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A Class of Nonassociative Algebras Including Flexible and Alternative Algebras, Operads and Deformations

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Abstract

There exists two types of nonassociative algebras whose associator satisfies a symmetric relation associated with a 1-dimensional invariant vector space with respect to the natural action of the symmetric group Σ_3. The first one corresponds to the Lie-admissible algebras and this class has been studied in a previous paper of Remm and Goze. Here we are interested by the second one corresponding to the third power associative algebras.

Keywords: Nonassociative algebras; Alternative algebras; Third power associative algebras; Operads

Introduction

Recently, we have classified for binary algebras, Cf. [1], relations of nonassociativity which are invariant with respect to an action of the symmetric group on three elements Σ_3 on the associator. In particular we have investigated two classes of nonassociative algebras, the first one corresponds to algebras whose associator A_μ satisfies

$$A_\mu \circ (Id - \tau_{12} - \tau_{23} - \tau_{13} + c + c^2) = 0, \tag{1}$$

and the second

$$A_\mu \circ (Id + \tau_{12} + \tau_{23} + \tau_{13} + c + c^2) = 0, \tag{2}$$

where τ_{ij} denotes the transposition exchanging i and j, c is the 3-cycle (1,2,3).

These relations are in correspondence with the only two irreducible one-dimensional subspaces of $\mathbb{K}[\Sigma_3]$ with respect to the action of Σ_3, where $\mathbb{K}[\Sigma_3]$ is the group algebra of Σ_3. In studies of Remm [1], we have studied the operadic and deformations aspects of the first one: the class of Lie-admissible algebras. We will now investigate the second class and in particular nonassociative algebras satisfying (2) with nonassociative relations in correspondence with the subgroups of Σ_3.

Convention: We consider algebras over a field $\mathbb{K}$ of characteristic zero.

G_i-p^3-associative Algebras

Definition

Let Σ_3 be the symmetric group of degree 3 and $\mathbb{K}$ a field of characteristic zero. We denote by $\mathbb{K}[\Sigma_3]$ the corresponding group algebra, that is the set of formal sums $\sum_{\sigma\in\Sigma_3} a_\sigma \sigma$, $a_\sigma \in \mathbb{K}$ endowed with the natural addition and the multiplication induced by multiplication in Σ_3, $\mathbb{K}$ and linearity. Let $\{G_i\}_{i=1,\cdots,6}$ be the subgroups of Σ_3. To fix notations we define

$$G_1 = \{Id\}, G_2 = <\tau_{12}>, G_3 = <\tau_{23}>, G_4 = <\tau_{13}>, G_5 = <c>, G_6 = \Sigma_3,$$

where $<\sigma>$ is the cyclic group subgroup generated by σ. To each subgroup G_i we associate the vector v_{G_i} of $\mathbb{K}[\Sigma_3]$:

$$v_{G_i} = \sum_{\sigma\in G_i} \sigma.$$

Lemma 1. *The one-dimensional subspace $\mathbb{K}\{v_{\Sigma_3}\}$ of $\mathbb{K}[\Sigma_3]$ generated by*

$$v_{G_6} = v_{\Sigma_3} = \sum_{\sigma\in\Sigma_3} \sigma$$

is an irreducible invariant subspace of $\mathbb{K}[\Sigma_3]$ with respect to the right action of Σ_3 on $\mathbb{K}[\Sigma_3]$.

Recall that there exists only two one-dimensional invariant subspaces of $\mathbb{K}[\Sigma_3]$, the second being generated by the vector $\sum_{\sigma\in\Sigma_3} \epsilon(\sigma)\sigma$ where $\epsilon(\sigma)$ is the sign of σ. As we have precised in the introduction, this case has been studied in literature of Remm [1].

Definition 2. *A G_i-p^3-associative algebra is a $\mathbb{K}$ -algebra $(\mathcal{A},\mu)$ whose associator*

$$A_\mu = \mu \circ (\mu \otimes Id - Id \otimes \mu)$$

satisfies

$$A_\mu \circ \Phi^{\mathcal{A}}_{v_{G_6}} = 0,$$

where $\Phi^{\mathcal{A}}_{v_{G_i}} : \mathcal{A}^{\otimes 3} \to \mathcal{A}^{\otimes 3}$ is the linear map

$$\Phi^{\mathcal{A}}_{v_{G_i}}(x_1 \otimes x_2 \otimes x_3) = \sum_{\sigma\in G_i} (x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}).$$

Let $\mathcal{O}(v_{G_i})$ be the orbit of v_{G_i} with respect to the right action

$$\begin{array}{ccc} \Sigma_3 \times \mathbb{K}[\Sigma_3] & \to & \mathbb{K}[\Sigma_3] \\ (\sigma, \sum_i a_i\sigma_i) & \mapsto & \sum_i a_i \sigma^{-1}\circ\sigma_i \end{array}$$

Then putting $F_{v_{G_i}} = K(\mathcal{O}(v_{G_i}))$ we have

$$\begin{cases} \dim F_{v_{G_1}} = 6, \\ \dim F_{v_{G_2}} = \dim F_{v_{G_3}} = \dim F_{v_{G_4}} = 3, \\ \dim F_{v_{G_5}} = 2, \\ \dim F_{v_{G_6}} = 1. \end{cases}$$

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Proposition 3. *Every G_i-p^3-associative algebra is third power associative.*

Recall that a third power associative algebra is an algebra $(\mathcal{A},\mu)$ whose associator satisfies $A_\mu(x, x, x) = 0$. Linearizing this relation, we obtain

$$A_\mu \circ \Phi^{\mathcal{A}}_{v_{\Sigma_3}} = 0.$$

Since each of the invariant spaces F_{G_i} contains the vector v_{Σ_3}, we deduce the proposition.

