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Asymptotically Bad Towers of Function Fields

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Abstract. In this paper we study general conditions to prove the infiniteness of the genus of certain towers of function fields over a perfect field. We show that many known examples of towers with infinite genus are particular cases of these conditions. In the case of tame towers we show that the infiniteness of their genus is actually equivalent to these conditions.

1. Introduction

The aim of this paper is to study, on the one hand, general sufficient conditions to show that the genus of certain towers $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields over a perfect field is infinite and, on the other hand, to give some criteria which can be useful in proving the infiniteness of the genus of recursive towers, an interesting case for their potential applications to coding theory. Several authors have dealt with these kind of problems in the case of recursive towers as can be seen in [4], [1], [7] and [2]. More structural approaches, in the sense that they hold for any kind of tower (recursive or not), were given in [3] and [5]. More precisely, it was proved in [3] that an abelian tower $\mathcal{F}$ (i.e. the extension F_i/F_0 is abelian for all $i > 0$) always has infinite genus, while in [5] a Galois tower $\mathcal{F}$ (i.e. the extension F_i/F_0 is Galois for all $i > 0$) was shown to have infinite genus if the number of ramified places of F_0 in the tower is infinite. We will work in the general setting of towers $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields F_i over a perfect field K with some additional properties (see Section 2 for precise definitions). Roughly speaking the type of towers we will consider in this paper are towers in the sense of Garcia and Stichtenoth (see Chapter 7 of [8]) without asking the condition that $g(F_i)$, the genus of F_i, tends to infinity as i tends to infinity. The reason in doing so is that there is no need of this condition when proving that a given tower has infinite genus. Once this is proved we have that $g(F_i)$ must tend to infinity as i tends to infinity so that the considered tower is, in fact, a tower in the sense of Garcia and Stichtenoth, which is, additionally, asymptotically bad.

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In the case of recursive towers (i.e. each extension F_{i+1}/F_i is defined by using some of the roots of a bivariate polynomial with coefficients in K) there is a simple sufficient condition for bad asymptotic behavior, namely that the recursive tower is skew in the sense that the degree of the polynomial defining the tower is not the same in each variable (see [6] for details). Unfortunately this condition is not necessary because there are examples of non skew asymptotically bad towers. Interestingly, when a tower is non skew, it seems equally hard to prove that the tower is asymptotically good or asymptotically bad.

The organization of the paper is as follows. In Section 2 we give the basic definitions and we establish the notation to be used throughout the paper. In Section 3 we prove our main results in the general setting of towers of function fields over a perfect field K and in Section 4 we give examples of asymptotically bad towers showing on the one hand that many known examples are particular cases of our general results and, on the other hand, that all can be obtained in a unified way. In particular, we prove that the infiniteness of the genus of a tame tower is equivalent to the conditions given in Proposition 3.2. The key in this general and unified treatment of the bad asymptotic behavior of towers is the existence of a particular divisor which is far from being obvious in some of the given examples.

2. Notation and Definitions

In this work we shall be concerned with *towers* which means that $\mathcal{F} = (F_0, F_1, \ldots)$ is an infinite sequence of function fields over K, where for each index $i \geq 0$ the field F_i is a proper subfield of F_{i+1}, the field extension F_{i+1}/F_i is finite and separable and K is the full field of constants of each field F_i (i.e. K is algebraically closed in each F_i). If the genus $g(F_i) \to \infty$ as $i \to \infty$ we shall say that $\mathcal{F}$ is a *tower in the sense of Garcia and Stichtenoth.*

Following [8] (see also [6]), one way of constructing towers of function fields over K is to be given by a bivariate polynomial

$$H \in K[X, Y],$$

and a transcendental element x_0 over K. In this situation a tower $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields over K is defined as

(i) $F_0 = K(x_0)$, and
(ii) $F_{i+1} = F_i(x_{i+1})$, where $H(x_i, x_{i+1}) = 0$ for $i \geq 0$.

A suitable choice of the bivariate polynomial H must be made in order to have towers. When the choice of H satisfies all the required conditions we shall say that the tower $\mathcal{F}$ constructed in this way is a *recursive tower* of function fields over K. Note that for a recursive tower $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields over K we have that

$$F_i = K(x_0, \ldots, x_i) \quad \text{for } i \geq 0,$$

where $\{x_i\}_{i=0}^{\infty}$ is a sequence of transcendental elements over K.

Associated to a recursive tower $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields F_i over K we have the so called *basic function field* $K(x, y)$, where x is transcendental over K and $H(x, y) = 0$.

For the sake of simplicity we shall say from now on that H defines the tower $\mathcal{F}$ or, equivalently, that the tower $\mathcal{F}$ is recursively defined by the equation $H(x, y) = 0$.

