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On Vassiliev Invariants of Degrees 2 and 3 for Torus Knots

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Abstract. We consider the **R**-valued Vassiliev invariants of degrees 2 and 3 normalized by the conditions that they take values 0 on the unknot and 1 on the trefoil. We give certain answers for a problem due to N. Okuda about these two invariants. Moreover, we prove a conjecture due to Simon Willerton concerning the degree-3 Vassiliev invariant in the case of torus knots.

1. Introduction

The present work is motivated by [Oh, §2.4 Vassiliev invariants and crossing numbers]. As to the Vassiliev invariants of degrees 2 and 3, N. Okuda [Ok] posed the following problem:

PROBLEM 1.1 (cf. [Oh, Problem 2.10]). *Let K be a knot and n the crossing number of a diagram D of K, and $v_2(K)$ and $v_3(K)$ the primitive Vassiliev invariants of degrees* 2 *and* 3 *of K, respectively. Then, describe the following set*

$$\mathcal{S}(K, D) := \left\{ \left(\frac{v_2(K)}{n^2}, \frac{v_3(K)}{n^3} \right) \in \mathbf{R} \times \mathbf{R} \right\} .$$

The most ideal solution for this problem would be to give a precise function $f(x, y)$ such that

$$f\left(\frac{v_2(K)}{n^2}, \frac{v_3(K)}{n^3} \right) = 0 .$$

However so far a reasonable solution would be to give a domain as sharp as possible containing the set $\mathcal{S}(K, D)$. As to each of $v_2(K)$ and $v_3(K)$, N. Okuda [Ok] showed the following inequalities:

$$-\frac{n^2}{16} \leq v_2(K) \leq \frac{n^2}{8} ,$$

$$|v_3(K)| \leq \frac{n(n-1)(n-2)}{15} .$$

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Here the right-hand-side inequality of the first one is due to Polyak–Viro [PO]. It follows from these two inequalities that the set $\mathcal{S}(K, D)$ is contained in the rectangle

$$\left[-\frac{1}{16}, \frac{1}{8}\right] \times \left[-\frac{1}{15}, \frac{1}{15}\right].$$

Then, as pointed out in the second Remark right after Problem 2.10 in [Oh, pp. 404–405], it is a problem to describe the smallest domain containing the set $\mathcal{S}(K, D)$. In this paper we give a non-trivial domain (i.e., non-rectangle domain) containing the set $\mathcal{S}(K, D)$ in the case of torus knots.

THEOREM 1.2. *Let K be a torus knot and let n be the crossing number of a diagram D of K. Then we have*

$$\mathcal{S}(K, D) \subset \left\{(x, y) \in \mathbf{R}^2 \,\middle|\, \frac{8}{3}x^2 < |y| \leq \frac{1}{3}x\right\} \bigcup \{(0, 0) \in \mathbf{R}^2\}. \tag{1}$$

As to $v_3(L)$, S. Willerton [W2] made the following conjecture:

CONJECTURE 1.3 ([W2] and cf. [Oh, Conjecture 2.11]). *Let v_3 be as above. If a knot K has a diagram with n crossings, then*

$$|v_3(K)| \leq \left[\frac{n(n^2-1)}{24}\right],$$

where $[x]$ denotes the Gauss symbol of x.

In this paper we show that the above conjecture is correct in the case of torus knots.

2. Primitive Vassiliev invariants and torus knots

In [V] V. A. Vassiliev introduced what is now called the *Vassiliev invariant of a knot*, using the cohomology of the complement of the knot. In [G] M.N. Goussarov redefined or independently defined the Vassiliev invariant more axiomatically.

A Vassiliev invariant v is called *primitive* if it is additive under the connected sum of knots K_1, K_2, that is, $v(K_1 + K_2) = v(K_1) + v(K_2)$. Let v_2 and v_3 be the $\mathbf{R}$-valued Vassiliev invariants of degrees 2 and 3 of K, respectively, normalized by the conditions that $v_2(K) = v_2(\overline{K})$ and $v_3(K) = -v_3(\overline{K})$ for any K and its mirror image $\overline{K}$ and that they take 0 on the unknot and 1 on the trefoil.

PROPOSITION 2.1 ([W2]). *Let $J_K(t)$ be the Jones polynomial of K and let $J_K^{(m)}(t)$ denote its m-th derivative with respect to t. Then $v_2(K)$ and $v_3(K)$ are described using the derivatives of the Jones polynomial as follows:*

$$v_2(K) = -\frac{1}{6}J_K^{(2)}(1),$$

$$v_3(K) = -\frac{1}{36}\big(J_K^{(3)}(1) + 3J_K^{(2)}(1)\big).$$

Let K be a (p,q)-torus knot and let n be the crossing number of a diagram of K. Then it is known that $n \geq min\{|p(q-1)|,\ |q(p-1)|\}$ (see [M]). A (p,q)-torus knot is trivial if and only if either p or q is equal to 1 or -1. The Vassiliev invariant of a trivial knot is 0. Therefore, since we deal with non-trivial knots, from now on we assume that $|p| \geq 2$ and $|q| \geq 2$. Moreover, we know that the Jones polynomial $J_K(t)$ of the (p,q)-torus knot K is expressed by:

$$J_K(t) = \frac{t^{\frac{(p-1)(q-1)}{2}}(1 - t^{p+1} - t^{q+1} + t^{p+q})}{1-t^2}.$$

REMARK 2.2 (cf. [W2, §4. Torus Knots, p. 292]). M. Alvarez and J. M. F. Labastida [AL] obtained the above two formula for $v_2(K)$ and $v_3(K)$ in a different way.