Remark. An important class of third power associative algebras is the class of power associative algebras, that is, algebras such that any element generates an associative subalgebra.

What are G_i-p^3-associative algebras?

(1) If $i=1$, since $v_{G_1} = Id$, the class of G_1-p^3-associative algebras is the full class of associative algebras.

(2) If $i=2$, the associator of a G_2 - p^3-associative algebra $\mathcal{A}$ satisfies

$$A_\mu(x_1, x_2, x_3) + A_\mu(x_2, x_1, x_3) = 0$$

and this is equivalent to

$$A_\mu(x, x, y),$$

for all $x, y \in \mathcal{A}$.

(3) If $i = 3$, the associator of a G_3-p^3-associative algebra $\mathcal{A}$ satisfies

$$A_\mu(x_1, x_2, x_3) + A_\mu(x_1, x_3, x_2) = 0,$$

that is,

$$A_\mu(x, y, y),$$

for all $x, y \in \mathcal{A}$.

Sometimes G_2-p^3-associative algebras are called left-alternative algebras, G_3-p^3-associative algebras are right-alternative algebras. An alternative algebra is an algebra which satisfies the G_2 and G_3-p^3-associativity.

(4) If $i = 4$, we have $A_\mu(x, y, x)$ for all $x, y \in \mathcal{A}$, and the class of G_3-p^3-associative algebras is the class of flexible algebras.

(5) If $i = 5$, the class of G_5-p^3-associative algebras corresponds to G_5-associative algebras [2].

(6) If $i = 6$, the associator of a G_6-p^3-associative algebra satisfies

$$A_\mu(x_1,x_2,x_3) + A_\mu(x_2,x_1,x_3) + A_\mu(x_3,x_2,x_1)$$
$$+A_\mu(x_1,x_3,x_2) + A_\mu(x_2,x_3,x_1)A_\mu(x_3,x_1,x_2) = 0.$$

If we consider the symmetric product $x \star y = \mu(x,y) + \mu(y,x)$ and the skew-symmetric product $[x,y] = \mu(x,y) - \mu(y,x)$, then the G_6-p^3-associative identity is equivalent to

$$[x \star y, z] + [y \star z, x] + [z \star x, y] = 0.$$

Definition 4. *A* ([,],$\star$)*-admissible-algebra is a* $\mathbb{K}$*-vector space* $\mathcal{A}$ *provided with two multiplications:*

(a) *a symmetric multiplication* $\star$,

(b) *a skew-symmetric multiplication* [,] *satisfying the identity*

$$[x \star y, z] + [y \star z, x] + [z \star x, y] = 0$$

for any $x, y \in \mathcal{A}$.

Then a G_6 - p^3-associative algebra can be defined as ([,], $\star$)-admissible algebra.

Remark: Poisson algebras. A $\mathbb{K}$-Poisson algebra is a vector space $\mathcal{P}$ provided with two multiplications, an assocative commutative one $x \bullet y$ and a Lie bracket $[x, y]$, which satisfy the Leibniz identity

$$[x \bullet y, z] - x \bullet [y,z] - [x,z] \bullet y = 0$$

In studies of Remm [3], it is shown that these conditions are equivalent to provide $\mathcal{P}$ with a nonassociative multiplication $x \cdot y$ satisfying

$$x\cdot(y\cdot z) = (x\cdot y)\cdot z - \frac{1}{3}\{(x\cdot z)\cdot y + (y\cdot z)\cdot x - (y\cdot x)\cdot z - (z\cdot x)\cdot y\}$$

If we denote by $A_{\cdot}(x,y,z) = x\cdot(y\cdot z)$ and $A^L_{\cdot}(x,y,z) = (x\cdot y)\cdot z$ then the previous identity is equivalent to

$$A^R_{\cdot} \circ \Phi^{\mathcal{P}}_{w_1} + A^L_{\cdot} \circ \Phi^{\mathcal{P}}_{w_2} = 0$$

where $w_1 = 3Id$ and $w_2 = -3Id + \tau_{23} + c_1 - c_2 - \tau_{12}$. In fact the class of Poisson algebras is a subclass of a family of nonassociative algebras defined by conditions on the associator. The product satisfies

$$A_{\cdot}(x,y,z) + A_{\cdot}(y,z,x) + A_{\cdot}(z,x,y) = 0$$

and

$$A_{\cdot}(x,y,z) + A_{\cdot}(z,y,x) = 0,$$

so it is a subclass of the class of algebras which are flexible and G_5-p^3-associative [1].