A tower $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields over a perfect field K of positive characteristic is called *tame* if the ramification index $e(Q|P)$ of any place Q of F_{i+1} lying above a place P of F_i is relatively prime to the characteristic of K for all $i \geq 0$. Otherwise the tower $\mathcal{F}$ is called *wild*.

The following definitions are important when dealing with the asymptotic behavior of a tower. This concept can be defined when K is a finite field. Let $\mathcal{F} = (F_0, F_1, \ldots)$ be a tower of function fields over a finite field $\mathbf{F}_q$ with q elements. The *splitting rate* $\nu(\mathcal{F})$ and the *genus* $\gamma(\mathcal{F})$ of $\mathcal{F}$ over F_0 are defined, respectively, as

$$\nu(\mathcal{F}) := \lim_{i\to\infty} \frac{N(F_i)}{[F_i : F_0]}, \quad \gamma(\mathcal{F}) := \lim_{i\to\infty} \frac{g(F_i)}{[F_i : F_0]},$$

where $N(F_i)$ denotes the number of rational places (i.e. places of degree 1) of F_i. If there is an index $i_0 \geq 0$ such that $g(F_i) \geq 2$ for every $i \geq i_0$, then the *limit* $\lambda(\mathcal{F})$ of $\mathcal{F}$ is defined as

$$\lambda(\mathcal{F}) := \lim_{i\to\infty} \frac{N(F_i)}{g(F_i)}.$$

It can be seen that all the above limits exist and that $\lambda(\mathcal{F}) \geq 0$ (see [8, Chapter 7]).

We shall say that a tower $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields over $\mathbf{F}_q$ is *asymptotically good* if $\nu(\mathcal{F}) > 0$ and $\gamma(\mathcal{F}) < \infty$. If either $\nu(\mathcal{F}) = 0$ or $\gamma(\mathcal{F}) = \infty$ we shall say that $\mathcal{F}$ is *asymptotically bad*. It is easy to check that $\mathcal{F}$ is asymptotically good if and only if $\lambda(\mathcal{F}) > 0$.

From the well-known Hurwitz genus formula (see [8, Theorem 3.4.13]) we see that the condition $g(F_i) \geq 2$ for $i \geq i_0$ in the definition of $\lambda(\mathcal{F})$ implies that $g(F_i) \to \infty$ as $i \to \infty$. Hence, when we speak of the limit of a tower of function fields we are actually speaking of the limit of a tower in the sense of Garcia and Stichtenoth (see [8, Section 7.2]).

In this paper we will concentrate on the genus $\gamma(\mathcal{F})$ of a tower $\mathcal{F}$ over a perfect field K. Notice that the finiteness of the field K is essential for the definition of the splitting rate of a tower but is not needed in the definition of the genus of a tower. Throughout this work the set of places of a function field F over K will be denoted by $\mathbf{P}(F)$.

3. Bad towers

As was mentioned in the introduction, a simple and useful condition implying that $H \in \mathbf{F}_q[x, y]$ does not give rise to an asymptotically good recursive tower $\mathcal{F}$ of function fields over $\mathbf{F}_q$ is that $\deg_x H \neq \deg_y H$. With this situation in mind we shall say that a recursive tower $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields over a perfect field K defined by a polynomial $H \in K[x, y]$ is *non skew* if $\deg_x H = \deg_y H$. In the skew case (i.e. $\deg_x H \neq \deg_y H$) we might have that $[F_{i+1} : F_i] \geq 2$ for all $i \geq 0$ and even that $g(F_i) \to \infty$ as $i \to \infty$ but, nevertheless, $\mathcal{F}$ will be asymptotically bad. What happens is that if $\deg_y H > \deg_x H$ then the splitting rate $\nu(\mathcal{F})$ is zero (this situation makes sense in the case $K = \mathbf{F}_q$) and if

$\deg_x H > \deg_y H$ the genus $\gamma(\mathcal{F})$ is infinite (see [6] for details). Therefore the search for conditions for bad asymptotic behavior must be focused on non skew towers.

Since nothing else on the polynomial H seems to help to decide if a tower defined by H is asymptotically bad, the natural step in the search for conditions for bad asymptotic behavior of a tower $\mathcal{F} = (F_0, F_1, \ldots)$ is to look for upper bounds for $N(F_i)$ or lower bounds for $g(F_i)$ with the hope of proving that the given tower has either zero splitting rate or infinite genus. In the present work we will focus in the latter case. The starting point in all the known results in this direction is the following proposition involving the different of F'/F which is a divisor of F' defined as

$$\operatorname{Diff}(F'/F) = \sum_{P \in \mathbf{P}(F)} \sum_{P'|P} d(P'|P)P',$$

where $d(P'|P)$ is the different exponent of P' over P (see [8, Section 3.4] for its definition and basic properties). We recall that $d(P'|P) \geq e(P'|P) - 1$ with equality if and only if the ramification index $e(P'|P)$ is not divisible by the characteristic of K (see Dedekind's different theorem [8, Theorem 3.51]). From now on K will denote a perfect field and we recall that K is assumed to be the full field of constants of each function field F_i of any given tower $\mathcal{F}$ over K.