Hence the Vassiliev invariants of degrees 2 and 3 for a $(p,\ q)$-torus knot K are respectively given by:

$$v_2(K) = -\frac{1}{6}J_K^{(2)}(1) = \frac{(p^2-1)(q^2-1)}{24},$$

$$v_3(K) = -\frac{1}{36}\big(J_K^{(3)}(1) + 3J_K^{(2)}(1)\big) = \frac{pq(p^2-1)(q^2-1)}{144}.$$

3. Results

THEOREM 3.1. *Let K be a non-trivial torus knot and let n be the crossing number of a diagram D of K. Then we have*

$$\frac{8}{3}\left(\frac{v_2(K)}{n^2}\right)^2 < \left|\frac{v_3(K)}{n^3}\right| \leq \frac{1}{3}\left(\frac{v_2(K)}{n^2}\right). \tag{2}$$

PROOF. We prove the above inequalities in the case of $q \leq p$, which implies that $n \geq |q(p-1)|$.

$$\begin{aligned}\left|\frac{v_3(K)}{n^3}\right| &= \left|\frac{pq(p^2-1)(q^2-1)}{144n^3}\right| \\ &= \frac{1}{6}\frac{(p^2-1)(q^2-1)}{24n^2}\frac{|pq|}{n^2}n \\ &\geq \frac{1}{6}\frac{v_2(K)}{n^2}\frac{(p^2-1)(q^2-1)}{24n^2}\frac{24|pq|}{(p^2-1)(q^2-1)}|q(p-1)| \\ &\geq 4\left(\frac{v_2(K)}{n^2}\right)^2\frac{|p(p-1)|}{p^2-1}\frac{q^2}{q^2-1}.\end{aligned}$$

Since $|p| \geq 2$, we have that the minimum of $\frac{|p(p-1)|}{(p^2-1)}$ is $\frac{2}{3}$ when $p = 2$. Moreover note that $|q| \geq 2$, therefore $\frac{q^2}{q^2-1} > 1$.

$$\begin{aligned}\left|\frac{v_3(K)}{n^3}\right| &= \left|\frac{pq(p^2-1)(q^2-1)}{144n^3}\right| \\ &= \frac{1}{6}\frac{(p^2-1)(q^2-1)}{24n^2}\left|\frac{pq}{n}\right| \\ &\leq \frac{1}{6}\frac{v_2(K)}{n^2}\left|\frac{pq}{|q(p-1)|}\right| \\ &\leq \frac{1}{6}\frac{v_2(K)}{n^2}\left|\frac{p}{p-1}\right|.\end{aligned}$$

Note that $|p| \geq 2$, therefore the maximum of $|\frac{p}{p-1}|$ is 2 when $p = 2$. Hence we obtain the following relation:

$$\frac{8}{3}\left(\frac{v_2(K)}{n^2}\right)^2 < \left|\frac{v_3(K)}{n^3}\right| \leq \frac{1}{3}\left(\frac{v_2(K)}{n^2}\right).$$

In the case of $q \leq p$, we exchange p and q in the above proof and we get the same result. $\square$

Let K be a torus knot. The above inequalities (2) imply that the set $\mathcal{S}(K, D)$ is contained in the domain

$$\left\{(x, y) \in \mathbf{R}^2 \;\middle|\; \frac{8}{3}x^2 < |y| \leq \frac{1}{3}x\right\} \bigcup \{(0,0) \in \mathbf{R}^2\}.$$

In particular, we get the following.

COROLLARY 3.2. *Let the situation be as above.*

$$0 \leq \frac{v_2(K)}{n^2} \leq \frac{1}{8}, \quad -\frac{1}{24} \leq \frac{v_3(K)}{n^3} \leq \frac{1}{24}.$$

REMARK 3.3. In [W2, Proposition 4.1, p. 292] Willerton showed the following inequalities for a torus knot K (he uses T instead of K):

$$\frac{2}{3}v_2(K)^3 + \frac{1}{3}v_2(K)^2 \leq v_3(K)^2 \leq \frac{8}{9}v_2(T)^3 + \frac{1}{9}v_2(K)^2.$$