The Operads G_i -$p^3\mathcal{A}ss$ and their Dual

For each $i \in \{1,\cdots,6\}$, the operad for G_i-p^3-associative algebras will be denoted by G_i-$p^3\mathcal{A}ss$. The operads $\{G_i - p^3\mathcal{A}ss\}_{i=1,\cdots,6}$ are binary quadratic operads, that is, operads of the form $\mathcal{P} = \Gamma(E) / (R)$, where $\Gamma(E)$ denotes the free operad generated by a Σ_2-module E placed in arity 2 and (R) is the operadic ideal generated by a Σ_3-invariant subspace R of $\Gamma(E)(3)$. Then the dual operad $\mathcal{P}^!$ is the quadratic operad $\mathcal{P}^! := \Gamma(E^\vee)/(R^\perp)$, where $R^\perp \subset \Gamma(E^\vee)(3)$ is the annihilator of $R \subset \Gamma(E)(3)$ in the pairing

$$\begin{cases} <(x_i\cdot x_j)\cdot x_k, (x_{i'}\cdot x_{j'})\cdot x_{k'}>=0, \text{ if } \{i,j,k\} \neq \{i',j',k'\}, \\ <(x_i\cdot x_j)\cdot x_k, (x_i\cdot x_j)\cdot x_k>=(-1)^{\varepsilon(\sigma)}, \\ \qquad \text{with } \sigma = \begin{pmatrix} i & j & k \\ i' & j' & k' \end{pmatrix} \text{if } \{i,j,k\} = \{i',j',k'\}, \\ <x_i\cdot(x_j\cdot x_k), x_{i'}\cdot(x_{j'}\cdot x_{k'})>=0, \text{ if } \{i,j,k\} \neq \{i',j',k'\}, \\ <x_i\cdot(x_j\cdot x_k), x_i\cdot(x_j\cdot x_k)>=-(-1)^{\varepsilon(\sigma)}, \\ \qquad \text{with } \sigma = \begin{pmatrix} i & j & k \\ i' & j' & k' \end{pmatrix} \text{if } (i,j,k) \neq (i',j',k'), \\ <(x_i\cdot x_j)\cdot x_k, x_{i'}\cdot(x_{j'}\cdot x_{k'})>=0, \end{cases} \quad (3)$$

and $(R^\perp)$ is the operadic ideal generated by $R^\perp$. For the general notions of binary quadratic operads [4,5]. Recall that a quadratic operad $\mathcal{P}$ is Koszul if the free $\mathcal{P}$-algebra based on a $\mathbb{K}$-vector space V is Koszul, for any vector space V. This property is conserved by duality and can be studied using generating functions of $\mathcal{P}$ and of $\mathcal{P}^!$ [4,6]. Before studying the Koszulness of the operads G_i-$p^3\mathcal{A}ss$, we will compute the homology of an associative algebra which will be useful to look if G_i-$p^3\mathcal{A}ss$ are Koszul or not.

Let A_2 the two-dimensional associative algebra given in a basis $\{e_1, e_2\}$ by $e_1e_1 = e_2$, $e_1e_2 = e_2e_1 = e_2e_2 = 0$. Recall that the Hochschild homology of an associative algebra is given by the complex $(C_n(\mathcal{A},\mathcal{A}), d_n)$ where $C_n(\mathcal{A},\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^{\otimes n}$ and the differentials $d_n : C_n(\mathcal{A},\mathcal{A}) \to C_{n-1}(\mathcal{A},\mathcal{A})$ are given by

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$$d_n(a_0,a_1,\cdots,a_n) = (a_0a_1,a_2,\cdots,a_n)+\sum_{i=1}^{n-1}(-1)^i(a_0,a_1,\cdots,a_ia_{i+1},\cdots,a_n) +(-1)^n(a_na_0,a_1,\cdots,a_{n-1}).$$

Concerning the algebra A_2, we have

$$d_1(e_i,e_j)=e_ie_j-e_je_i=0,$$

for any i, j = 1,2. Similarly we have

$$\begin{cases} d_2(e_1,e_1,e_1)=2(e_2,e_1)-(e_1,e_2), \\ d_2(e_1,e_1,e_2)=d_2(e_1,e_2,e_1)=-d_2(e_2,e_1,e_1)=(e_2,e_2) \end{cases}$$

and 0 in all the other cases. Then dim $Im\ d_2 = 2$ and dim $Ker\ d_1 = 4$. Then $H_1(A_2, A_2)$ is isomorphic to A_2. We have also

$$\begin{cases} d_3(e_1,e_1,e_1,e_1) &= -(e_1,e_2,e_1)+(e_1,e_1,e_2), \\ d_3(e_1,e_1,e_1,e_2) &= (e_2,e_1,e_2)-(e_1,e_2,e_2), \\ d_3(e_1,e_1,e_2,e_1) &= -d_2(e_2,e_1,e_1,e_1)=(e_2,e_2,e_1)-(e_2,e_1,e_2), \\ d_3(e_1,e_2,e_1,e_1) &= (e_1,e_2,e_2)-(e_2,e_2,e_1), \\ d_3(e_1,e_1,e_2,e_2) &= -d_3(e_1,e_2,e_2,e_1)=-d_3(e_2,e_1,e_1,e_2)=d_3(e_2,e_2,e_1,e_1) \\ &= (e_2,e_2,e_2) \end{cases}$$

and $d_3 = 0$ in all the other cases. Then dim $Im\ d_3 = 4$ and dim $Ker\ d_2 = 6$. Thus $H_2(A_2, A_2)$ is non trivial and A_2 is not a Koszul algebra.