PROPOSITION 3.1. *Let $\mathcal{F} = (F_0, F_1, \ldots)$ be a tower of function fields over K. Suppose that there is a subtower $\mathcal{F}' = (F_{s_1}, F_{s_2}, \ldots)$ such that the series*

$$\sum_{i=1}^{\infty} \frac{\deg \operatorname{Diff}(F_{s_i}/F_{s_{i-1}})}{[F_{s_i} : F_0]}\,, \tag{3.1}$$

where F_{s_0} denotes F_0, is divergent. Then $\gamma(\mathcal{F}) = \infty$, which implies that $\mathcal{F}$ is an asymptotically bad tower of function fields over K. Reciprocally, if $\gamma(\mathcal{F}) = \infty$ then the series (3.1) *is divergent for any subtower $(F_{s_1}, F_{s_2}, \ldots)$ of $\mathcal{F}$. In particular, if $\mathcal{F}$ is an asymptotically bad tower of function fields over $\mathbf{F}_q$ and $\nu(\mathcal{F}) > 0$ then the series* (3.1) *is divergent for any subtower $(F_{s_1}, F_{s_2}, \ldots)$.*

PROOF. The proposition follows easily from the fact that the genus $\gamma(\mathcal{F})$ of the tower $\mathcal{F}$ can be written as

$$\gamma(\mathcal{F}) = \lim_{i\to\infty} \frac{g(F_{s_i})}{[F_{s_i} : F_0]} = g(F_0) - 1 + \frac{1}{2}\sum_{i=1}^{\infty} \frac{\deg \operatorname{Diff}(F_{s_i}/F_{s_{i-1}})}{[F_{s_i} : F_0]}. \tag{3.2}$$

This can be easily proved by induction. In fact, using Hurwitz genus formula (see [8, Theorem 3.4.13]) for the extension F_{s_1}/F_0 we have that

$$2(g(F_{s_1}) - 1) = [F_{s_1} : F_0](2g(F_0) - 2) + \deg \operatorname{Diff}(F_{s_1}/F_0)$$

and hence

$$\frac{g(F_{s_1})-1}{[F_{s_1}:F_0]} = g(F_0)-1+\frac{1}{2}\frac{\deg \mathrm{Diff}(F_{s_1}/F_0)}{[F_{s_1}:F_0]}.$$

Now suppose that

$$\frac{g(F_{s_i})-1}{[F_{s_i}:F_0]} = g(F_0)-1+\frac{1}{2}\sum_{j=1}^{i}\frac{\deg \mathrm{Diff}(F_{s_j}/F_{s_{j-1}})}{[F_{s_j}:F_0]},$$

where F_{s_0} denotes F_0. Then by Hurwitz genus formula for the extension $F_{s_{i+1}}/F_{s_i}$ we have

$$2(g(F_{s_{i+1}})-1) = [F_{s_{i+1}}:F_{s_i}](2g(F_{s_i})-2)+\deg \mathrm{Diff}(F_{s_{i+1}}/F_{s_i})$$

and therefore

$$\begin{aligned}
\frac{g(F_{s_{i+1}})-1}{[F_{s_{i+1}}:F_0]} &= \frac{g(F_{s_i})-1}{[F_{s_i}:F_0]}+\frac{1}{2}\frac{\deg \mathrm{Diff}(F_{s_{i+1}}/F_{s_i})}{[F_{s_{i+1}}:F_0]}\\
&= g(F_0)-1+\frac{1}{2}\sum_{j=1}^{i}\frac{\deg \mathrm{Diff}(F_{s_j}/F_{s_{j-1}})}{[F_{s_j}:F_0]}+\frac{1}{2}\frac{\deg \mathrm{Diff}(F_{s_{i+1}}/F_{s_i})}{[F_{s_{i+1}}:F_0]}\\
&= g(F_0)-1+\frac{1}{2}\sum_{j=1}^{i+1}\frac{\deg \mathrm{Diff}(F_{s_j}/F_{s_{j-1}})}{[F_{s_j}:F_0]}.
\end{aligned}$$

