Modules Over Color Hom-Poisson Algebras

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Abstract
In this paper we introduce color Hom-Poisson algebras and show that every color Hom-associative algebra has a non-commutative Hom-Poisson algebra structure in which the Hom-Poisson bracket is the commutator bracket. Then we show that color Poisson algebras (respectively morphism of color Poisson algebras) turn to color Hom-Poisson algebras (respectively morphism of Color Hom-Poisson algebras) by twisting the color Poisson structure. Next we prove that modules over color Hom–associative algebras A extend to modules over the color Hom–Lie algebras L(A), where L(A) is the color Hom-Lie algebra associated to the color Hom-associative algebra A. Moreover, by twisting a color Hom–Poisson module structure map by a color Hom-Poisson algebra endomorphism, we get another one.

Keywords: Color hom-associative algebras; Color hom-Lie algebras; Homomorphism; Formal deformation; Hom-modules; Modules over color Hom-Lie algebras; Modules over color Hom-Poisson algebras

Introduction
Color Hom-Poisson algebras are generalizations of Hom-Poisson algebras introduced in [1], where they emerged naturally in the study of 1-parameter formal deformations of commutative Hom-associative algebras. Color Hom-Poisson algebras generalize, on the one hand, color Hom-associative [2,3] and color Hom-Lie algebras [2,3] which have been recently investigated by various authors. On the other hand, they generalize Hom-Lie superalgebras [4]. These structures are well known to physicists and to mathematicians studying differential geometry and homotopy theory. The cohomology theory of Lie superalgebras [5] has been generalized to the cohomology of Hom-Lie superalgebras in [6]. A cohomology of color Lie algebras was introduced and investigated in [7], and the representations of color Lie algebras were explicitly described in [8]. Modules over Poisson algebras receive various definitions [9,10] we will use them introduced in [9]. The aim of this paper is to study color Hom-Poisson algebras and modules over color Hom-Poisson algebras.

The paper is organized as follows. In section 4, we recall some basic notions related to color Hom-associative algebras and color Hom-Lie algebras. In section 5, we define color Hom-Poisson algebras and point out that to any color Hom-associative algebra ones can associate a color Hom-Lie algebra. Next, starting from a color Poisson algebra and Poisson algebra morphism we get another one by twisting the associative product and Lie bracket. In section 6, we introduce modules over color Hom-Poisson algebras and prove that starting from a color Hom-Poisson module we get another one by twisting the module structure map by a color Hom-Poisson algebra endomorphism, we get another one.

Preliminaries
Let G be an abelian group. A vector space V is said to be a G-graded if, there exist a family \( V_a \) of vector subspaces of V such that
\[
V = \bigoplus_{a \in G} V_a
\]
An element \( x \) \( \in V \) is said to be homogeneous of degree \( a \) \( \in G \) if \( x \in V_a \). We denote H(V) the set of all homogeneous elements in V.

Let \( V = \bigoplus_{a \in G} V_a \) and \( V' = \bigoplus_{a \in G} V'_a \) be two G-graded vector spaces. A linear mapping \( f : V \rightarrow V' \) is said to be homogeneous of degree b if \( f(V_a) \subseteq V_{a+b}, \forall a \in G \). If f is homogeneous of degree zero i.e. \( f(V_a) \subseteq V_a \) holds for any \( a \in G \) then f is said to be even.

An algebra \( (A, \mu) \) is said to be G-graded if its underlying vector space is G-graded i.e. \( A = \bigoplus_{a \in G} A_a \) and if furthermore \( \mu(A_a, A_b) \subseteq A_{a+b} \) for all \( a, b \in G \).

Let \( A' \) be another G-graded algebra. A morphism \( f : A \rightarrow A' \) of G-graded algebras is by definition an algebra morphism from A to \( A' \) which is, in addition an even mapping.

Definition
Let \( G \) be an abelian group. A map \( \epsilon : G \times G \rightarrow K^* \) is called a skew-symmetric bicharacter on \( G \) if the following identities hold,
1. \( \epsilon(a,b) = \epsilon(b,a) = 1 \)
2. \( \epsilon(a+b+c) = \epsilon(a,c) \epsilon(b,c) \)
3. \( \epsilon(a,b) \epsilon(b,a) = 1 \)

Remark that \( \epsilon(0,a) = \epsilon(a,0) = 1, \epsilon(a,a) = \pm 1 \) for all \( a \in G \).