Now we will study all the operads G_i-$p^3\mathcal{A}ss$.

The operad (G_1-$p^3\mathcal{A}ss$)

Since G_1-$p^3\mathcal{A}ss$=$\mathcal{A}ss$, where $\mathcal{A}ss$ denotes the operad for associative algebras, and since the operad $\mathcal{A}ss$ is selfdual, we have

$$(G_1-p^3\mathcal{A}ss)^! = \mathcal{A}ss^! = G_1-p^3\mathcal{A}ss.$$

We also have

$$\widetilde{G_1-p^3\mathcal{A}ss} = \widetilde{\mathcal{A}ss} = \mathcal{A}ss,$$

where $\widetilde{\mathcal{P}}$ is the maximal current operad of $\mathcal{P}$ defined in [7,8].

The operad (G_2-$p^3\mathcal{A}ss$)

The operad G_2-$p^3\mathcal{A}ss$ is the operad for left-alternative algebras. It is the quadratic operad $\mathcal{P} = \Gamma(E) / (R)$, where the Σ_3-invariant subspace R of $\Gamma(E)(3)$ is generated by the vectors

$$(x_1\cdot x_2)\cdot x_3 - x_1\cdot(x_2\cdot x_3)+(x_2\cdot x_1)\cdot x_3 - x_2\cdot(x_1\cdot x_3).$$

The annihilator $R^\perp$ of R with respect to the pairing (3) is generated by the vectors

$$\begin{cases} (x_1\cdot x_2)\cdot x_3 - x_1\cdot(x_2\cdot x_3), \\ (x_1\cdot x_2)\cdot x_3 + (x_2\cdot x_1)\cdot x_3. \end{cases} \qquad (4)$$

We deduce from direct calculations that dim $R^\perp = 9$ and

Proposition 5. *The* $(G_2$-$p^3\mathcal{A}ss)^!$ *-algebras are associative algebras satisfying*

$abc = -bac.$

Recall that $(G_2\mathcal{A}ss)^!$-algebras are associative algebras satisfying

$abc = bac.$

and this operad is classically denoted $\mathcal{P}erm$.

Theorem 6. *The operad* $(G_2$-$p^3\mathcal{A}ss)^!$ *is not Koszul* [9].

Proof. It is easy to describe $(G_2$-$p^3\mathcal{A}ss)^!(n)$ for any n. In fact $(G_2$-$p^3\mathcal{A}ss)^!(4)$ corresponds to associative elements satisfying

$$x_1x_2x_3x_4 = -x_2x_1x_3x_4 = -x_2(x_1x_3)x_4 = x_1x_3x_2x_4 = -x_1x_2x_3x_4$$

and $(G_2$-$p^3\mathcal{A}ss)^!(4) = \{0\}$. Let $\mathcal{P}$ be $(G_2$-$p^3\mathcal{A}ss)$. The generating function of $\mathcal{P}^! = (G_2$-$p^3\mathcal{A}ss)^!$ is

$$g_{\mathcal{P}^!}(x) = \sum_{a\geq 1}\frac{1}{a!}\dim(G_2-p^3\mathcal{A}ss)^!(a)x^a = x+x^2+\frac{x^3}{2}.$$

But the generating function of $\mathcal{P} = (G_2$-$p^3\mathcal{A}ss)$ is

$$g_{\mathcal{P}}(x) = x+x^2+\frac{3}{2}x^3+\frac{5}{2}x^4+O(x^5)$$

and if $(G_2$-$p^3\mathcal{A}ss)$ is Koszul, then the generating functions should be related by the functional equation

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-x)) = x$$

and it is not the case so both $(G_2$-$p^3\mathcal{A}ss)$ and $(G_2$-$p^3\mathcal{A}ss)^!$ are not Koszul.

By definition, a quadratic operad $\mathcal{P}$ is Koszul if any free $\mathcal{P}$-algebra on a vector space V is a Koszul algebra. Let us describe the free algebra $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ when dim V =1 and 2.

A $(G_2$-$p^3\mathcal{A}ss)^!$-algebra $\mathcal{A}$ is an associative algebra satisfying

$xyz = -yxz,$

for any $x, y, z \in \mathcal{A}$. This implies $xyzt = 0$ for any $x, y, z \in \mathcal{A}$. In particular we have

$$\begin{cases} x^3 = 0, \\ x^2y = 0, \end{cases}$$

for any $x, y \in \mathcal{A}$. If dim V =1, $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ is of dimension 2 and given by

$$\begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_2e_2 = 0. \end{cases}$$

