}_N)$, and this homomorphism is determined by the relation between the right and left module structures of the bimodule of q-differential 1-forms $\mathfrak{C}_N^{\bar{1}}$. Let us denote this homomorphism by $R: \mathfrak{P}_N \to \mathfrak{M}_2(\mathfrak{P}_N)$, i.e. for $f \in \mathfrak{P}_N$ we have

$$R(f) = \begin{pmatrix} r_{11}(f) & r_{12}(f) \\ r_{21}(f) & r_{22}(f) \end{pmatrix}$$

and

$$f\,dx = dx\, r_{11}(f) + dy\, r_{21}(f), \tag{24}$$

$$f\,dy = dx\, r_{12}(f) + dy\, r_{22}(f). \tag{25}$$

Since $\mathfrak{P}_N$ is the algebra of polynomial functions in two variables x, y which are subjected to the commutation relation $xy = q\,yx$ and the relations $x^N = 1$, $y^n = 1$ it is sufficient to find the explicit formula for the homomorphism R applied to coordinates x, y. Taking firstly $f = x$ and then $f = y$ in (24), (25) we find

$$R(x) = \begin{pmatrix} qx & 0 \\ 0 & q^2 x \end{pmatrix}, \quad R(y) = \begin{pmatrix} q^{-1} y & q^{-1}(q-1)x^{-1}y^2 \\ (q-1)x & q^{-1}(q^2-q+1)y \end{pmatrix}. \tag{26}$$

Putting the entries of these matrices into the relations (24), (25) we get

$$x dx = q\, dx x, \quad y dx = \frac{1}{q} dx y + (q-1)\, dy x, \tag{27}$$

$$x dy = q^2\, dy x, \quad y dy = \frac{q-1}{q} dx x^{-1} y^2 + \frac{q^2-q+1}{q} dy y. \tag{28}$$

The partial derivatives induced by the N-differential d are defined by

$$df = dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y}.$$

It can be shown [20] that the partial derivatives satisfy the twisted Leibniz rule

$$\frac{\partial (fh)}{\partial x} = \frac{\partial f}{\partial x} h + r_{11}(f) \frac{\partial f}{\partial x} + r_{12}(f) \frac{\partial f}{\partial x},$$

$$\frac{\partial (fh)}{\partial y} = h + r_{21}(f) \frac{\partial f}{\partial y} + r_{22}(f) \frac{\partial f}{\partial y}.$$

Using the twisted Leibniz rule and (26) we calculate

$$\frac{\partial x^k}{\partial x} = [k]_q x^{k-1}, \qquad \frac{\partial x^k}{\partial y} = 0$$

$$\frac{\partial y^l}{\partial x} = \frac{[l]_q (q^{l-1}-1)}{q^{l-1}(q+1)} x^{-1} y^l, \quad \frac{\partial y^l}{\partial y} = \frac{[l]_q (q^l+1)}{q^{l-1}(q+1)} y^{l-1}$$

Using these formulae we can calculate the partial derivatives of any polynomial function $f \in \mathfrak{P}_N$ which can be written as $f = \sum_{k,l=0}^{N-1} \zeta_{kl} x^k y^l$. For instant the derivative with respect to x of f is

$$\frac{\partial f}{\partial x} = \sum_{k,l=0}^{N-1} ([k]_q + q^{k-l+1}[l]_q \frac{q^{l-1}-1}{q+1})\, \zeta_{kl} x^{k-1} y^l.$$

We remind that the set of generators $\{x,u,y,v\}$ of the generalized Clifford algebra $\mathfrak{C}_N$ has been split into two parts, where the first part $\{x, y\}$ generates the algebra of polynomial functions $\mathfrak{P}_N$, and the second part $\{u, v\}$ generates the N-differential calculus $\left(\mathfrak{P}_N, d, \mathfrak{C}_N\right)$. We already have proved that any 1- form $\theta = u\,f + v\,h$ can be uniquely expressed in terms of the differentials dx, dy. Now if we take x, y, dx, dy as the generators for the algebra $\mathfrak{C}_N$ then we may divide the algebraic relations between the new generators into three parts. The first part contains the relations

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$$xy = q\,yx, \quad x^N = y^N = 1,$$

which determine the structure of the algebra $\mathfrak{P}_N$ of polynomial functions on the reduced quantum plane. The second part consists of the relations (27),(28) between coordinates *x, y* and their differentials

$$xdx = q\,dx\,x, \quad ydx = \frac{1}{q}\,dx\,y + (q-1)\,dy\,x,$$

$$xdy = q^2\,dy\,x, \quad ydy = \frac{q-1}{q}\,dx\,x^{-1}y^2 + \frac{q^2-q+1}{q}\,dy\,y.$$

The third part will contain the relations between the differentials *dx, dxy*. It is evident that the commutation relation $uv = qvu$ will give us a quadratic relation for the differentials *dx, dy*, and the relations $u^N = v^N = 1$will give rise to two relations of degree *N* with respect to differentials.

Proposition 3.11: *The commutation relation uv = qvu for the generators u, v written in terms of the differentials dx, dy takes on the form*

$$dxdy = \gamma_1\,dydx + \gamma_2\,(dx)^2x^{-1}y + \gamma_3\,(dy)^2y^{-1}x, \tag{29}$$

where

$$\gamma_1 = \frac{1+q^4}{2}, \quad \gamma_2 = \frac{q-1}{2q}, \quad \gamma_3 = \frac{(1-q^3)q}{2}. \tag{30}$$

This proposition can be proved by straightforward computation with the help of the formulae (21), (22) and the relations (27), (28).

The relation (29) allows us to choose the ordered set of monomials *B*,

where

$$\mathcal{B} = \{(dx)^k, dy(dx)^{k-1}, \ldots, (dy)^{k-1}dx, (dy)^k\}, \quad 2 \le k \le N-1,$$

as the basis for the right $\mathfrak{P}_N$-module of *q*-differential *k*-forms $\mathfrak{C}_N^k$. For example the right $\mathfrak{P}_N$-module of *q*-differential 2-forms $\mathfrak{C}_N^2$ is spanned by the monomials $(dx)^2, dydx, (dy)^2$. Hence any *q*-differential k-form θ on the reduced quantum plane can be uniquely expressed as follows:

$$\theta = (dx)^k f_0 + dy(dx)^{k-1} f_1 + \ldots (dy)^{k-1}dx\,f_{k-1} + (dy)^k f_k,$$

where $f_0, f_1, \ldots, f_k \in \mathfrak{P}_N$ are polynomial functions. The peculiar property of the *q*-analog of exterior calculus on a reduced quantum plane is an appearance of the higher order differentials of coordinates, and this gives us a possibility to construct one more basis for the module of 2-forms. Indeed as $dk \neq 0$for *k* running integers from 2 to $N-1$ we have the set of higher order differentials of coordinates d^2x, $d^2y, \ldots, d^{N-1}x$, $d^{N-1}y$, and we can use these higher order differentials to construct a basis for $\mathfrak{C}_N^k$. In this paper we shall describe a case of the module of *q*-differential 2-forms. The elements ω, ω_1, $\omega-\omega_1$ can be written as *q*-differential 1-forms as follows:

$$\omega = \frac{1}{1-q}dx\,x^{-1}, \quad \omega_1 = \frac{1}{1-q}dx\,x^{-1} - dy\,y^{-1}, \quad \omega - \omega_1 = dy\,y^{-1}. \tag{31}$$