Where, 0 is the identity of \( G \). If x and y are two homogeneous elements of degree \( a \) and \( b \) respectively and \( \epsilon \) is a skew-symmetric bicharacter, then we shorten the notation by writing \( \epsilon(x, y) \) instead of \( \epsilon(a, b) \).

Definition
A color Hom-associative algebra is a quadruple \( (A, \mu, \epsilon, \alpha) \) consisting of a G-graded vector space A, an even bilinear map \( \mu : A \times A \rightarrow A \) and an even linear map such \( \alpha : A \rightarrow A \) that

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\[ \mu (a(x), a(y)) = a (\mu(x, y)) \]  
\[ \mu(a(x), \mu(y, z)) = \mu(\mu(x, y), a(z)) \] (1) (2)

If in addition \( \mu(x, y) = \epsilon(x, y) \mu(y, x) \) the color Hom-associative algebra \((A, \mu, \epsilon, \alpha)\) is said to be a \(\epsilon\)-commutative color Hom-associative algebra.

**Remark**

When \(\alpha = \text{Id} \) we recover the classical associative color algebra.

Recall that the Hom-associator, \(aS\), of a Hom-algebra \(A\) is defined as:

\[ aS : A\otimes A\otimes A \to A \]

Observe that \(aS = 0\) when \(A\) is a color-Hom-associative algebra.

**Definition**

Let \((A, \mu, \epsilon, \alpha)\) and \((A', \mu', \epsilon', \alpha')\) be two color Hom-associative algebras.

An even linear map \(f : A \to A'\) is then said to be a morphism of color Hom-associative algebras if \(f \circ \alpha = \alpha' \circ f\) and

\[ f(\mu(x, y)) = \mu'(f(x), f(y)) \]

For all \(x, y \in A\).

**Lemma**

((17)) Let \((A, \mu, \epsilon, \alpha)\) be a color associative algebra and \(\alpha\) be an even linear map \(\alpha : A \to A\).

If \(\mu(x, y) = \epsilon(x, y) \mu(y, x)\) is a \(\epsilon\)-commutative color Hopf algebra, then \(\alpha = \text{Id}\) is a color Hom-associative algebra.

**Definition**

((17)) A color Hom-Poisson algebra consists of a G-graded vector space \(A\), an even bilinear homomorphism \(\mu : A\otimes A \to A\) and an even linear map \(\alpha : A \to A\).

We recover Poisson algebras ([6, 5]) when \(\alpha = \text{Id}\) and \(\epsilon \equiv 1\).

We need the following lemma in Proposition 6.1.

**Lemma**

If \((A, \mu, \epsilon, \alpha)\) is a \(\epsilon\)-commutative color Hom-Poisson algebra, then \((A, -\mu, -\epsilon, \alpha)\) is also a \(\epsilon\)-commutative color Hom-Poisson algebra.

The following theorem is the color version of ([11], Proposition 4.6).

**Theorem**

Let \((A, \mu, \epsilon, \alpha)\) be a color Hom-associative algebra.

Then \((A, \alpha \mu, \epsilon, \alpha)\) is a color Hom-Poisson algebra.

**Proof**

According to Lemma 4.2, it remains to prove the color Hom-Leibniz identity 7. For any \(x, y, z \in H(A)\)

\[ \{\alpha(x), \mu(y, z)\} - \mu(\{x, y\}, \alpha(z)) - \epsilon(x, y)\mu(\alpha(y), \{x, z\}) = \]

\[ = \epsilon(x, \mu(y, z)) \mu(\alpha(y), \{x, z\}) - \epsilon(x, y) \mu(\alpha(y), \{x, z\}) = \]

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This finishes the proof.

**Corollary**

Let \((A, \mu, \epsilon, \alpha)\) be a color associative algebra and \(\alpha\) an even color algebra endomorphism. Then \((A, \mu \alpha, \epsilon, \alpha)\)
\[ \mu = a \circ \mu, \{ \} = \mu - \varepsilon(\cdot, \cdot)\beta_{\mu} \] is a color Hom-Poisson algebra.