In fact if $V = \mathbb{K}\{e_1\}$ thus in $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ we have $e_1^3 = 0$. We deduce that $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V) = A_2$ and $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ is not Koszul. It is easy to generalize this construction. If dim $V = n$, then $\dim\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V) = \frac{n(n^2+n+2)}{2}$ and if $\{e_1,\cdots,e_n\}$ is a basis of V then $\{e_i, e_i^2, e_ie_j, e_le_me_p\}$ for $i, j = 1,\cdots,n$ and $l, m, p = 1,\cdots,n$ with $m > l$, is a basis of $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$. For example, if $n = 2$, the basis of $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ is

$$\{v_1=e_1, v_2=e_2, v_3=e_1^2, v_4=e_2^2, v_5=e_1e_2, v_6=e_2e_1, v_7=e_1e_2e_1, v_8=e_1e_2^2\}$$

and the multiplication table is

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	v_3	v_5	0	v_8	0	v_7	0	0
v_2	v_6	v_4	$-v_7$	0	$-v_8$	0	0	0
v_3	0	0	0	0	0	0	0	0
v_4	0	0	0	0	0	0	0	0
v_5	v_7	v_8	0	0	0	0	0	0
v_6	$-v_7$	$-v_8$	0	0	0	0	0	0
v_7	0	0	0	0	0	0	0	0
v_8	0	0	0	0	0	0	0	0

For this algebra we have

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$$\begin{cases} d_1(v_1, v_2) = v_5 - v_6, \\ \frac{1}{2} d_1(v_1, v_6) = v_7 = -d_1(v_1, v_5) = d_1(v_2, v_3), \\ \frac{1}{2} d_1(v_2, v_5) = -v_8 = d_1(v_6, v_2) = -d_1(v_1, v_4), \end{cases}$$

and $Ker\, d_1$ is of dim 64. The space $Im\, d_2$ doesn't contain in particular the vectors (v_i, v_i) for $i = 1, 2$ because these vectors v_i are not in the derived subalgebra. Since these vectors are in $Ker\, d_1$ we deduce that the second space of homology is not trivial.

Proposition 7. *The current operad of* G_2-$p^3\mathcal{A}ss$ *is*

$\widetilde{G_2 - p^3\mathcal{A}ss} = \mathcal{P}erm.$

This is directly deduced of the definition of the current operad [7].

The operad (G_2 - $p^3\mathcal{A}ss$)

It is defined by the module of relations generated by the vector

$$(x_1x_2)x_3 - x_1(x_2x_3) + (x_1x_3)x_2 - x_1(x_3x_2),$$

and R is the linear span of

$$\begin{cases} (x_1x_2)x_3 - x_1(x_2x_3), \\ (x_1x_2)x_3 + (x_1x_3)x_2. \end{cases}$$

Proposition 8. *A* $(G_2\ p^3\mathcal{A}ss)^!$*-algebra is an associative algebra* $\mathcal{A}$ *satisfying*

$abc = -acb,$

for any $a, b, c \in \mathcal{A}$.

Since $(G_3$-$p^3\mathcal{A}ss)^!$ is basically isomorphic to $(G_2$-$p^3\mathcal{A}ss)^!$ we deduce that $(G_3$-$p^3\mathcal{A}ss)$ is not Koszul.

The operad (G_4-$p^3\mathcal{A}ss$)

Remark that a $(G_4$-$p^3\mathcal{A}ss)$-algebra is generally called flexible algebra. The relation

$$A_\mu(x_1, x_2, x_3) + A_\mu(x_3, x_2, x_1) = 0$$

is equivalent to $A_\mu(x, y, x) = 0$ and this denotes the flexibility of $(\mathcal{A}, \mu)$.

Proposition 9. *A* $(G_4$-$p^3\mathcal{A}ss)^!$*-algebra is an associative algebra satisfying*

$abc = -cba.$

This implies that dim $(G_4$-$p^3\mathcal{A}ss)^!$ (3) = 3. We have also $x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} = (-1)^{\varepsilon(\sigma)} x_1x_2x_3x_4$ for any $\sigma \in \Sigma_4$. This gives dim $(G_4$-$p^3\mathcal{A}ss)^!$ (4) = 1. Similarly

$$\begin{aligned} x_1x_2(x_3x_4x_5) &= -x_3(x_4x_5x_2)x_1 = x_1(x_4x_5(x_2x_3)) = -x_1x_2(x_3x_5x_4) \\ &= (x_1x_2(x_4x_5))x_3 = -(x_4x_5)(x_2x_1)x_3 = (x_3x_2x_1)x_4x_5 \\ &= -x_1x_2x_3x_4x_5 \end{aligned}$$

(the algebra is associative so we put some parenthesis just to explain how we pass from one expression to an other). We deduce $(G_4$-$p^3\mathcal{A}ss)^!$ (5) = {0} and more generally $(G_4$-$p^3\mathcal{A}ss)^!(a) = \{0\}$ for $a \geq 5$.

The generating function of $(G_4$ - $p^3\mathcal{A}ss)^!$ is

$$f(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{12}.$$

Let $\mathcal{F}_{(G_4 - p^3\mathcal{A}ss)^!}(V)$ be the free $(G_4$ - $p^3\mathcal{A}ss)^!$-algebra based on the vector space V. In this algebra we have the relations

$$\begin{cases} a^3 = 0, \\ aba = 0, \end{cases}$$

for any $a, b \in V$. Assume that dim$V$ =1. If $\{e_1\}$ is a basis of V, then $e_1^3 = 0$ and $\mathcal{F}_{(G_4 - p^3\mathcal{A}ss)^!}(V) = \mathcal{F}_{(G_2 - p^3\mathcal{A}ss)^!}(V)$. We deduce that $\mathcal{F}_{(G_4 - p^3\mathcal{A}ss)^!}(V)$ is not a Koszul algebra.