Differentiating ω we obtain

$$d\omega = [\omega, \omega]_q = \frac{1}{q(1-q)}(dx)^2x^{-2}. \tag{32}$$

where $d\omega$ is the *q*-differential 2-form. Now we can write the second order differential d^2x as a *q*-differential 2-form as follows:

$$d^2x = (1-q)d(\omega x) = (1-q)(d(\omega)x + q\,\omega dx) = \frac{1+q}{q}(dx)^2x^{-1}. \tag{33}$$

Expressing the second order differential d^2y in terms of $(dx)^2$, *dy dx*, $(dy)^2$ we prove the following proposition:

Proposition 3.12: *The second order differential d^2y can be written as follows:*

$$d^2y = \frac{1}{1-q}\left(\frac{1}{q^2}dxdy - q\,dydx\right)x^{-1}. \tag{34}$$

Proof: As $dy = (\omega - \omega_1)y$, $y\omega = \omega_1 y$ we have

$$d^2y = [\omega, (\omega-\omega_1)y] = (\omega(\omega-\omega_1) - q\,(\omega-\omega_1)\omega_1)y.$$

Now applying the formula (31) and the multiplication rules (27), (28) we get the expression (34).

The Propositions 3.11, 3.12 show that we can change the basis $\mathcal{B}_2$ by the basis $\mathcal{B}_2'$ in the right $\mathfrak{P}_N$-module of *q*-differential 2-forms $\mathfrak{C}_N^2$, where

$$\mathcal{B}_2 = \{(dx)^2, dydx, (dy)^2\}, \quad \mathcal{B}'_2 = \{d^2x, dydx, d^2y\}.$$

We point out that from the Proposition 3.12 it follows that the relation (29) can be written by means of the second order differential d^2y in a more symmetric form

$$dxdy = q^3\,dydx + q^2(1-q)\,d^2y\,x. \tag{35}$$

We end this section by considering the structure of algebra of *q*-differential forms on a reduced quantum plane at cubic root of unity, i.e in the case of $N = 3$. In this case we have the algebra of *q*-differentials forms on a reduced quantum plane at cubic root of unity with differential *d* satisfying $d^3 = 0$. It can be verified that now the right hand sides of the formulae (33), (34) are the 1-closed *q*-differential 2-forms, i.e.

$$d\left((dx)^2x^{-1}\right) = 0, \qquad d\left(\frac{1}{q^2}dxdy\,x^{-1} - q\,dydx\,x^{-1}\right) = 0.$$

The last term in the relation (29) vanishes because *q* is a primitive cube root of unity and satisfies $q^3 - 1 = 0$. Making use of the relation $1 + q + q^2 = 0$ we can write the coefficient γ_1 as follows:

$$\gamma_1 = \frac{1+q^4}{2} = \frac{1+q}{2} = -\frac{q^2}{2}.$$

Hence the relation for the differentials (29) with respect to the basis $\mathcal{B}_2$ takes on the form

$$dxdy = -\frac{q^2}{2}\,dydx + \frac{q^2(1-q)}{2}(dx)^2x^2y. \tag{36}$$

The above relation is similar to the anticommutativity of differentials in a classical algebra of differential forms on a plane and meaning this analogy we can say that the algebra of *q*-differential forms on a reduced quantum plane at cubic root of unity with differential satisfying $d^3 = 0$ may be viewed as a deformation of the classical algebra of differential forms with parameter $\frac{q^2}{2}$ and the additional term proportional to the form $(dx)^2x^2y$.

N-connection form and its curvature

In this section we propose a generalization of a concept of connection form by means of a notion of *q*-GDA [15-17]. We begin this section with an algebra of polynomials in two variables which we will use later in this section to prove propositions describing the structure of the curvature of a connection form.

Let $\mathfrak{d}$, $a_1, a_2, \ldots, a_n, \ldots$ be the set of variables.

Definition 4.1: *The algebra of polynomials* $\mathfrak{P}[\mathfrak{d}, a]$ *with coefficients in* $\mathbb{C}$ *generated by the variables* $\mathfrak{d}$, $a_1, a_2, \ldots, a_n, \ldots$ *is said to be the algebra of connection form if the variables are subjected to the relations*

$$\mathfrak{d}a_i = qa_i\mathfrak{d} + a_{i+1}, \qquad i = 1,2,\ldots \tag{37}$$

where $q \neq 0$ *is a complex number.*

Let $\mathfrak{P}[\mathfrak{d}]$ be the subalgebra of $\mathfrak{P}[\mathfrak{d}, a]$ generated by the variable $\mathfrak{d}$ and $\mathfrak{P}[\mathfrak{d}]$ be the subalgebra of $\mathfrak{P}[\mathfrak{d}, a]$ freely generated by the variables $a_1, a_2, \ldots, a_n, \ldots$. We define the $\mathbb{N}$-graduation of the algebra of connection form $\mathfrak{P}[\mathfrak{d}, a]$ by assigning degree one to the generator $\mathfrak{d}$ and degree i to a generator a_i. It should be mentioned that this $_{\mathbb{N}}$-graded structure of $\mathfrak{P}[\mathfrak{d}, a]$ induces the $\mathbb{N}$-graded structures of the subalgebras $\mathfrak{P}[\mathfrak{d}], \mathfrak{P}[a]$.

Definition 4.2: *The algebra of connection form* $\mathfrak{P}_N[\mathfrak{d}, a]$ *is said to be the algebra of N-connection form if q is a primitive N th root of unity, and the variable* $\mathfrak{d}$ *obeys the additional relation* $\mathfrak{d}^N = 0$. *The algebra of N-connection form will be denoted by* $\mathfrak{P}_N[\mathfrak{d}, a]$.

Proposition 4.3: *The algebra of N-connection form* $\mathfrak{P}_N[\mathfrak{d}, a]$ *is an algebra of polynomials over* $\mathbb{C}$ *generated by finite set of variables* $\mathfrak{d}$ $a_1, a_2, \ldots, a_{N-1}$ *which are subjected to the relations.*

$$\begin{aligned}
&\mathfrak{d}a_1 = qa_1\mathfrak{d} + a_2,\\
&\mathfrak{d}a_2 = q^2 a_2\mathfrak{d} + a_3,\\
&\quad\ldots\\
&\mathfrak{d}a_{N-2} = q^{N-1} a_{N-2}\mathfrak{d} + a_{N-1},\\
&\mathfrak{d}a_{N-1} = a_{N-1}\mathfrak{d},\\
&\mathfrak{d}^N = 0.
\end{aligned} \tag{38}$$

Proof: It is suffice to show that in the case of the algebra of *N*-connection form we have $a_n = 0$ for any integer $n \geq N$. Indeed making use of the commutation relations (37) we can express any variable a_n, $n \geq 2$ in the terms of $\mathfrak{d}$, a_1 as polynomial

$$a_n = \sum_{k+m=n} (-1)^m q^{\frac{(m+1)m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \mathfrak{d}^k a_1 \mathfrak{d}^m. \tag{39}$$

Now assuming $n = N$ we see that the first and last terms in (39) vanish because of $\mathfrak{d}^N = 0$, other terms are zero because of vanishing of *q*-binomial coefficients provided *q* is a primitive *Nth* root of unity. Hence for any integer $n \geq N$ we have $a_n = 0$. In what follows we will denote by $\mathfrak{P}_N[a]$ the subalgebra of $\mathfrak{P}_N[\mathfrak{d}, a]$ generated by $a_1, a_2, \ldots, a_{N-1}$.