**Proof**

The proof follows from Lemma 4.1 and Theorem 3.1.

**Lemma**

Let \((A, \mu \{ \}, \varepsilon, \alpha)\) be a color Poisson algebra and \(\alpha\) be an even color Poisson algebra endomorphism. Then \((A, \mu_{\alpha} \{ \}, \varepsilon_{\alpha}, \alpha)\) is a color Hom-Poisson algebra.

**Proof**

By Lemma 4.1 and ([3] Example 1.2), we only need to prove the color Hom-Leibniz identity. For any \(x, y, z \in H(A)\),

\[ [\alpha(x), \mu_{\alpha}(y, z)] = [\mu_{\alpha}(x, y), \alpha(z)] - [\mu_{\alpha}(x, \alpha(z)), y] + [\mu_{\alpha}(\alpha(x), y), z] = 0, \]

This completes the proof.

**Theorem**

Let \((A, \mu \{ \}, \varepsilon, \alpha)\) be a color Hom-Poisson algebra and \(\beta: A \to A\) be an even color Poisson algebra endomorphism. Then \((A, \mu_{\beta} \{ \}, \varepsilon, \beta \circ \alpha)\) is a color Hom-Poisson algebra.

Moreover, suppose that \((\tilde{A}, \mu', \{ \}, \varepsilon', \alpha')\) is a color Poisson algebra and \((A, \mu \{ \}, \varepsilon, \alpha)\) is an even color Poisson algebra endomorphism. If \(f: A \to \tilde{A}\) is a color Poisson algebra morphism that satisfies \(f \circ \beta = \alpha \circ f\), then

\[ f(A, \mu_{\beta} \{ \}, \varepsilon, \beta \circ \alpha) \to (A', \mu_{\beta}' \{ \}, \varepsilon, \alpha') \]

is a color Hom-Poisson algebra homomorphism.

**Proof**

It is straightforward to show that \((A, \mu_{\beta} \{ \}, \varepsilon, \beta \circ \alpha)\) is a color Hom associative algebra and \((A, \mu \{ \}, \varepsilon, \alpha)\) is a color Hom-Leibniz algebra ([3] Theorem 1.1). The proof of the color Hom-Leibniz identity is similar to that of Lemma 5.2. For the second assertion, we have

\[ f(\mu_{\beta}(x, y)) = f(\mu(\beta(x), \beta(y))) = \mu'(f(\beta(x)), f(\beta(y))) = \mu'(f(\alpha(x)), f(\alpha(y)) = \mu''(f(x), y)). \]

We have a similar proof for the color Hom-Poisson bracket.

**Corollary 5.2**

Let \((A, \mu \{ \}, \varepsilon, \alpha)\) be a color Hom-Poisson algebra. Then

\[ A^\alpha = (A, \{ \}^{(\alpha)} = A^{\alpha} \circ \{ \} = A^{\alpha} \circ \mu, \varepsilon, \alpha^{(\alpha)} = \alpha \circ \{ \}, \varepsilon_{\alpha}, \alpha) \]

is a color Hom-Poisson algebra for each integer \(n \geq 0\). We finish this section by studying deformations by composition of color Hom-Poisson algebras.

**Definition 5.2**

Let \((A, \mu \{ \}, \varepsilon, \alpha)\) be a color Hom-Poisson algebra. A one parameter formal deformation of \(A\) is given by \(K[t]\)-bilinear maps \(\mu_{\theta}: \mathcal{A}[t][\otimes] \to \mathcal{A}[t]\) and \(\{ \}_\theta: \mathcal{A}[t][\otimes] \to \mathcal{A}[t]\) of the form \(\mu = \sum_{n=0}^\infty \theta^n \mu_{\theta^n}\) and \(\{ \} = \sum_{n=0}^\infty \theta^n \{ \}_{\theta^n}\) where each \(\mu_{\theta^n}\) and \(\{ \}_{\theta^n}\) are \(K\)-bilinear maps and (extended to \(K[[t]]\)-bilinear maps), and such that for all following conditions be satisfied.