Proposition 10. *The operad for flexible algebra is not Koszul.*

Let us note that, if dim V=2 and $\{e_1,e_2\}$ is a basis of V, then $\mathcal{F}_{(G_4 - p^3\mathcal{A}ss)^!}(V)$ is generated by $\{e_1, e_2, e_1^2, e_2^2, e_1e_2, e_2e_1, e_1e_2^2, e_1^2e_2, e_2e_1^2, e_2^2e_1, e_1^2e_2^2, e_2^2e_1^2\}$ and is of dimension 12.

Proposition 11. *We have*

$\widetilde{G_4 - p^3\mathcal{A}ss} = (G_4 - \mathcal{A}ss)^!.$

This means that a $\widetilde{G_4 - p^3\mathcal{A}ss}$ is an associative algebra $\mathcal{A}$ satisfying $abc = cba$, for any $a, b, c \in \mathcal{A}$.

The operad (G_5-$p^3\mathcal{A}ss$)

It coincides with $(G_5$-$\mathcal{A}ss)$ and this last has been studied in studies of Remm [2].

The operad (G_6-$p^3\mathcal{A}ss$)

A $(G_6$-$p^3\mathcal{A}ss)$-algebra $(\mathcal{A}, \mu)$ satisfies the relation

$$\begin{aligned} &A_\mu(x_1, x_2, x_3) + A_\mu(x_2, x_1, x_3) + A_\mu(x_3, x_2, x_1) \\ &+ A_\mu(x_1, x_3, x_2) + A_\mu(x_2, x_3, x_1) + A_\mu(x_3, x_1, x_2) = 0. \end{aligned}$$

The dual operad $(G_6$ - $p^3\mathcal{A}ss)^!$ is generated by the relations

$$\begin{cases} (x_1x_2)x_3 = x_1(x_2x_3), \\ (x_1x_2)x_3 = (-1)^{\varepsilon(\sigma)}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}, \text{ for all } \sigma \in \Sigma_3. \end{cases}$$

We deduce

Proposition 12. *A* $(G_6$-$p^3\mathcal{A}ss)^!$*-algebra is an associative algebra* $\mathcal{A}$ *which satisfies*

$abc = -bac = -cba = -acb = bca = cab,$

for any $a, b, c \in \mathcal{A}$. In particular

$$\begin{cases} a^3 = 0, \\ aba = aab = baa = 0. \end{cases}$$

Lemma 13. *The operad* $(G_6$ - $p^3\mathcal{A}ss)^!$ *satisfies* $(G_6$ - $p^3\mathcal{A}ss)^!(4) = \{0\}$.

Proof. We have in $(G_6$-$p^3\mathcal{A}ss)^!(4)$ that

$$x_1(x_2x_3)x_4 = x_2(x_3x_4)x_1 = -x_1(x_3x_4x_2) = x_1x_3x_2x_4 = -x_1x_2x_3x_4$$

so $x_1x_2x_3x_4 = 0$. We deduce that the generating function of $(G_6$-$p^3\mathcal{A}ss)^!$ is

$$f^!(x) = x + x^2 + \frac{x^3}{6}.$$

If this operad is Koszul the generating function of the operad $(G_6$ - $p^3\mathcal{A}ss)$ should be of the form

$$f(x) = x + x^2 + \frac{11}{6}x^3 + \frac{25}{6}x^4 + \frac{127}{12}x^5 + \cdots$$

But if we look the free algebra generated by V with dim$V$ =1, it satisfies $a^3 = 0$ and coincides with $\mathcal{F}_{(G_2 - p^3\mathcal{A}ss)^!}(V)$. Then $(G_6$-$p^3\mathcal{A}ss)$ is not Koszul.

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Proposition 14. *We have*

$$\overline{G \quad p^3\mathcal{A}ss} \quad \mathcal{L}ieAdm^!$$

that is the binary quadratic operad whose corresponding algebras are associative and satisfying

$$abc = acb = bac.$$

Cohomology and Deformations

Let $(\mathcal{A},\mu)$ be a $\mathbb{K}$-algebra defined by quadratic relations. It is attached to a quadratic linear operad $\mathcal{P}$. By deformations of $(\mathcal{A},\mu)$, we mean [10]

- A $\mathbb{K}^*$ non archimedian extension field of $\mathbb{K}$, with a valuation v such that, if A is the ring of valuation and $\mathcal{M}$ the unique ideal of A, then the residual field $A/\mathcal{M}$ is isomorphic to $\mathbb{K}$.
- The $A/\mathcal{M}$ vector space $\widetilde{\mathcal{A}}$ is $\mathbb{K}$-isomorphic to $\mathcal{A}$.
- For any $a, b \in \mathcal{A}$ we have that

$$\tilde{\mu}(a,b) - \mu(a,b)$$

belongs to the $\mathcal{M}$-module $\widetilde{\mathcal{A}}$ (isomorphic to $\mathcal{A} \otimes \mathcal{M}$).