Let us consider the algebra of connection form $\mathfrak{P}_N[\mathfrak{d}, a]$ and its subalgebra $\mathfrak{P}_N[a]$ generated by the set of variables $\{a_i\}_{i \geq 1}$. Let $\xi \in \mathfrak{P}[a]$ be a polynomial in variables $a_1, a_2, \ldots, a_n, \ldots$. The commutation relations of the algebra of connection form (37) show that for any polynomial $\xi(a) \in \mathfrak{P}[a]$ the product $\mathfrak{d}\xi 9a0 \in \mathfrak{P}[\mathfrak{d}, a]$ can be represented in the form

$$\mathfrak{d}\xi(a) = \tilde{\xi}(a)\mathfrak{d} + \xi_{\mathfrak{d}}(a),$$

where $\tilde{\xi}(a), \xi_{\mathfrak{d}}(a) \in \mathfrak{P}[a]$ the uniquely determined polynomials. We define the linear operator $\hat{\mathfrak{d}} : \mathfrak{P}[a] \to \mathfrak{P}[b]$ associated to the variable $\mathfrak{d}$ by the formula

$$\hat{\mathfrak{d}}(\xi) = \xi_{\mathfrak{d}},$$

where ξ any polynomial of $\mathfrak{P}[a]$.

Theorem 4.4: *The linear operator* $\hat{\mathfrak{d}} : \mathfrak{P}[a] \to \mathfrak{P}[b]$ *is the graded q-differential of the algebra* $\mathfrak{P}[a]$, *i.e.* $\hat{\mathfrak{d}}$ *satisfies the graded q-Leibniz rule with respect to N-graded structure of* $\mathfrak{P}[a]$. *Particularly*

$$\hat{\mathfrak{d}}(1) = 0, \hat{\mathfrak{d}}(a_i) = a_{i+1}, \hat{\mathfrak{d}}(a_i a_j) = a_{i+1}a_j + q^i a_i a_{j+1}.$$

In the case of the algebra of N-connection form $\mathfrak{P}_N[\mathfrak{d}, a]$ *the graded q-differential* $\hat{\mathfrak{d}} : \mathfrak{P}[a] \to \mathfrak{P}[b]$ *is the N-differential, i.e.* $\hat{\mathfrak{d}}^N = 0$, *and the algebra* $\mathfrak{P}_N[a]$ *is the q-GDA.*

Now for any integer $n \geq 1$ we define the polynomials $p^{(n)} \in \mathfrak{P}[\mathfrak{d}, a], f_a^{(n)} \in \mathfrak{P}[a]$ and the operator $\hat{\mathfrak{d}}_a : \mathfrak{P}[a] \to \mathfrak{P}[b]$ of degree one by

$$p^{(n)} = (\mathfrak{d} + a_1)^n, \quad f_a^{(k+1)} = \hat{\mathfrak{d}}_a^k(a_1), \quad \hat{\mathfrak{d}}_a(p) = \hat{\mathfrak{d}}(p) + a_1 p, \quad \forall p \in \mathfrak{P}[a]. \tag{40}$$

For the first values of *k* the straightforward computation of polynomials $f_a^{(k)}$ by means of recurrent relation *f* $f_a^{(k+1)} = \hat{\mathfrak{d}}_a(f_a^{(k)})$ gives

$$f_a^{(2)} = a_2 + a_1^2, \tag{41}$$

$$f_a^{(3)} = a_3 + a_2 a_1 + [2]_q a_1 a_2 + a_1^3, \tag{42}$$

$$\begin{aligned}
f_a^{(4)} &= a_4 + a_3 a_1 + [3]_q a_1 a_3 + [3]_q a_2^2\\
&\quad + a_2 a_1^2 + [3]_q a_1^2 a_2 + [2]_q a_1 a_2 a_1 + a_1^4,
\end{aligned} \tag{43}$$

$$\begin{aligned}
f_a^{(5)} &= a_5 + a_4 a_1 + [4]_q a_1 a_4 + [4]_q a_3 a_2\\
&\quad + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q a_2 a_3 + a_3 a_1^2 + [3]_q a_2^2 a_1 + [4]_q a_2 a_1 a_2\\
&\quad + [2]_q [4]_q a_1 a_2^2 + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q a_1^2 a_3 + [3]_q a_1 a_3 a_1\\
&\quad + [2]_q a_1 a_2 a_1^2 + [3]_q a_1^2 a_2 a_1 + a_2 a_1^3 + [4]_q a_1^3 a_2 + a_1^5.
\end{aligned} \tag{44}$$

We associate the linear operator $\hat{p}^{(n)} : \mathfrak{P}[a] \to \mathfrak{P}[a]$ to the polynomial $p^{(n)}$ replacing the variable $\mathfrak{d}$ in $p^{(n)}$ by the associated graded *q*-differential $\hat{\mathfrak{d}}$. Evidently $\hat{p}^{(n)} = \hat{d}_a^n$ and $f_a^{(n+1)} = \hat{p}^{(n)}(a_1)$. Our aim now is to find a power expansion for polynomials $p^{(n)}$ with respect to variables $\mathfrak{d}$, $a_1, a_2, \ldots, a_n$. It is obvious that making use of the commutation relations (37) we can rearrange the factors in each summand of this expansion by removing all $\mathfrak{d}$'s to the right.

Lemma 4.5: *Each polynomial* $p^{(n)}$ *can be expanded with respect to variables of the algebra of connection form* $\mathfrak{P}[\mathfrak{d}, a]$ *as follows*

$$p^{(n)} = \sum_{k+m=n} \begin{bmatrix} n \\ k \end{bmatrix}_q f_a^{(m)} \mathfrak{d}^k = \mathfrak{d}^n + [n]_q f_a^{(1)} \mathfrak{d}^{n-1} + \ldots + [n]_q f_a^{(n-1)} \mathfrak{d} + f_a^{(n)},$$

where $f_a^{(n)} = \hat{p}^{(n-1)}(a_1)$. *In the case of the algebra of N-connection form* $\mathfrak{P}_N[\mathfrak{d}, a]$ *the operator* $\hat{p}^{(N)} = \hat{\mathfrak{d}}_a^N : \mathfrak{P}_N[a] \to \mathfrak{P}_N[a]$ *induced by the polynomial* $p^{(n)}$ *is the operator of multiplication by* $f_a^{(N)}$.

Let Ω_q be a $\mathbb{Z}_N$-graded *q*-differential algebra with *N*-differential *d*, where *q* is a *Nth* primitive root of unity, and $\mathfrak{A} = \Omega_q^0$ be the subalgebra of elements of degree zero.

Definition 4.6: *We will call an element of degree one $A = \Omega_q^1$ a N-connection form in a q-GDA Ω_q. The linear operator of degree one $d_A = d + A : \Omega_i \to \Omega_{i+1}$ will be referred to as a covariant N-differential induced by a N-connection form A.*

Since d is a N-differential which means that $d^n \neq 0$ for $1 \leq n \leq N - 1$ if we successively apply it to a N-connection form A we get the sequence of elements $A, dA, d^2A, \ldots, d^{N-1}A$, where. Let us denote by $\Omega_q[A]$ the graded subalgebra of Ω_q generated by these elements $A, dA, d^2A, \ldots, d^{N-1}A$. For any integer $n = 1, 2, \ldots, N$ we define the polynomial $F_A^{(n)} \in \Omega_q[A]$ by the formula $F_A^{(n)} \in d_A^{n-1}(A)$.