Since the limit $g(F_n)/[F_n:F_0]$ always exists and $[F_{s_{i+1}}:F_0]\to\infty$ as $i\to\infty$, we have that

$$\begin{aligned}
\gamma(\mathcal{F}) &= \lim_{n\to\infty}\frac{g(F_n)}{[F_n:F_0]}\\
&= \lim_{n\to\infty}\frac{g(F_n)-1}{[F_n:F_0]}\\
&= \lim_{i\to\infty}\frac{g(F_{s_i})-1}{[F_{s_i}:F_0]}\\
&= g(F_0)-1+\frac{1}{2}\lim_{i\to\infty}\sum_{j=1}^{i}\frac{\deg \mathrm{Diff}(F_{s_j}/F_{s_{j-1}})}{[F_{s_j}:F_0]}\\
&= g(F_0)-1+\frac{1}{2}\sum_{j=1}^{\infty}\frac{\deg \mathrm{Diff}(F_{s_j}/F_{s_{j-1}})}{[F_{s_j}:F_0]},
\end{aligned}$$

which is the desired result. Finally, if $\gamma(\mathcal{F})=\infty$ the reciprocal result follows easily from (3.2). □

Proposition 3.1 was used in [4], [7], [1] and [2], where the authors give different conditions to prove that some non skew recursive towers are asymptotically bad. However, not stated explicitly in the just mentioned articles, the infiniteness of the genus of the given towers is deduced from the following simple result.

PROPOSITION 3.2. *Suppose that $\mathcal{F} = (F_0, F_1, \ldots)$ is a tower of function fields over K and that there exist positive functions $c_1(t)$ and $c_2(t)$, defined for $t \geq 0$, and a divisor $B_i \in \mathcal{D}(F_i)$ such that for each $i \geq 1$*

(a) $\deg B_i \geq c_1(i)[F_i : F_0]$ *and*
(b) $\sum_{P \in \mathrm{supp}(B_i)} \sum_{Q|P} d(Q|P) \deg Q \geq c_2(i)[F_{i+1} : F_i] \deg B_i$,

where the inner sum runs over all places Q of F_{i+1} lying above P. If the series

$$\sum_{i=1}^{\infty} c_1(i) c_2(i) \tag{3.3}$$

is divergent then $\gamma(\mathcal{F}) = \infty$.

PROOF. From (a) and (b) we immediately see that

$$\begin{aligned} \deg \mathrm{Diff}(F_{i+1}/F_i) &= \sum_{P \in \mathbf{P}(F_i)} \sum_{Q|P} d(Q|P) \deg Q \\ &\geq \sum_{P \in supp(B_i)} \sum_{Q|P} d(Q|P) \deg Q \\ &\geq c_2(i)[F_{i+1} : F_i] \deg B_i \\ &\geq c(i)\,[F_{i+1} : F_0]\,, \end{aligned} \tag{3.4}$$

where $c(i) = c_1(i) c_2(i)$. Therefore

$$\sum_{i=1}^{\infty} \frac{\deg \mathrm{Diff}(F_{i+1}/F_i)}{[F_{i+1} : F_0]} \geq \sum_{i=1}^{\infty} c(i)\,,$$

and it follows from Proposition 3.1 that $\gamma(\mathcal{F}) = \infty$. □

REMARK 3.3. Condition (a) in the above proposition can be replaced by the following one when $\mathcal{F}$ is non skew and recursively defined by the equation $H(x, y) = 0$, where $H(x, y)$, as a polynomial with coefficients in $K(y)$, is irreducible in $K(y)[x]$.

(a') $\deg B_j \geq c_1(j) \cdot \deg(b(x_j))^j$, where $b \in K(T)$ is a rational function and $(b(x_j))^j$ denotes either the pole divisor or the zero divisor of $b(x_j)$ in F_j and $F_j = F_{j-1}(x_j)$.

Then the same result holds, i.e., $\gamma(\mathcal{F}) = \infty$. This is so because

$$\begin{aligned} \deg B_i &\geq c_1(i) \cdot \deg(b(x_i))^i \\ &= c_1(i)\,[F_i : K(b(x_i))] \\ &\geq c_1(i)\,[F_i : K(x_i)] \end{aligned}$$

$$= c_1(i)\,[F_i : F_0]\,, \qquad \text{for } i \geq 1\,.$$

Note that

$$[F_i : K(x_i)] = (\deg_y H)^i = (\deg_x H)^i = [F_i : F_0]$$

since the tower $\mathcal{F}$ is non skew.

REMARK 3.4. It is easy to see that Proposition 3.2 and Remark 3.3 also hold if the conditions are expressed in terms of a subtower $\mathcal{F}' = (F_{s_1}, F_{s_2}, \ldots)$ of $\mathcal{F}$.

Recall that a tower $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields over a perfect field K of positive characteristic is called *tame* if the ramification index $e(Q|P)$ of any place Q of F_{i+1} lying above a place P of F_i is relatively prime to the characteristic of K for all $i \geq 0$. We show next that the genus of some particular class of tame towers $\mathcal{F} = (F_0, F_1, \ldots)$ depends only on the number of ramified places from $\bar{F}_i$ to $\bar{F}_{i+1}$, where $\bar{F}_i = F_i \cdot K'$ and K' is an algebraic closure of K.