The most important example concerns the case where A is $\mathbb{K}[[t]]$, the ring of formal series. In this case $\mathcal{M} = \left\{\sum_{i\geq 1} a_i t^i, a_i \in K\right\}$, $\mathbb{K}^* = \mathbb{K}((t))$ the field of rational fractions. This case corresponds to the classical Gerstenhaber deformations. Since A is a local ring, all the notions of valued deformations coincides [11].

We know that there exists always a cohomology which parametrizes deformations. If the operad $\mathcal{P}$ is Koszul, this cohomology is the "standard"-cohomology called the operadic cohomology. If the operad $\mathcal{P}$ is not Koszul, the cohomology which governs deformations is based on the minimal model of $\mathcal{P}$ and the operadic cohomology and deformations cohomology differ [12].

In this section we are interested by the case of left-alternative algebras, that is, by the operad $(G_2\text{-}p^3\mathcal{A}ss)$ and also by the classical alternative algebras.

Deformations and cohomology of left-alternative algebras

A $\mathbb{K}$-left-alternative algebra $(\mathcal{A},\mu)$ is a $\mathbb{K}$-$(G_2\text{-}p^3\mathcal{A}ss)$-algebra. Then satisfies

$$A_\mu(x_1,x_2,x_3) + A_\mu(x_2,x_1,x_3) = 0.$$

A valued deformation can be viewed as a $\mathbb{K}[[t]]$ -algebra $(A \otimes \mathbb{K}[[t]], \mu_t)$ whose product μ_t is given by

$$\mu_t = \mu + \sum_{i\geq 1} t^i \varphi_i.$$

The operadic cohomology: It is the standard cohomology $H^*_{(G_2-p^3\mathcal{A}ss)}(\mathcal{A},\mathcal{A})_{st}$ of the $(G_2\text{-}p^3\mathcal{A}ss)$ -algebra $(\mathcal{A},\mu)$. It is associated to the cochains complex

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^2_{st}} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^3_{st}} \cdots$$

where $\mathcal{P} = (G_2\text{-}p^3\mathcal{A}ss)$ and

$$\mathcal{C}^p_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} = Hom(\mathcal{P}^!(p) \otimes_{\Sigma_p} \mathcal{A}^{\otimes p}, \mathcal{A}).$$

Since $(G_2\text{-}p^3\mathcal{A}ss)^!(4) = 0$, we deduce that

$$H^p_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} = 0 \text{ for } p \geq 4,$$

because the cochains complex is a short sequence

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^2_{st}} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{0} 0.$$

The coboundary operator are given by

$$\begin{cases} \delta^1 f(a,b) & = & f(a)b + af(b) - f(ab), \\ \delta^2\varphi(a,b,c) & = & \varphi(ab,c) + \varphi(ba,c) - \varphi(a,bc) - \varphi(b,ac) \\ & & \varphi(a,b)c + \varphi(b,a)c - a\varphi(b,c) - b\varphi(a,c). \end{cases}$$

The deformations cohomology: The minimal model of $(G_2\text{-}p^3\mathcal{A}ss)$ is a homology isomorphism

$$(G_2 - p^3\mathcal{A}ss, 0) \xrightarrow{\rho} (\Gamma(E), \partial)$$

of dg-operads such that the image of ∂ consists of decomposable elements of the free operad $\Gamma(E)$. Since $(G_2\text{-}p^3\mathcal{A}ss)(1) = \mathbb{K}$, this minimal model exists and it is unique. The deformations cohomology $H^*(\mathcal{A},\mathcal{A})_{defo}$ of $\mathcal{A}$ is the cohomology of the complex

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^1} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^2} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^3} \cdots$$

where

$$\begin{cases} \mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(\mathcal{A},\mathcal{A}), \\ \mathcal{C}^k_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(\oplus_{q\geq 2} E_{k-2}(q) \otimes_{\Sigma_q} \mathcal{A}^{\otimes q}, \mathcal{A}). \end{cases}$$

The Euler characteristics of $E(q)$ can be read off from the inverse of the generating function of the operad $(G_2\text{-}p^3\mathcal{A}ss)$

$$g_{G_2-p^3\mathcal{A}ss}(t) = t + t^2 + \frac{3}{2}t^3 + \frac{5}{2}t^4 + \frac{53}{12}t^5$$

which is

$$g(t) = t - t^2 + \frac{t^3}{2} + \frac{13}{3}t^5 + O(t^6).$$

We obtain in particular

$$(\quad (4)) = 0.$$

Each one of the modules $E(p)$ is a graded module $(E_*(p))$ and

$$\chi(E(p)) = \dim E_0(p) - \dim E_1(p) + \dim E_2(p) + \cdots$$

We deduce

- $E(2)$ is generated by two degree 0 bilinear operation $\mu_2 : V \cdot V \to V$,
- $E(3)$ is generated by three degree 1 trilinear operation $\mu_3 : V^{\otimes 3} \to V$,
- $E(4) = 0$.