\[ \mu_\theta(\alpha(x), \alpha(y)) = \alpha(\mu_\theta(x, y)), \]

\[ \mu_\theta(\alpha(x), \mu_\theta(y, z)) = \mu_\theta(\mu_\theta(x, y), \alpha(z)), \]

\[ \alpha \circ \{ \}_{\theta} = \{ \}_{\theta} \circ \alpha_{\theta^2}, \]

\[ \{ x, y \} = -\varepsilon(x, y)\{ y, x \}, \]

\[ [\alpha(x), \mu_\theta(y, z)] = \mu_\theta((x, y), \alpha(z)) + \varepsilon(x, y)\mu_\theta(\alpha(y), \{ y, z \}). \]

The deformation is said to be of order \(k\) if \(\mu_\theta = \sum_{i=0}^k \theta^i \mu_{\theta^i}\) and \(\{ \}_{\theta^i} = \sum_{i=0}^k \theta^i \{ \}_{\theta^i}\).

**Proposition 5.1**

Let \((A, \mu \{ \}, \alpha)\) be a color Poisson algebra and \(\alpha\) an even color Poisson algebra endomorphism of the form \(\alpha = \alpha_0 + \sum_{k=1}^\infty \theta^k \alpha_k\) where \(\alpha_k\) are endomorphism of \(A\) (as color Poisson algebra), \(t\) is a parameter in \(K\) and \(k\) is an integer. Let \(\mu, \alpha, \mu, \alpha\) and \(\{ \}, \varepsilon, \alpha\) then \((A, \mu \{ \}, \varepsilon, \alpha)\) is a color Hom-Poisson algebra which is a deformation of the color Poisson algebra \((A, \mu \{ \}, \varepsilon, \alpha)\) viewed as a color Hom-Poisson algebra \((A, \mu \{ \}, \varepsilon, \alpha)\).

**Proof**

The proof follows from Theorem 2.

As in the case of Poisson algebras ([10,12,13]), the cohomology of color Hom-Poisson algebras is described by the cohomology of the underlying color Hom-Lie algebras ([3]).

**Modules Over Color Hom-Poisson Algebras**

**Definition 6.1**

Let \(G\) be an abelian group. A Hom-module is a pair \((M, \mu_M)\) in which \(M\) is a \(G\)-graded vector space and \(\mu_M : M \otimes M \to M\) is an even linear map.

**Definition 6.2**

Let \((A, \mu \{ \}, \varepsilon, \alpha)\) be a color Hom-associative algebra. An A-module is a Hom-module \((M, \mu_M)\) together with a bilinear map \(\mu_M : A \otimes M \to M\) called structure map, such that

\[ \mu_M(A, \mu_M) \subseteq M_{\alpha \beta}, \]

\[ \alpha_M \circ \mu_M = \mu_M \circ (\alpha_M \otimes \alpha_M), \]

\[ \mu_M(\alpha_M \otimes \mu_M) = \mu_M \circ (\mu_M \otimes \alpha_M) \]

Twisting a module structure map by an algebra endomorphism, we get another one as stated in the following Lemma.

**Lemma 6.1**

Let \((A, \mu \{ \}, \varepsilon, \alpha)\) be a color Hom-associative algebra and \(M\) an A-module with structure map \(\mu_M : A \otimes M \to M\). Define the map

\[ \mu_M = \mu_M \circ (\alpha_M \otimes \text{Id}_M) : A \otimes M \to M. \]

Then \(\mu_M\) is the structure map of another A-module structure on \(M\).
Proof
The proof is similar to that of ([14], Lemma 4.5).

Definition 6.3
([13]) Let \((L, \ldots, e, \alpha)\) be a color Hom-Lie algebra and \((M, \alpha_M)\) a Hom-module. An \(L\)-module on \(M\) consists of a \(K\)-bilinear map \(\mu_M : A \otimes M \rightarrow M\) such that

\[
\mu_M(A_x, M_m) \subseteq M_{\alpha_{\alpha}(x)(\alpha_M(m))},
\]

(19)

\[
\mu_M((x, y), \alpha_M(m)) = \mu_M(\alpha_M(x), \alpha_M(y), m))
\]

(20)

\[
- \epsilon(x, y)\mu_M(\alpha(y), \alpha_M(x), m))
\]

(21)

for any \(m \in H(M), x, y \in H(L)\).