PROPOSITION 3.5. *Let $\mathcal{F} = (F_0, F_1, \ldots)$ be a tame tower of function fields over a perfect field K of positive characteristic. Suppose that either each extension F_{i+1}/F_i is Galois or that there exists a constant $M \geq 2$ such that $[F_{i+1} : F_i] \leq M$ for any $i \geq 0$. Let*

$$R_i = \{P \in \mathbf{P}(\bar{F}_i) \;:\; P \text{ is ramified in } \bar{F}_{i+1}\}\,,$$

where $\bar{F}_i = F_i \cdot K'$ with K' an algebraic closure of K and let $r_i = \#R_i$. Then

$$g(F_0) - 1 + \frac{1}{2M'}\sum_{i=0}^{\infty}\frac{r_i}{[F_i : F_0]} \leq \gamma(\mathcal{F}) \leq g(F_0) - 1 + \frac{1}{2}\sum_{i=0}^{\infty}\frac{r_i}{[F_i : F_0]}\,, \qquad (3.5)$$

where $M' = 2$ if each extension F_{i+1}/F_i is Galois and $M' = M$ otherwise.

PROOF. In the case of a tame tower the Dedekind's different theorem implies that $1 \leq e(Q|P)/2 \leq d(Q|P) \leq e(Q|P)$ if $e(Q|P) > 1$. We can assume that K is algebraically closed so that $\bar{F}_i = F_i$ for $i \geq 0$ and every place is of degree one and all the inertia (or relative) degrees are one. Then we have

$$\begin{aligned}\deg \operatorname{Diff}(F_{i+1}/F_i) &= \sum_{P\in R_i}\sum_{\substack{Q\in\mathbf{P}(F_{i+1})\\ Q|P}} d(Q|P)\deg Q\\ &\leq \sum_{P\in R_i}\sum_{\substack{Q\in\mathbf{P}(F_{i+1})\\ Q|P}} e(Q|P) = [F_{i+1} : F_i]r_i\,.\end{aligned}$$

On the other hand if the extension F_{i+1}/F_i is Galois we clearly have that

$$\sum_{P\in R_i}\sum_{\substack{Q\in\mathbf{P}(F_{i+1})\\ Q|P}} d(Q|P)\deg Q \geq \frac{1}{2}[F_{i+1} : F_i]r_i\,.$$

If F_{i+1}/F_i is not Galois some of the different exponents $d(Q|P)$ could be zero but not all of them because $P \in R_i$. Then

$$\sum_{P\in R_i}\sum_{\substack{Q\in\mathbf{P}(F_{i+1})\\ Q|P}} d(Q|P)\deg Q \geq r_i \geq \frac{1}{M}[F_{i+1}:F_i]r_i\,,$$

so that we always have

$$\sum_{P\in R_i}\sum_{\substack{Q\in\mathbf{P}(F_{i+1})\\ Q|P}} d(Q|P)\deg Q \geq \frac{1}{M'}[F_{i+1}:F_i]r_i\,, \tag{3.6}$$

and thus (3.5) readily follows. □

Now we prove the following partial converse of the situation described in Proposition 3.2.

THEOREM 3.6. *Let $\mathcal{F}=(F_0,F_1,\ldots)$ be a tame tower of function fields over K of positive characteristic such that $\gamma(\mathcal{F})=\infty$. Suppose that either each extension F_{i+1}/F_i is Galois or that there exists a constant $M\geq 2$ such that $[F_{i+1}:F_i]\leq M$ for any $i\geq 0$. Then for each $i\geq 1$ there exist a divisor $B_i\in\mathcal{D}(F_i)$ and functions $c_1(t)$ and $c_2(t)$ such that Conditions* (a) *and* (b) *of Proposition* 3.2 *hold and the series $\sum_{i=1}^{\infty}c_1(i)c_2(i)$ is divergent.*

PROOF. We may assume that K is algebraically closed. Let R_i be as in Proposition 3.5 and let $r_i=\#R_i$. Since $\gamma(\mathcal{F})=\infty$ then by (3.5) we have that the series

$$\sum_{i=0}^{\infty}\frac{r_i}{[F_i:F_0]}\,,$$

is divergent. Let

$$B_i=\sum_{P\in R_i}P.$$

We have

$$\deg B_i=\sum_{P\in R_i}\deg P=r_i=c_1(i)[F_i:F_0]\,,$$

where $c_1(i)=r_i[F_i:F_0]^{-1}$. By (3.6) it is easy to see that Condition (b) of Proposition 3.2 holds with $c_2(i)=1/M'$ where M' is as in Proposition 3.5. Therefore

$$\sum_{i=1}^{\infty}c_1(i)c_2(i)\geq\frac{1}{M'}\sum_{i=1}^{\infty}\frac{r_i}{[F_i:F_0]}=\infty\,,$$

as desired. □

4. Examples

In some examples of this section we shall use the following convention: a place defined by a monic and irreducible polynomial $f \in K[T]$ in a rational function field $K(x)$ will be denoted by $P_{f(x)}$.