Now we will use the algebra of N-connection form $\mathfrak{P}_N[\mathfrak{d}, a]$ to study the structure of a *kth* power of the covariant N-differential d_A. Indeed it is easy to see that we can identify a N-differential d in Ω_q with the N-differential $\hat{\mathfrak{d}}$ in $\mathfrak{P}_N[a]$, a N-connection form A in Ω_q with the variable a_1 in $\mathfrak{P}_N[a]$ and $d^nA \in \Omega_q$ with $a_{n+1} \in \mathfrak{P}_N[a]$. Consequently from Theorem 4.5 we obtain

Theorem 4.7: *For any integer $1 \leq n \leq N$ the nth power of the covariant N-differential d_A can be expanded as follows*

$$(d_A)^n = \sum_{k+m=n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_A^{(m)} d^k = d^n + [n]_q F_A^{(1)} d^{n-1} + \ldots + [n]_q F_A^{(n-1)} d + F_A^{(n)},$$

where $F_A^{(n)} = (d_A)^{n-1}(A)$. Particularly if $n = N$ then the Nth power of the covariant N-differential d_A is the operator of multiplication by the element $F_A^{(N)}$ of degree zero.

The Theorem 4.7 allows us to define the curvature of a N-connection form A as follows

Definition 4.8: The N-curvature form of a N-connection form A is the element of degree zero $F_A^{(N)} \in \mathfrak{A}$.

It is easy to see that in the particular case of a graded differential algebra $(N = 2, q = -1)$ with differential d satisfying $d^2 = 0$ the above definition yields a connection form A and its curvature $F_A^{(2)} = dA + A^2$ as elements of degree respectively one and two of a graded differential algebra [13].

Proposition 4.9: *For any N-connection form A in a q-GDA Ω_q the N- curvature form $F_A^{(N)}$ satisfies the Bianchi identity*

$$dF_A^{(N)} + [A, F_A^{(N)}]_q = 0. \tag{45}$$

Proof: Indeed we have

$$\begin{aligned} dF_A^{(N)} + [A, F_A^{(N)}]_q &= dF_A^{(N)} + AF_A^{(N)} + q^N F_A^{(N)} A \\ &= (d_A) F_A^{(N)} - F_A^{(N)} A \\ &= (d_A)\big((d_A)^{N-1} A\big) - F_A^{(N)} A \\ &= (d_A)^N A - F_A^{(N)} A = F_A^{(N)} A - F_A^{(N)} A = 0. \end{aligned}$$

From (41)–(44) we obtain the expressions for N-curvature form

$$F_A^{(2)} = dA + A^2, \tag{46}$$

$$F_A^{(3)} = d^2A + dAA + [2]_q AdA + A^3, \tag{47}$$

$$\begin{aligned} F_A^{(4)} &= d^3A + (d^2A)A + [3]_q A(d^2A) + [3]_q (dA)^2 \\ &\quad + dAA^2 + [3]_q A^2 dA + [2]_q AdAA + A^4 \end{aligned}, \tag{48}$$

$$\begin{aligned} F_A^{(5)} &= d^4A + (d^3A)A + [4]_q A(d^3A) + [4]_q (d^2A)dA \\ &\quad + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q dA(d^2A) + (d^2A)A^2 + [3]_q (dA)^2 A \\ &\quad + [4]_q dAAdA + [2]_q [4]_q A(dA)^2 + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q A^2 d^2 A \\ &\quad + [3]_q Ad^2AA + [2]_q AdAA^2 + [3]_q A^2 dAA \\ &\quad + dAA^3 + [4]_q A^3 dA + A^5. \end{aligned} \tag{49}$$

N-connection on Modules

This section is devoted to the definition and the study of a connection in non-commutative geometry. Our main purpose is to describe a concept of N-connection on a module which may be considered as a generalization of a classical connection [15-17]. In our approach we generalize a concept of Ω-connection on modules proposed in reference [9]. In order to have an algebraic model of differential forms with values in vector bundle we introduce a left module over the subalgebra of elements of degree zero of a q-GDA. We study the structure of a N-connection, define its curvature and prove the Bianchi identity. We show that every projective module admits a N-connection. In a last part of this section we study the local structure of a N-connection. Assuming that left module is a finitely generated free module we introduce a matrix of N-connection and the curvature matrix. Finally we find the expressions for components of the curvature in terms of the components of a N-connection.

Let A be an unital associative $\mathbb{C}$-algebra and Ω_q be an N-differential calculus over A, i.e. Ω_q is a q-GDA with N-differential d and $\mathfrak{A} = \Omega_q^0$. Let ε be a left $\mathfrak{A}$ -module. Considering algebra Ω_q as the $(\mathfrak{A}, \mathfrak{A})$-bimodule we take the tensor product $\mathfrak{F} = \Omega_q \otimes_{\mathfrak{A}} \varepsilon$ of modules which clearly has the structure of left $\mathfrak{A}$ -module. Taking into account that an algebra Ω_q can be viewed as the direct sum of $(\mathfrak{A}, \mathfrak{A})$-bimodules Ω_q^i we can split the left $\mathfrak{A}$ -module $\mathfrak{F}$ into the direct sum of the left A -modules $\mathfrak{F}^i = \Omega_q^i \otimes_{\mathfrak{A}} \varepsilon$ [21], i.e. $\mathfrak{F} = \oplus_i \mathfrak{F}^i$, which means that $\mathfrak{F}$ inherits the graded structure of algebra Ω_q, and $\mathfrak{F}$ is the graded left $\mathfrak{A}$ -module. It is worth noting that the left $\mathfrak{A}$ submodule $\mathfrak{F}^0 = \mathfrak{A} \otimes_{\mathfrak{A}} \varepsilon$ of elements of degree zero is isomorphic to a left $\mathfrak{A}$ -module ε, where isomorphism $\varphi : \varepsilon \to \mathfrak{F}^0$ can be defined by $\varphi(\xi) = e \otimes_{\mathfrak{A}} \xi$, where e is the identity element of algebra $\mathfrak{A}$. Since an algebra Ω_q can be considered as the $(\Omega_q, \mathfrak{A})$ -bimodule the left $\mathfrak{A}$ -module $\mathfrak{F}$ can be also viewed as the left Ω_q-module [21] and we will use this structure to describe a concept of N- connection.

The tensor product $\mathfrak{F}$ is also the vector space over $\mathbb{C}$ where this vector space is the tensor product of the vector spaces Ω_q and ε. It is evident that $\mathfrak{F}$ is a graded vector space, i.e. $\mathfrak{F} = \oplus_i \mathfrak{F}^i$, where $\mathfrak{F}^i = \Omega_q^i \otimes_{\mathbb{C}} \varepsilon$. Let us denote the $\mathfrak{F}$ by $\mathfrak{L}(\mathfrak{F})$. The structure of the graded vector space of $\mathfrak{F}$ induces the structure of a graded vector space on $\mathfrak{L}(\mathfrak{F})$, and we shall denote the subspace of homogeneous linear operators of degree i by $\mathfrak{L}^i(\mathfrak{F})$.