Remark 6.1
When \(\alpha = \text{Id}_M\) and \(\alpha = \text{Id}_L\), we recover the definition of Lie modules ([15-17]).

The following statement is the Lie analogue of Lemma 6.1.

Lemma 6.2
Let \((L, \ldots, e, \alpha)\) be a color Hom-Lie algebra and \((M, \alpha_M)\) an \(L\)-module with structure map \(\mu_M = A \otimes M \rightarrow M\). Define the map

\[
\tilde{\mu}_M = \mu_M \circ (A^2 \otimes \text{Id}_M) : L \otimes M \rightarrow M
\]

(22)

Then \(\tilde{\mu}_M\) is the structure map of another \(L\)-module structure on \(M\).

Proof
Equalities 19 and 20 are proved as in Lemma 6.1. Now, we prove

21 for \(\tilde{\mu}_M\). For any \(x, y \in L, m \in M\)

\[
\tilde{\mu}_M((x, y), \alpha_M(m)) = \mu_M(\alpha(x), \alpha(y), \alpha_M(m))
\]

(23)

\[
= \mu_M(\alpha_M(x), \alpha_M(y), \alpha_M(m))
\]

(24)

\[
- \epsilon(x, y)\mu_M(\alpha_M(x), \alpha_M(y), m))
\]

(25)

Hence the conclusion holds.

The following result shows that \(A\)-modules extend to \(L(A)\)-modules with samemodule structure map.

Theorem 6.1
Let \((A, \mu, \epsilon, \alpha)\) be a color Hom-associative algebra and \((M, \alpha_M)\) an \(A\)-module with structure map \(\mu_M\). Then, \(M\) is an \(L(A)\)-module with structure map \(\tilde{\mu}_M\).

Proof
In fact, it suffices to show the relation 21. For any \(x, y \in H(A), m \in H(M)\), we have

\[
\tilde{\mu}_M((x, y), \alpha_M(m)) = \mu_M(\alpha(x), \alpha(y), \alpha_M(m))
\]

(26)

\[
= \mu_M(\mu_M(x, y), \alpha_M(m))
\]

(27)

\[
= \mu_M(\mu_M(x, y), \alpha_M(m))
\]

(28)

This is similar to the relation 21 for \(\tilde{\mu}_M\).

Now we define modules for color Hom-Poisson algebras.

Definition 6.4
Let \((A, \mu, \ldots, e, \alpha)\) be a color commutative color Hom-Poisson algebra and \((M, \alpha_M)\) a Hom-module.
A color Hom-Poisson module structure on $M$ consists of two $K$-bilinear maps $\mu_M: A \otimes M \to M$ and $\lambda_M: A \otimes M \to M$ such that

(i) $M$ is an $A$-module and an $L$-module,

(ii) And for any $x, y \in H(A), m \in H(M)$,

\[ \lambda_M(\alpha(x), \mu_M(y, m)) = \mu_M(\{x, y\}, \alpha_M(m)) + \varepsilon(x, y)\mu_M(\alpha(y), \lambda_M(x, m)), \quad (23) \]

\[ \lambda_M(\mu(x, y), \alpha_M(m)) = \mu_M(\alpha(x), \lambda_M(y, m)) + \varepsilon(x, y)\mu_M(\alpha(y), \lambda_M(x, m)). \quad (24) \]

When $\alpha = \text{Id}_A, \alpha_M = \text{Id}_M$ and $\varepsilon = 1$

We recover the definition of modules over Poisson algebras ([9]).

**Example 6.1**

(i) Any module over a $\varepsilon$ commutative color Hom-associative algebra (resp. color Hom-Lie algebra) can be seen as a module over a $\varepsilon$ commutative color Hom-Poisson algebra with the trivial color Hom-Lie bracket (resp. trivial color Hom-associative product).

(ii) Any $\varepsilon$ commutative color Hom-Poisson algebra is a module over itself.