Considering the action of Σ_n on $E(n)$ we deduce that $E(2)$ is generated by a binary operation of degree 0 whose differential satisfies

$$\partial(\mu_2) = 0,$$

$E(3)$ is generated by a trilinear operation of degree one such that

$$\partial(\mu_3) = \mu_2 \circ_1 \mu_2 - \mu_2 \circ_2 \mu_2 + \mu_2 \circ_1 (\mu_2 \cdot \tau_{12}) - (\mu_2 \circ_2 \mu_2) \cdot \tau_{12}.$$

(we have $(\mu_2 \circ_2 \mu_2) \cdot \tau_{12}(a,b,c) = b(ac)$)

Since $E(4) = 0$ we deduce

Proposition 15. *The cohomology* $H^*(\mathcal{A},\mathcal{A})_{defo}$ *which governs deformations of right-alternative algebras is associated to the complex*

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^1} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^2} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^3} \mathcal{C}^4_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \to \cdots$$

with

$$\begin{aligned} &\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 1}, V), \\ &\mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 2}, V), \\ &\mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 3}, V), \\ &\mathcal{C}^4_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 5}, V) \oplus \cdots \oplus Hom(V^{\otimes 5}, V), \end{aligned}$$

In particular any 4*-cochains consists of* 5*-linear maps.*

Alternative algebras

Recall that an alternative algebra is given by the relation

$$A_\mu(x_1,x_2,x_3) = -A_\mu(x_2,x_1,x_3) = A_\mu(x_2,x_3,x_1).$$

Theorem 16. *An algebra* $(\mathcal{A},\mu)$ *is alternative if and only if the associator satisfies*

$$A_\mu \circ \Phi_v^{\mathcal{A}} = 0,$$

with $v = 2Id + \tau_{12} + \tau_{13} + \tau_{23} + c_1$.

Proof. The associator satisfies $A_\mu \circ \Phi_{v_1}^{\mathcal{A}} = A_\mu \circ \Phi_{v_2}^{\mathcal{A}}$ with $v_1 = Id + \tau_{12}$ and $v_2 = Id + \tau_{23}$. The invariant subspace of $\mathbb{K}[\Sigma_3]$ generated by v_1 and v_2 is of dimension 5 and contains the vector $\sum_{\sigma\in\Sigma_3}\sigma$. From literature of Remm [1], the space is generated by the orbit of the vector v.

Proposition 17. *Let* $\mathcal{A}lt$ *be the operad for alternative algebras. Its dual is the operad for associative algebras satisfying*

$$abc - bac - cba - acb + bca + cab = 0.$$

Remark. The current operad $\widetilde{\mathcal{A}lt}$ is the operad for associative algebras satisfying $abc = bac = cba = acb = bca$, that is, 3-commutative algebras so

$$\widetilde{\mathcal{A}lt} = \mathcal{L}ie\mathcal{A}dm^!.$$

In literature of Dzhumadil'daev and Zusmanovich [9], one gives the generating functions of $\mathcal{P} = \mathcal{A}lt$ and $\mathcal{P}^! = \mathcal{A}lt^!$

$$g_{\mathcal{P}}(x) = x + \frac{2}{2!}x^2 + \frac{7}{3!}x^3 + \frac{32}{4!}x^4 + \frac{175}{5!}x^5 + \frac{180}{6!}x^6 + O(x^7),$$

$$g_{\mathcal{P}^!}(x) = x + \frac{2}{2!}x^2 + \frac{5}{3!}x^3 + \frac{12}{4!}x^4 + \frac{15}{5!}x^5.$$

and conclude to the non-Koszulness of $\mathcal{A}lt$.

The operadic cohomology is the cohomology associated to the complex

$$(\mathcal{C}^p_{\mathcal{A}lt}(\mathcal{A},\mathcal{A})_{st} = (Hom(\mathcal{A}lt^!(p)\otimes_{\Sigma_p}\mathcal{A}^{\otimes p},\mathcal{A}),\delta_{st}).$$

Since $\mathcal{A}lt^!(p) = 0$ for $p \geq 6$ we deduce the short sequence

$$\mathcal{C}^1_{\mathcal{A}lt}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{A}lt}(\mathcal{A},\mathcal{A})_{st} \to \cdots \to \mathcal{C}^5_{\mathcal{A}lt}(\mathcal{A},\mathcal{A})_{st} \to 0.$$

But if we compute the formal inverse of the function $-g_{\mathcal{A}lt}(-x)$ we obtain

$$x + x^2 + \frac{5}{6}x^3 + \frac{1}{2}x^4 + \frac{1}{8}x^5 - \frac{11}{72}x^6 + O(x^7).$$

Because of the minus sign it can not be the generating function of the operad $\mathcal{P}^! = \mathcal{A}lt^!$. So this implies also that both operad are not Koszul. But it gives also some information on the deformation cohomology. In fact if $\Gamma(E)$ is the free operad associated to the minimal model, then

$$\dim \chi(E(2)) = -2,$$

$$\dim \chi(E(3)) = -5,$$

$$\dim \chi(E(4)) = -12,$$

$$\dim \chi(E(5)) = -15,$$

$$\dim \chi(E(6)) = +110.$$

Since $\chi(E(6)) = \sum_i (-1)^i \dim E_i(6)$, the graded space $E(6)$ is not concentred in degree even. Then the 6-cochains of the deformation cohomology are 6-linear maps of odd degree.

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