From which one can get the following inequalities, by dividing them throughout by n^6:

$$\frac{2}{3}\left(\frac{v_2(K)}{n^2}\right)^3 + \frac{1}{3n^2}\left(\frac{v_2(K)}{n^2}\right)^2 \leq \left(\frac{v_3(K)}{n^3}\right)^2 \leq \frac{8}{9}\left(\frac{v_2(T)}{n^2}\right)^3 + \frac{1}{9n^2}\left(\frac{v_2(K)}{n^2}\right)^2.$$

Hence, we get the following inequalities:

$$\frac{2}{3}\left(\frac{v_2(K)}{n^2}\right)^3 < \left(\frac{v_3(K)}{n^3}\right)^2 \le \frac{8}{9}\left(\frac{v_2(T)}{n^2}\right)^3 + \frac{1}{9}\left(\frac{v_2(K)}{n^2}\right)^2 .$$

Therefore we can see that the set $\mathcal{S}(K, D)$ is contained in the following domain:

$$\left\{(x, y) \in \mathbf{R}^2 \,\middle|\, \frac{2}{3}x^3 < y^2 < \frac{8}{9}x^3 + \frac{1}{9}x^2\right\}.$$

It follows from the above inequalities $\frac{8}{3}x^2 < |y| < \frac{1}{3}x$ (just before Corollary 3.2) and $\frac{2}{3}x^3 < y^2 < \frac{8}{9}x^3 + \frac{1}{9}x^2$ (at the end of Remark 3.3) that we can get the following corollary:

COROLLARY 3.4. *Let the situation be as above.*

$$\mathcal{S}(K, D) \subset \left\{(x, y) \in \mathbf{R}^2 \,\middle|\, \frac{2}{3}x^3 < y^2 \le \frac{1}{9}x^2, 0 < x \le \frac{3}{32}\right\}$$
$$\bigcup \left\{(x, y) \in \mathbf{R}^2 \,\middle|\, \frac{8}{3}x^2 < |y| \le \frac{1}{3}x, \frac{3}{32} \le x \le \frac{1}{8}\right\} \bigcup \{(0, 0) \in \mathbf{R}^2\} .$$

As to the Vassiliev invariant v_3, we get the following inequality, i.e. the aforementioned Willerton's conjecture [W2] (c.f. [Oh, Conjecture 2.11]).

THEOREM 3.5. *Let K be a torus knot and let n be the crossing number of a diagram of K. Then we have*

$$\left|v_3(K)\right| \le \left[\frac{n(n^2 - 1)}{24}\right].$$

PROOF. We prove the above inequality in the case of $q \le p$. First we show the following inequality:

$$\left|\frac{p(p + 1)(q^2 - 1)}{6}\right| \le \left|q(p - 1)\right|^2 - 1. \tag{3}$$

To show this we observe the following:

$$\begin{aligned} 6\big(\left|q(p - 1)\right|^2 - 1\big) &- p(p + 1)(q^2 - 1) \\ &= 5p^2q^2 - 13pq^2 + 6q^2 + p^2 + p - 6 \\ &= q^2(p - 2)(5p - 2) + (p - 2)(p + 3) \\ &= (p - 2)\big((q^2(5p - 2) + p + 3\big) . \end{aligned}$$

Note that $|p| \ge 2$, therefore

$$\frac{p(p + 1)(q^2 - 1)}{6} \le \left|q(p - 1)\right|^2 - 1 .$$

Moreover note that $|q| \geq 2$, therefore $p(p+1)(q^2-1) > 0$. Thus we obtain the above inequality (3).

Next, we observe the following:

$$\left|v_3(K)\right| = \left|\frac{pq(p^2-1)(q^2-1)}{144}\right| \tag{4}$$

$$\leq \frac{|q(p-1)|}{24}\left|\frac{p(p+1)(q^2-1)}{6}\right|. \tag{5}$$

By the inequality (3), we obtain

$$\left|v_3(K)\right| \leq \frac{|q(p-1)|}{24}\left(\left|q(p-1)\right|^2 - 1\right).$$

Hence

$$\left|v_3(K)\right| \leq \frac{n\left(n^2-1\right)}{24}.$$

In the case of $q \leq p$, we exchange p and q in the above proof and we get the same result. □

REMARK 3.6. As remarked in [W2, §2], the degree-3 Vassiliev invariant v_3 satisfies that for any knot K with the crossing number n

$$\left|v_3(K)\right| \leq \frac{n(n-1)(n-2)}{4},$$

which was obtained in [W1] using Domergue and Donato's integration [DD]. For any n we do have that $\frac{n(n-1)(n-2)}{15} < \frac{n(n-1)(n-2)}{4}$. Thus, Okuda's inequality is sharper than Willerton's inequality. However, if $n > 8$, we have that

$$\frac{n(n^2-1)}{24} < \frac{n(n-1)(n-2)}{15}.$$

Here, we note that the equality holds for $n = 7$. Thus our inequality, i.e. the inequality conjectured by Willerton, is sharper for $n \geq 7$, although the knots considered in the present paper are torus knots.

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