**Example 6.2**

Let $(V, \mu_V, \lambda_V, \alpha_V)$ and $(W, \mu_W, \lambda_W, \alpha_W)$ be two modules over the $\varepsilon$ Commutative color Hom-Poisson algebra $(A, \mu, \{\cdot,\cdot\}, \varepsilon, \alpha)$

Then the direct product $M = V \times W$ is a module over $A$ with structure maps $\mu_M: A \otimes M \to M, \lambda_M: A \otimes M \to M$ and $\alpha_M: M \to M$

Defined by

$\mu_M((x, y), (v, w)) = (\mu_V(x, v), \mu_W(x, w)), \lambda_M((x, y), (v, w)) = (\lambda_V(x, v), \lambda_W(x, w))$ and $\alpha_M((x, y), (v, w)) = (\alpha_V(x, v), \alpha_W(x, w))$ for any $x \in H(A), v \in H(V)$ and $w \in H(W)$.

**Proposition 6.1**

If $(M, \mu_M, \lambda_M, \alpha_M)$ is a module over the $\varepsilon$ commutative color Hom-Poisson algebra $(A, \mu, \{\cdot,\cdot\}, \varepsilon, \alpha)$ then $(M, \mu_M^*, \lambda_M^*, \alpha_M^*)$ is also a module.

Over the $\varepsilon$ commutative color Hom-Poisson algebra $(A, \mu, \{\cdot,\cdot\}, \varepsilon, \alpha)$

**Proof**

The proof comes from Definition 6.4 and Lemma 5.1.

**Theorem 6.2**

Let $(A, \mu, \{\cdot,\cdot\}, \varepsilon, \alpha)$ be an $\varepsilon$ commutative color Hom-Poisson algebra and $(M, \mu_M^*, \lambda_M^*, \alpha_M^*)$ color Hom-Poisson module. Then

$\mu_M^* = \mu_M \circ (\alpha^* \otimes I_{\lambda_M^*}): A \otimes M \to M$,

$\lambda_M^* = \lambda_M \circ (\alpha^* \otimes I_{\lambda_M^*}): A \otimes M \to M$.

Define another color Hom-Poisson module structure on $M$.

**Proof**

We know that $\mu_M$ is a structure of another $A$-module structure map on $M$ (Lemma 6.1) and $\lambda_M$ is a structure of another $L$-module structure map on $M$ (Lemma 6.2). Show relations 23 and 24 for $\mu_M$ and $\lambda_M$. For all $x, y \in H(A)$ and $m \in H(M)$

\[ \tilde{\lambda}_M(\alpha(x), \tilde{\mu}_M(y, m)) = \tilde{\lambda}_M(\alpha(x), \tilde{\mu}_M(y \otimes m)) \]

\[ = \tilde{\lambda}_M(\alpha(x), \tilde{\mu}_M(\alpha^*(y), \lambda_M(x, m))) \]

\[ = \tilde{\lambda}_M(\alpha(x), \tilde{\mu}_M(\alpha^*(y) \otimes m)) \]

\[ = \lambda_M(\alpha^*(y), \lambda_M(\alpha(x), m)) \quad (by \ 23) \]

\[ = \lambda_M(\alpha^*(y), \lambda_M(\alpha(x), m)) \quad (by \ 24) \]

\[ + \varepsilon(x, y)\mu_M(\alpha(y), \lambda_M(\alpha^*(x), m)) \]

\[ + \varepsilon(x, y)\mu_M(\alpha(y), \lambda_M(\alpha^*(x) \otimes m)) \]

\[ = \tilde{\mu}_M(\alpha(x), \tilde{\lambda}_M(y \otimes m)) + \varepsilon(x, y)\tilde{\mu}_M(\alpha(y), \tilde{\lambda}_M(x \otimes m)). \]

Hence equations 23 and 24 hold for $\mu_M$ and $\lambda_M$. This completes the proof.

**Corollary 6.3**

Let $(A, \mu, \{\cdot,\cdot\}, \varepsilon, \alpha)$ be an $\varepsilon$ commutative color Poisson algebra and $(M, \mu_M^*, \lambda_M^*, \alpha_M^*)$ a module over the color Hom-Poisson algebra $(A, \mu, \{\cdot,\cdot\}, \varepsilon, \alpha)$. Then $\mu_M, \lambda_M$ define another color Hom-Poisson module structure on $M$.

**References**