Definition 5.1: *A N-connection on the left Ω_q-module $\mathfrak{F}$ is a linear operator $\nabla_q : \mathfrak{F} \to \mathfrak{F}$ of degree one satisfying the condition*

$$\nabla_q(\omega \otimes_{\mathfrak{A}} \xi) = d\omega \otimes_{\mathfrak{A}} \xi + q^{|\omega|} \omega \nabla_q(\xi), \tag{50}$$

where $\omega \in \Omega_q^k$, $\xi \in \varepsilon$, and $|\omega|$ is the degree of the homogeneous element of algebra Ω_q.

Similarly one can define a N-connection on right modules. If ε^R

is a right $\mathfrak{A}$-module, a N-connection on $\mathfrak{G}=\varepsilon^R\otimes_{\mathfrak{A}}\Omega_q$ is a linear map $\nabla_q:\mathfrak{G}\to\mathfrak{G}$ of degree one such that $\nabla_q(\xi\otimes_{\mathfrak{A}}\omega)=\xi\otimes_{\mathfrak{A}} d\omega+q^{\omega}\nabla_q(\xi)\omega$ for any $\xi\in\varepsilon^R$ and homogeneous element $\omega\in\Omega_q$.

Making use of the previously introduced notations we can write $\nabla_q\in\mathfrak{L}^1(\mathfrak{F})$. Let us note that if $N=2$ then $q=-1$, and in this particular case the Definition 5.1 gives us the notion of a classical connection. Hence a concept

of a N-connection can be viewed as a generalization of a classical connection. Let ε be a left A -module. The set of all homomorphisms of ε into A has the structure of the dual module of the left A -module ε and is denoted by ε^*. It is evident that ε^* is a right A -module.

Definition 5.2: *A linear map* $\nabla_q^*:\varepsilon^*\to\varepsilon^*\otimes_{\mathfrak{A}}\Omega_q^1$ *defined as follows*

$$\nabla_q^*(\eta)(\xi)=d(\eta(\xi))-\eta(\nabla_q(\xi)),$$

where $\xi\in\varepsilon$, $\eta\in\varepsilon^*$ *and* ∇_q *is a N-connection on ε, is said to be the dual connection of* ∇_q.

It is easy to verify that ∇^*_q has a structure of N-connection on the right module ε^*. Indeed, for any $f\in\mathfrak{A},\eta\in\varepsilon^*,\xi\in\varepsilon$ we have

$$\nabla_q^*(\eta f)(\xi)=d(\eta f(\xi))-(\eta f)(\nabla_q\xi)=d(\eta(\xi)f)-\eta(\nabla_q\xi)f$$
$$=d(\eta(\xi))f+\eta(\xi)\otimes_{\mathfrak{A}}df-\eta(\nabla_q\xi)f=\eta(\xi)\otimes_{\mathfrak{A}}df+\nabla_q^*(\eta(\xi))f.$$

In order to define a Hermitian structure on a right A -module ε we assume A to be a graded q-differential algebra with involution * such that the largest linear subset contained in the convex cone $C\in\mathfrak{A}$ generated by a^*a is equal to zero, i.e. $C\cap(-C)=0c$. The right A -module ε is called a Hermitian module if ε is endowed with a sesquilinear map $h:\varepsilon\times\varepsilon\to\mathfrak{A}$ which satises

$$h(\xi\omega,\xi\omega')=\omega\ h(\xi,\xi')\omega',\quad\forall\omega,\omega'\in\quad,\forall\xi,\xi'\in\varepsilon,$$
$$h(\xi,\xi)\in C,\quad\forall\xi\in\varepsilon\ \text{ and }\ h(\xi,\xi)=0\Rightarrow\xi=0.$$

We have used the convention for sesquilinear map to take the second argument to be linear, therefore we define a Hermitian structure on right modules. In a similar manner one can define a Hermitian structure on left modules.

Definition 5.3: *A N-connection* ∇_q *on a Hermitian right A -module ε is said to be consistent with a Hermitian structure of ε if it satisfies*

$$dh(\xi,\xi')=h(\nabla_q(\xi),\xi')+h(\xi,\nabla_q(\xi')).$$

where $\xi,\xi'\in\varepsilon$.

Our next aim to define a notion of a curvature of N-connection. We start with the following

Proposition 5.4: *The N-th power of any N-connection* ∇_q *is the endomorphism of degree N of the left* Ω_q*-module* $\mathfrak{F}$.

Proof. It suffices to verify that for any homogeneous element $\omega\in\Omega_q$ an endomorphism $\nabla_q\in\mathfrak{L}^1(\mathfrak{F})$ satisfies $\nabla_q^N(\omega\otimes_{\mathfrak{A}}\xi)=\omega\nabla_q^N(\xi)$. Let us expand the k-th power of ∇_q as follows

$$\nabla_q^k(\omega\otimes_{\mathfrak{A}}\xi)=\sum_{m=0}^{k}q^{m|\omega|}\begin{bmatrix}k\\m\end{bmatrix}_q d^{k-m}\omega\nabla_q^m(\xi),\tag{51}$$

where $\begin{bmatrix}k\\m\end{bmatrix}_q$ are the q-binomial coefficients. Since d is the N-differential of a graded q-differential algebra Ω_q we have $d^N\omega=0$. Taking into account that $\begin{bmatrix}N\\m\end{bmatrix}_q=0$ for $1\le m\le N-1$ we see that in the case of $k=N$ the expansion (51) takes the following

$$\nabla_q^N(\omega\otimes_{\mathfrak{A}}\xi)=q^{N|\omega|}\omega\nabla_q^N(\xi)=\omega\nabla_q^N(\xi)\tag{52}$$

and this proves that ∇_q^N is the endomorphism of the left Ω_q-module $\mathfrak{F}$.

This proposition allows us to define the curvature of N-connection as follows

Definition 5.5: *The endomorphism* $F=\nabla_q^N$ *of degree N of the left* Ω_q*-module* $\mathfrak{F}$ *is said to be the curvature of a N-connection* ∇_q.

The graded vector space $\mathfrak{L}(\mathfrak{F})$ can be endowed with the structure of a graded algebra if one takes the product $A\circ B$ of two linear operators A, B of the vector space $\mathfrak{F}$ as an algebra multiplication. We can extend a N- connection ∇_q to the linear operator on the vector space $\mathfrak{L}(\mathfrak{F})$ by means of the graded q-commutator as follows

$$\nabla_q(A)=[\nabla_q,A]_q=\nabla_q\circ A-q^{|A|}A\circ\nabla_q,\tag{53}$$

where A is a homogeneous linear operator. Obviously ∇_q is the linear operator of degree one on the vector space $\mathfrak{L}(\mathfrak{F})$, i.e. $\nabla_q:\mathfrak{L}^i(\mathfrak{F})\to\mathfrak{L}^{i+1}(\mathfrak{F})$, and ∇_q satisfies the graded q-Leibniz rule with respect to the algebra structure of $\mathfrak{L}(\mathfrak{F})$. It follows from the definition of the curvature of a N-connection that F can be viewed as the linear operator of degree N on the vector space $\mathfrak{F}$, i.e. $F\in\mathfrak{L}^N(\mathfrak{F})$. Consequently one can act on F by ∇_q, and it holds that

Proposition 5.6 : *For any N-connection* $\nabla\nabla_q$ *the curvature F of this connec- tion satisfies the Bianchi identity* $\nabla_q(F)=0$.