In the case of a wild tower $\mathcal{F} = (F_0, F_1, \ldots)$ over K, the infiniteness of its genus can be proved by showing that for infinitely many indices $i \geq 1$ there is a place Q of F_{i+1} such that the different exponent $d(Q|P)$ satisfies

$$d(Q|P) \geq c_i[F_i : F_0], \tag{4.1}$$

where P is the place of F_i lying under Q and $c_i > 0$ is such that the series

$$\sum_{i=1}^{\infty} \frac{c_i}{[F_{i+1} : F_i]}, \tag{4.2}$$

is divergent. In this case it is easy to check that the divisor

$$B_i = d(Q|P)P \in \mathcal{D}(F_i),$$

satisfies Conditions (a) and (b) of Proposition 3.2 with $c_1(i) = c_i$ and $c_2(i) = [F_{i+1} : F_i]^{-1}$. An example of the situation just described was given in [4] and generalized in the following example.

EXAMPLE 1. Let $\mathcal{F} = (F_0, F_1, \ldots)$ be a non skew recursive tower of function fields over a perfect field K of characteristic p defined by the polynomial

$$H = (T^{p+1} + T)f(S) - S^{p+1} \in K[S, T],$$

where $f \in K[S]$ is a polynomial of degree $p + 1 - r$ with $\gcd(p + 1, r) = 1$ and such that $f(0) \neq 0$. We will show that $\mathcal{F}$ is an asymptotically bad tower over K.

The basic function field is $K(x, y)$ with

$$y^{p+1} + y = \frac{x^{p+1}}{f(x)}.$$

Let Q be a zero of y in $K(x, y)$. Then Q lies above P_y and, since $v_Q(y) > 0$, we have

$$v_Q(y^{p+1} + y) = v_Q(y) + v_Q(y^p + 1) = v_Q(y),$$

and if $P = Q \cap K(x)$ then

$$0 < v_Q(y^{p+1} + y) = e(Q|P)((p + 1)v_P(x) - v_P(f(x))).$$

Therefore $v_P(x) > 0$ (the inequality $\deg f < p+1$ is used to rule out the possibility $v_P(x) \leq 0$) and then $P = P_x$, which implies that Q lies over P_x. Now let Q' be a zero of $y^p + 1$ in $K(x, y)$ and let $R = Q' \cap K(x)$. Then $v_{Q'}(y) = 0$ and

$$0 < v_{Q'}(y^p + 1) = v_{Q'}(y^{p+1} + y) = e(Q'|R)((p + 1)v_R(x) - v_R(f(x))).$$

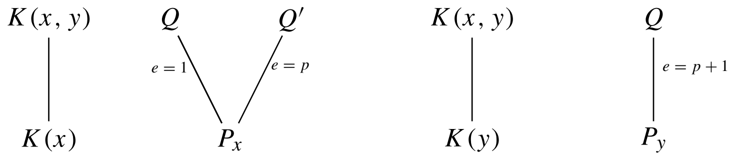

FIGURE 1. Ramification of P_x and P_y in $K(x, y)$

This implies that $v_R(x) > 0$, and then $R = P_x$ so that Q' lies above P_x. Then $v_{P_x}(f(x)) = 0$ and using that we are in characteristic p and that $v_{Q'}(y) = 0$ we have

$$pv_{Q'}(y + 1) = v_{Q'}(y^{p+1} + y) = (p + 1)e(Q'|P_x)v_{P_x}(x) = (p + 1)e(Q'|P_x)\,.$$

Since $[K(x, y) : K(x)] = p + 1$, the above equality forces to have $e(Q'|P_x) = p$ and then $e(Q|P_x) = 1$.

On the other hand, since Q is a place of $K(x, y)$ lying above P_y, we have

$$(p + 1)v_Q(x) - v_Q(f(x)) = e(Q|P_y)(v_{P_y}(y) + v_{P_y}(y^p + 1)) = e(Q|P_y) \geq 1\,,$$

and this implies that $v_Q(x) > 0$. Hence $v_Q(f(x)) = 0$ and then

$$p + 1 \leq (p + 1)v_Q(x) = e(Q|P_y) \leq p + 1\,,$$

so that $e(Q|P_y) = p + 1$. Summarizing the above results, we have that if Q is a zero of y in $K(x, y)$ then Q lies above P_y in $K(y)$ and above P_x in $K(x)$, $e(Q|P_y) = p + 1$ (hence P_y is tamely and totally ramified in $K(x, y)$ and $\gcd(e(Q|P_y), p) = 1$), $e(Q|P_x) = 1$ (hence $\gcd(e(Q|P_y), e(Q|P_x)) = 1$) and if Q' is a zero of $y^p + 1$ in $K(x, y)$ then Q' lies above P_x in $K(x)$ and $e(Q'|P_x) = p$ so that $Q'|P_x$ is wildly ramified and $\gcd(e(Q'|P_x), e(Q|P_y)) = 1$ (see Figure 1).