Proof : We have

$$\nabla_q(F)=[\nabla_q,F]_q=\nabla_q\circ F-q^N F\circ\nabla_q=\nabla_q^{N+1}-\nabla_q^{N+1}=0.$$

The following theorem shows that not every left A-module admits a N-connection [14]. In analogy with the theory of Ω-connection [9] we can prove that there is an N-connection on every projective module, and for this we need the following proposition.

Proposition 5.7 : *If* ε = A⊗*V is a free* A*-module, where V is a* $\mathbb{C}$*-vector space, then* $\nabla_q=d\otimes I_v$ *is N-connection on ε and this connection is flat, i.e. its curvature vanishes.*

Proof : Indeed, $\nabla_q:\mathfrak{A}\otimes V\to\Omega_q^1\otimes(\mathfrak{A}\otimes V)$ and

$$\nabla_q(f(g\otimes v))=(d\otimes I_V)(f(g\otimes v))=d(fg)\otimes v=(dfg)\otimes v+f(dg\otimes v)=df\otimes_A(g\otimes v)+f\nabla_q(g\otimes v),$$

Where $f,g\in$ A, $v\in V$. Because of $d^N=0$ and q is the primitive N th root of unity, we get

$$\nabla_q^N(f(g\otimes v))=\sum_{k+m=N}\begin{bmatrix}N\\m\end{bmatrix}_q d^k f(d^m g\otimes v)=0,$$ i. e. the curvature of such a N-connection vanishes.

Theorem 5.8 : *Every projective module admits a N-connection.*

Proof : Let $\mathcal{P}$ be a projective module. From the theory of modules it is known that a module $\mathcal{P}$ is projective if and only if there exists a module $\mathbb{N}$ such that $\varepsilon=\mathcal{P}\otimes_{\mathbb{N}}$ is a free module. It is well known that free left $\mathfrak{A}$-module ε can be represented as the tensor product $A\otimes V$, where V is a $\mathbb{C}$-vector space. A linear map $\nabla_q=\pi\circ(d\otimes I_V):P\to\Omega^1_q\otimes_{\mathfrak{A}}P$ is a N-connection on a projective module $\mathcal{P}$, where $d\otimes I_v$ is a N-connection on a left $\mathfrak{A}$-module ε, π is the projection on the first sum and in the direct sum $\mathcal{P}\otimes_{\mathbb{N}}$ and $\pi(\omega\otimes_{\mathfrak{A}}(g\otimes v))=\omega\otimes_{\mathfrak{A}}\pi(g\otimes v)=\omega\otimes_{\mathfrak{A}}m$, where $\omega\in\Omega^1_q$, $g\in$A, $v\in V$, $m\in\mathcal{P}$. Taking into account Proposition 5.7 we get

$$\nabla_q(fm)=\pi((d\otimes I_V)(fm))=\pi(df\otimes_{\mathfrak{A}}m+fdm)=$$

$$=df\otimes_{\mathfrak{A}}\pi(m)+f\nabla_q(m)=df\otimes_{\mathfrak{A}}m+f\nabla_q(m)$$

where $f\in$ A, $m\in\mathcal{P}$

It is well known that a connection on the vector bundle of finite rank over a finite dimensional smooth manifold can be studied locally by choosing a local trivialization of the vector bundle and this leads to the basis for the module of sections of this vector bundle. Let us concentrate now on an algebraic analog of the local structure of a N-connection ∇_q. For this purpose we assume ε to be a finitely generated free left $\mathfrak{A}$-module. Let $\mathfrak{e}=\{\mathfrak{e}_\mu\}_{\mu=1}^r$ be a basis for a left module ε. For any element $\xi\in\varepsilon$ we have $\xi=\xi^\mu\mathfrak{e}_\mu$.

As mentioned above $\varepsilon\cong\mathfrak{F}^0$ and the basis for a module ε induces the basis $\mathfrak{f}=\{\mathfrak{f}_\mu\}_{\mu=1}^r$, where $\mathfrak{f}_\mu=e\otimes_{\mathfrak{A}}\mathfrak{e}_\mu$, for the left $\mathfrak{A}$-module $\mathfrak{F}^0$. For any $\xi\in\mathfrak{F}^0$ we have $\xi=\xi^\mu\mathfrak{f}_\mu$. Taking into account that $\mathfrak{F}^0\subset\mathfrak{F}$ and $\mathfrak{F}$ is the left Ω_q –module we can multiply the elements of the basis $\mathfrak{f}$ by elements of Ω_q. It is easy to see that if $\omega\in\Omega^i_q$ then for any μ we have $\omega\mathfrak{f}_\mu\in\mathfrak{F}^i$. Consequently we can express any element of the $\mathfrak{F}^i$ as a linear combination of $\mathfrak{f}_\mu$ with coefficients from Ω^i_q. Indeed let $\omega\otimes_{\mathfrak{A}}\xi$ be an element of $\mathfrak{F}^i=\Omega^i\otimes_{\mathfrak{A}}\varepsilon$. Then

$$\omega\otimes_{\mathfrak{A}}\xi=(\omega e)\otimes_{\mathfrak{A}}(\xi^\mu\mathfrak{e}_\mu)=(\omega e\xi^\mu)\otimes_{\mathfrak{A}}\mathfrak{e}_\mu$$

$$=(\omega\xi^\mu e)\otimes_{\mathfrak{A}}\mathfrak{e}_\mu=\omega\xi^\mu(\mathrm{e}\otimes_{\mathfrak{A}}\mathfrak{e}_\mu)=\omega^\mu f_\mu$$

where $\omega^\mu=\omega\xi^\mu\in\Omega^i_q$.

Let $\xi=\xi^\mu\mathfrak{f}_\mu\in\mathfrak{F}^0$. Obviously $\nabla_q(\xi)\in\mathfrak{F}^1$, and making use of (50) we can express the element $\nabla_q(\xi)$ as follows

$$\nabla_q(\xi)=\nabla_q(\xi^\mu\mathfrak{f}_\mu)=d\xi^\mu\otimes_{\mathfrak{A}}\mathfrak{f}_\mu+\xi^\mu\nabla_q(\mathfrak{f}_\mu)\cdot \tag{54}$$

Let $\mathfrak{W}_r(\Omega_q)$ be the vector space of the r x r-matrices whose entries are the elements of a q-GDA Ω_q. If each entry of a matrix $\Theta=(\theta^\nu_\mu)$ is an element of a homogeneous subspace Ω^i_q, i.e. $\theta^\nu_\mu\in\Omega^i_q$ then Θ will be refered to as a homogeneous matrix of degree i and shall denote the vector space of such matrices by $\mathfrak{W}^i_r(\Omega_q)$. Obviously $\mathfrak{W}_r(\Omega_q)=\oplus_i\mathfrak{W}^i_r(\Omega_q)$. The vector space $\mathfrak{W}_r(\Omega_q)$ of r x r-matrices becomes the associative unital graded algebra if we define the product of two matrices $\Theta=(\theta^\nu_\mu),\Theta'=(\theta'^\nu_\mu)\in\mathfrak{W}_r(\Omega_q)$ as follows

$$(\Theta\Theta')^\nu_\mu=\theta^\sigma_\mu\theta'^\nu_\sigma\ . \tag{55}$$

If $\Theta,\Theta'\in\mathfrak{W}_r(\Omega_q)$ are homogeneous matrices then we define the graded q-commutator by $[\Theta,\Theta']_q=\Theta\Theta'-q^{|\Theta||\Theta'|}\Theta'\Theta$. We extend the N –differential d of a q-GDA Ω_q to the algebra $\mathfrak{W}_r(\Omega_q)$ as follows $d\Theta=d(\theta^\nu_\mu)=(d\theta^\nu_\mu)$.