Let Q' be a zero of $x_{i+1}^p + 1$ in $K(x_i, x_{i+1})$, and let P' be a place of F_{i+1} lying above Q'. It is easy to see that P' lies above P_{x_j} for $j = 0, \ldots, i$. By Abhyankar's Lemma (see [8, Theorem 3.9.1]) we have the ramification as in Figure 2, where P is the place of F_i lying under P'.

The transitivity formula for the different exponent (see [8, Corollary 3.4.12]) implies that

$$\begin{aligned}
d(P'|P) &= d(P'|P_{x_i}) - e(P'|P)d(P|P_{x_i}) \\
&= e(P'|Q')d(Q'|P_{x_i}) + d(P'|Q') - e(P'|P)d(P|P_{x_i}) \\
&= (p + 1)^i d(Q'|P_{x_i}) + (p + 1)^i - 1 - p((p + 1)^i - 1) \\
&\geq (p + 1)^i p + ((p + 1)^i - 1)(1 - p) \\
&\geq (p + 1)^i p + (p + 1)^i (1 - p)
\end{aligned}$$

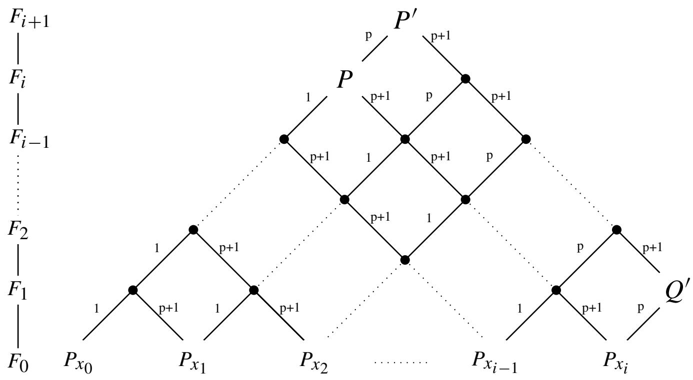

FIGURE 2. Ramification of P'

$$= (p+1)^i = [F_i : F_0],$$

where in the last equality we have used that the tower $\mathcal{F}$ is non skew. We have that (4.1) holds with $c_i = 1$ and clearly the series (4.2) is divergent because $[F_{i+1} : F_i] = p+1$ for all $i \geq 0$. Therefore $\gamma(\mathcal{F}) = \infty$.

We have thus that $\mathcal{F}$ is, in fact, an asymptotically bad tower over K.

As we mentioned above, a particular case of this example was presented in [4] with $p = 2$ and $f(S) = S + 1$.

EXAMPLE 2. In Theorem 2.1 of [1] the following result is proved. Let $\mathcal{F} = (F_0, F_1, \ldots)$ be an Artin-Schreier tower of function fields over a perfect field K of characteristic $p > 0$ defined by the polynomial

$$H = (T^p - T)b_2(S) - b_1(S) \in K[S, T],$$

where $b_1, b_2 \in K[S]$ are coprime polynomials of degree $\deg b_1 = p$ and $\deg b_2 = r < p$ with $\gcd(r, p) = 1$. Let $b(S) = b_1(S)/b_2(S)$ and let $\{x_i\}_{i=0}^{\infty}$ be a sequence of transcendental elements over K such that

$$F_0 = K(x_0) \quad \text{and} \quad F_{i+1} = F_i(x_{i+1}),$$

with $x_{i+1}^p - x_{i+1} = b(x_i)$ for $i \geq 0$. Suppose that there is a constant $C > 0$ such that for infinitely many indices $0 \leq r_1 < r_2 < \ldots$ we have that

$$\deg B_{r_j} \geq C \cdot \deg(b(x_{r_j}))_{\infty}^{r_j},$$

where

$$B_{r_j} := \sum_{P \in L_{r_j}} -v_P(b(x_{r_j}))P \quad \in \mathcal{D}(F_{r_j}),$$

and

$$L_j = \{P \in \mathbf{P}(F_j) : v_P(b(x_j)) < 0 \quad \text{and} \quad \gcd(v_P(b(x_j)), p) = 1\},$$

for all $j \geq 0$. Then the tower $\mathcal{F}$ is asymptotically bad.