Since any element of a left $\mathfrak{A}$-module $\mathfrak{F}^1$ can be expressed in terms of the basis $\mathfrak{f}=\{\mathfrak{f}_\mu\}_{\mu=1}^r$ with coefficients from Ω^1_q we have

$$\nabla_q(\mathfrak{f}_\mu)=\theta^\nu_\mu\mathfrak{f}_\nu, \tag{56}$$

where $\theta^\nu_\mu\in\Omega^1_q$. In analogy with the classical theory of connections in differential geometry of fibre bundles we introduce a notion of a matrix of N-connection.

Definition 5.9: A $r\times r$-matrix $\Theta=(\theta^\nu_\mu)$, whose entries θ^ν_μ are the elements of Ω^1_q i.e. $\Theta\in\mathfrak{W}^1_r(\Omega_q)$, is said to be a matrix of a N-connection ∇_q with respect to the basis $\mathfrak{f}$ of the left $\mathfrak{A}$-module $\mathfrak{F}^0$.

Using (54) and (56) we obtain

$$\nabla_q(\xi)=(d\xi^\mu+\xi^\nu\theta^\mu_\nu)\mathfrak{f}_\mu\cdot \tag{57}$$

In order to express the curvature F of a N-connection ∇_q in the terms of the entries of the matrix Θ of a N-connection ∇_q we should express the kth power of a N-connection ∇_q, where $1\le k\le N$, in the terms of the entries of the matrix Θ. It can be calculated that the kth power of ∇_q has the following form

$$\nabla^k_q\xi=\sum_{l=0}^k\begin{bmatrix}k\\l\end{bmatrix}_q d^{k-l}\xi^\mu\,\psi^{(l,k)\nu}_\mu\,\mathfrak{f}_\nu$$

$$=(d^k\xi^\mu\,\psi^{(0,k)\nu}_\mu+[k]_q d^{k-1}\xi^\mu\,\psi^{(1,k)\nu}_\mu+\ldots+\xi^\mu\,\psi^{(k,k)\nu}_\mu)\mathfrak{f}_\nu, \tag{58}$$

where $\psi^{(l,k)\nu}_\mu\in\Omega^l_q$ are polynomials on the entries θ^μ_ν of the matrix Θ of a N-connection ∇_q and their differentials. We can calculate the polynomials $\psi^{(l,k)\nu}_\mu$ by means of the following recursion formula

$$\psi^{(l,k)\nu}_\mu=d\psi^{(l-1,k)\nu}_\mu+q^{l-1}\psi^{(l-1,k)\sigma}_\mu\theta^\nu_\sigma, \tag{59}$$

or in the matrix form

$$\Psi^{(l,k)}=d\Psi^{(l-1,k)}+q^{l-1}\Psi^{(l-1,k)}\Theta, \tag{60}$$

where we begin with the polynomial $\psi^{(0,k)\nu}_\mu=\delta^\nu_\mu e\in\mathfrak{A}$, and e is the identity element of A $\subset\Omega_q$. For example the first four polynomials in the expansion (58) obtained with the help of the recursion formula (59) have the form

$$\psi^{(1,k)\nu}_\mu=\theta^\nu_\mu, \tag{61}$$

$$\psi^{(2,k)\nu}_\mu=d\theta^\nu_\mu+q\theta^\sigma_\mu\theta^\nu_\sigma, \tag{62}$$

$$\psi^{(3,k)\nu}_\mu=d^2\theta^\nu_\mu+(q+q^2)d\theta^\sigma_\mu\theta^\nu_\sigma+q^2\theta^\sigma_\mu d\theta^\nu_\sigma+q^3\theta^\tau_\mu\theta^\sigma_\tau\theta^\nu_\sigma\cdot \tag{63}$$

$$\begin{aligned}\psi^{(4,k)\nu}_\mu&=d^3\theta^\nu_\mu+(q+q^2+q^3)d^2\theta^\sigma_\mu\theta^\nu_\sigma+q^3\theta^\sigma_\mu d^2\theta^\nu_\sigma\\&+(q^3+q^4+q^5)d\theta^\tau_\mu\theta^\sigma_\tau\theta^\nu_\sigma+(q^4+q^5)\theta^\tau_\mu d\theta^\sigma_\tau\theta^\nu_\sigma\\&+q^5\theta^\tau_\mu\theta^\sigma_\tau d\theta^\nu_\sigma+(q^2+q^3+q^4)d\theta^\sigma_\mu d\theta^\nu_\sigma+q^6\theta^\tau_\mu\theta^\sigma_\tau\theta^\rho_\sigma\theta^\nu_\rho\cdot\end{aligned} \tag{64}$$

From (58) it follows that if $k=N$ then the first term $d^N\xi^\mu\,\psi^{(0,N)\nu}_\mu$ in this expansion vanishes because of the N-nilpotency of the N-differential d, and the next terms corresponding to the l values from 1 to $N-1$ also vanish because of the well known property of q-binomial coefficients $\begin{bmatrix}N\\l\end{bmatrix}_q=0$ provided q is a primitive Nth root of unity. Hence if $k=N$ then the formula (58) takes on the form

$$\nabla_q^N(\xi) = \xi^\mu \psi_\mu^{(N,N)\nu}\, \mathfrak{f}_\nu \,. \tag{65}$$

In order to simplify the notations and assuming that N is fixed we shall denote $\psi_\mu^\nu = \psi_\mu^{(N,N)\nu}$.

Definition 5.10 : *A* $(r\times r)$ *-matrix* $\Psi = (\psi_\mu^\nu)$, *whose entries are the elements of degree N of a q-GDA* Ω_q, *is said to be the curvature matrix of a N-* connection ∇_q.

Obviously $\Psi \in \mathfrak{M}_r^N(\Omega_q)$. In new notations the formula (65) can be written as follows $\nabla_q^N(\xi) = \xi^\mu\psi_\mu^\nu\, \mathfrak{f}_\nu$, and it clearly demonstrates that ∇_q^N is the endomorphism of degree N of the left Ω_q -module $\mathfrak{F}$.

Let us consider the form of the curvature matrix of a N-connection in three special cases when $N = 2$, $N = 3$ and $N = 4$. *If* $N = 2$ then $q = -1$, and Ω_q is a GDA with differential d satisfying $d^2 = 0$. This is a classical case, and if we assume that Ω_q is the algebra of differential forms on a smooth manifold M with exterior differential d and exterior multiplication $\wedge, \mathfrak{E}$ is the module of smooth sections of a vector bundle $\pi : EM$ over M, ∇_q is a connection on E, e is a local frame of a vector bundle E then Θ is the matrix of 1-forms of a connection ∇_q and the formula (62) gives the expression for the curvature 2-form

$$\psi_\mu^\nu = d\theta_\mu^\nu + q\theta_\mu^\sigma\theta_\sigma^\nu = d\theta_\mu^\nu - \theta_\mu^\sigma\wedge\theta_\sigma^\nu = d\theta_\mu^\nu + \theta_\sigma^\nu\wedge\theta_\mu^\sigma \,,$$

in which we immediately recognize the classical expression for the curvature.

If $N = 3$ then $q = \exp(\frac{2\pi i}{3})$ is the cubic root of unity satisfying the relations $q^3=1$, $1+q+q^2=0$. This is a first non-classical case of a q-connection, and the formula (63) gives the following expression for the curvature of a N-connection

$$\begin{aligned}\psi_\mu^\nu &= d^2\theta_\mu^\nu + (q+q^2)d\theta_\mu^\sigma\theta_\sigma^\nu + q^2\theta_\mu^\sigma d\theta_\sigma^\nu + q^3\theta_\mu^\tau\theta_\tau^\sigma\theta_\sigma^\nu\\ &= d^2\theta_\mu^\nu - d\theta_\mu^\sigma\theta_\sigma^\nu + q^2\theta_\mu^\sigma d\theta_\sigma^\nu + \theta_\mu^\tau\theta_\tau^\sigma\theta_\sigma^\nu\\ &= d^2\theta_\mu^\nu - (d\theta_\mu^\sigma\theta_\sigma^\nu - q^2\theta_\mu^\sigma d\theta_\sigma^\nu) + \theta_\mu^\tau\theta_\tau^\sigma\theta_\sigma^\nu\end{aligned}$$

It is useful to write the above expression for the curvature in a matrix form

$$\Psi = d^2\Theta - [d\Theta,\Theta]_q + \Theta^3 \,. \tag{66}$$

If $N = 4$ then $q = \mathrm{i}$ is the fourth root of unity satisfying relations $1+q^2 = 0$, $q^2= -1$. The expression (64) for curvature in this case takes on the form

$$\begin{aligned}\psi_\mu^\nu &= d^3\theta_\mu^\nu - (d^2\theta_\mu^\sigma\theta_\sigma^\nu - q^3\theta_\mu^\sigma d^2\theta_\sigma^\nu) + d\theta_\mu^\tau\theta_\tau^\sigma\theta_\sigma^\nu\\ &-(q^2+q^3)\theta_\mu^\tau d\theta_\tau^\sigma\theta_\sigma^\nu + q\theta_\mu^\tau\theta_\tau^\sigma d\theta_\sigma^\nu - qd\theta_\mu^\sigma d\theta_\sigma^\nu - \theta_\mu^\tau\theta_\tau^\sigma\theta_\sigma^\rho\theta_\rho^\nu \,.\end{aligned} \tag{67}$$

This expression can be put into a matrix form as follows

$$\Psi = d^3\Theta - \left[d^2\Theta,\, \Theta\right]_q + \left[\left[d\Theta,\, \Theta\right]_q,\Theta\right]_q - q\left(d\Theta\right)^2 - \Theta^4 \,. \tag{68}$$

It should be mentioned that in the case of N = 4 it holds that

$$\left[d\Theta,\, \Theta\right]_q = d\Theta\Theta - q^2\,\Theta\, d\Theta = d\Theta\Theta + \Theta\, d\Theta = \left[\Theta,\, d\Theta\right]_q \,, \tag{69}$$

and the graded q-commutator degenerates in the case of $d\Theta$, i.e. $[d\Theta, d\Theta]_q = d\Theta\, d\Theta - q^4 d\Theta\, d\Theta = 0$.

From the proposition 5.6 it follows that the curvature of a N –connection satisfies the Bianchi identity. Straight forward computation shows that in both cases of (66) and (68) this identity in terms of the matrices of the curvature and a connection takes on the form

$$d\Psi = \left[\Psi,\, \Theta\right]_q \,. \tag{70}$$

Acknowledgement

V. Abramov and O. Liivapuu gratefully acknowledge the financial support of their research by the Estonian Science Foundation under the research grant ETF 7427.

References

1. Kapranov MM. Ontheq -analog of homological algebra. Math.
2. Dubois-Violette M, Kerner R (1996) Universal q-differential calculus and q-analog of homological algebra. Acta Math Univ Comenian 65: 175-188.
3. Dubois-Violette M (1998) dN = 0: Generalized Homology. K-Theory 14: 371-404.
4. Dubois-Violette M (2000) Lectures on differentials, generalized differentials and on some examples related to theoretical physics. Math 1-36.
5. Mathai V, Quillen D (1986) Super connections Thom classes and equivariant differential forms. Topology 25: 85-110.
6. Connes AC (1980) Algebra set geometry differential. CR Acad Sci Paris 290: 599-604.
7. Cuntz J, Quillen D (19995) Algebra extensions and non-singularity. J Amer Math Soc 8: 251-289.
8. Dubois-Violette M, Masson T (1996) On the first order operators in bimodules. Lett Math Phys 37: 467-474.
9. Dubois-Violette M (2001) Lectures on graded differential algebras and non-commutative geometry. Math Phys Stud 23: 245-306.
10. Abramov V, Kerner R (2000) Exterior differentials of higher order and their covariant generalization. Journal of Mathematical Physics 41: 5598-5614.
11. Kerner R, Bramov V (1999) On certain realizations of q-deformed exterior differential calculus. Rep Math Phys 43: 179-194.
12. Abramov V (2007) Algebra Forms with Dn = 0 on Quantum Plane. Generalized Clifford Algebra Approach. Advances in Applied Clifford Algebras 17: 577-588.
13. Quillen D (1990) Chern-Simons Forms and Cyclic Cohomology. The Interface of Mathematics and Particle Physics 117-134.
14. Abramov V, Liivapuu O (2010) Generalization of connection based on a concept of graded q-differential algebra. Proc Estonian Acad Sci Phys Math 59: 256-264.
15. Abramov V (2006) Generalization of super connection in non-commutative geometry. Proc Estonian Acad Sci Phys Math 55: 3-15.
16. Abramov V, Liivapuu O (2006) Geometric approach to BRST-symmetry and ZN -generalization of super connection. Journal of Nonlinear Math Phys 13: 9-20.
17. Abramov V (2009) Graded q-differential algebra approach to q-connection. Generalized Lie Theory in Mathematics 71-79.
18. Abramov V (2006) On a graded q-differential algebra. Journal of Nonlinear Math Phys 13: 1-8.
19. Coquereaux R, Garcia AO, Trinchero R (2000) Differential calculus and connection on a quantum plane at a cubic root of unity. Reviews in Mathematical Physics 12: 227-285.
20. Borowiec A, Kharchenko VK (1995) First order optimum calculi. Bull Soc Sci Lett Lodz 45: 75-88.
21. Curtis C, Reiner I (1962) Representation Theory of Finite Groups and Associative Algebras. Inter science Publishers.

Citation: Abramov V, Liivapuu O (2015) *N*-Complex, Graded *q*-Differential Algebra and *N*-Connection on Modules. J Generalized Lie Theory Appl S1: 006. doi:10.4172/1736-4337.S1-006

This article was originally published in a special issue, **Algebra, Combinatorics and Dynamics** handled by Editor. Dr. Natalia Iyudu, Researcher School of Mathematics, The University of Edinburgh, UK