The last assumption of this result is just Condition (a') of Remark 3.3 with $c_1(j) = C$. Following the proof given by the authors in [1] we see that the divisor B_{r_j} satisfies

$$\sum_{P \in \mathrm{supp}(B_{r_j})} \sum_{P'|P} d(P'|P) \deg P' \geq \frac{1}{2}[F_{r_{j+1}} : F_{r_j}] \sum_{P \in \mathrm{supp}(B_{r_j})} -v_P(b(x_{r_j})) \deg P$$
$$= \frac{1}{2}[F_{r_{j+1}} : F_{r_j}] \deg B_{r_j} ,$$

which is Condition (b) of Proposition 3.2 with $c_2(j) = 1/2$. Thus in this example the existence and use of the divisor of Proposition 3.2 is obvious.

EXAMPLE 3. Let $\mathcal{F} = (F_0, F_1, \ldots)$ be a non skew recursive tower of function fields over K defined by a polynomial $H(S, T) \in K[S, T]$. Following [2], we define its dual tower $\mathcal{G} = (G_0, G_1, \ldots)$ as the recursive tower defined by $H(T, S)$. We identify the rational function field $F_0 = K(x_0)$ with $G_0 = K(y_0)$ by setting $x_0 = y_0$. Then we have that $F_0 = G_0$ and

$$F_n = K(x_0, \ldots, x_n) \quad \text{with } H(x_i, x_{i+1}) = 0, \text{ and}$$
$$G_n = K(y_0, \ldots, y_n) \quad \text{with } H(y_{i+1}, y_i) = 0$$

for all $n \geq 1$ and $0 \leq i \leq n-1$. Note that the function fields F_i and G_i are K-isomorphic for all $i \geq 0$.

For $P \in \mathbf{P}(F_0) = \mathbf{P}(G_0)$ we define the set

$$\varepsilon(P, \mathcal{F}) := \sup_{n \geq 1}\{e(Q_n|P) : Q_n \in \mathbf{P}(F_n) \text{ and } Q_n|P\}.$$

We will show that if

$$\varepsilon(P, \mathcal{F}) \neq \varepsilon(P, \mathcal{G}) ,$$

then $\gamma(\mathcal{F}) = \infty$ so that $\mathcal{F}$ is actually a tower which is asymptotically bad. In order to prove this, we follow the proof given in [2] by considering $\mathcal{F}$ as a tower over an algebraic closure K' of K (recall that the genus of a tower and the ramification indices do not change in constant field extensions). Hence any place of each F_i is of degree one. We also have that

$$[F_{n+1} : F_n] = [G_{n+1} : G_n] = m ,$$

for all $n \geq 1$, where $1 < m = \deg_S H = \deg_T H$.

Without loss of generality we can assume that $\varepsilon(P, \mathcal{F}) > \varepsilon(P, \mathcal{G})$. Then $e_1 = \varepsilon(P, \mathcal{G})$ is a positive integer. By definition of $\varepsilon(P, \mathcal{G})$ there exist a positive integer n and a place $Q_1 \in \mathbf{P}(G_n)$ such that

(i) $e(Q_1|P) = e_1$,

(ii) Q_1 splits completely in G_l/G_n for all $l \geq n$.

Since $\varepsilon(P, \mathcal{F}) > \varepsilon(P, \mathcal{G})$, there exists a positive integer k such that there is a place $Q_2 \in \mathbf{P}(F_k)$ lying above P with

$$e_2 = e(Q_2|P) > e_1 .$$

Proceeding like in [2, Theorem 3.3] it can be proven that there are at least m^{l-n} places of F_l that ramify in F_{l+k}, for $l \geq n$.

Now let us consider the sequence $\{r_j\}_{j\geq 0}$ such that $r_j = n + jk$. We know that in each extension F_{r_j} there are at least

$$m^{r_j-n} = m^{n+jk-n} = m^{jk}$$

places, $P_1, P_2, \ldots, P_{m^{jk}}$ which ramify in the extension $F_{r_{j+1}}$. Let

$$B_{r_j} = \sum_{i=1}^{m^{jk}} P_i .$$

Then

$$\deg B_{r_j} = \sum_{i=1}^{m^{jk}} \deg P_i = m^{jk} = \frac{1}{m^n}[F_{r_j} : F_0] ,$$

which is Condition (a) of Proposition 3.2 with $c_1(j) = \frac{1}{m^n}$. Also

$$\begin{aligned}\sum_{P\in\operatorname{supp} B_{r_j}} \sum_{P'|P} d(P'|P) \deg P' &\geq \sum_{i=1}^{m^{jk}} (e(P'|P) - 1)\\ &\geq m^{jk}\\ &= \frac{1}{m^k}[F_{r_{j+1}} : F_{r_j}] \deg B_{r_j} ,\end{aligned}$$

which is Condition (b) of Proposition 3.2 with $c_2(j) = m^{-k}$. Then $\mathcal{F}$, as a tower over K', has infinite genus by Proposition 3.2. Therefore $\mathcal{F}$ has infinite genus as a tower over K and then $\mathcal{F}$ is, in fact, an asymptotically bad tower over K.

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