

Nonlinear Schrödinger Equations with Steep Magnetic Well

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Abstract. We study the nonlinear magnetic Schrödinger equation, $-(\nabla - i\lambda A)^2 u = f(x, |u|^2)u$ on \mathbf{R}^N , where $N \geq 2$ and the nonlinearity is super-linear and subcritical. The vector potential A and the associated magnetic field are assumed to vanish on a common bounded open set Ω . It is shown that the equation above has more and more solutions which are localized near Ω as $\lambda \rightarrow \infty$.

1. Introduction and results

In this paper, we study the nonlinear Schrödinger equation with magnetic field,

$$-(\nabla - i\lambda A(x))^2 u(x) = f(x, |u(x)|^2)u(x) \quad (1.1)$$

on \mathbf{R}^N , where $\lambda > 0$ is a large parameter and $N \geq 2$. We are concerned with the existence of multiple solutions which are localized near the interior of the zero set of the magnetic potential as $\lambda \rightarrow \infty$.

This kind of the “steep-potential-well” problem for equations of the form $-\Delta u + (\lambda V(x) + W(x))u = f(x, u)$ as $\lambda \rightarrow \infty$, where the interior of the zero set $V^{-1}(0)$ is assumed to be nonempty, has been studied by many authors under various conditions on the potentials and the nonlinearity (See, e.g., [Ba-Wa], [Ba-Wa2], [Ba-Pa-Wa], [Si], [De-Di], [Cl-Di], [Di-Ta], [Al], [Al-So], [Al-Mo-So], and the references therein). In [Ta1], [Ta2], and [Ta3], Tang considers the corresponding problem for the equation with magnetic field, $-(\nabla - iA(x))^2 u + (\lambda V(x) + W(x))u = f(u)$.

Our result gives a magnetic analog of these works, replaced the steep scalar potential well by the steep magnetic well. The proof is based on the argument used in [Ba-Pa-Wa] and [Ba-Wa2].

We fix some notation. By “ $A + B =: C + D$ ”, we mean that C and D are defined by A and B , respectively. We denote by $|\cdot|$ the standard Euclidean norm. For any metric space

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M , we set $B_r(x) = \{y \in M | \text{dist}(x, y) \leq r\}$. For any subset S of M , we set $\text{dist}(x, S) = \inf_{y \in S} \text{dist}(x, y)$ and set $S_\delta^+ = \{x \in M | \text{dist}(x, S) \leq \delta\}$ for $\delta > 0$.

For any subset Ω of a topological space X , we denote the closure, the complement, the interior, and the boundary of Ω by $\overline{\Omega}$, Ω^c (or $X \setminus \Omega$), $\text{int}(\Omega)$, and $\partial\Omega$, respectively.

We denote by $F(M, M')$ the space of all M' -valued F -functions on M , and we denote $F(\mathbf{R}^N, \mathbf{C})$ by $F(\mathbf{R}^N)$. We denote by $C_0^\infty(\mathbf{R}^N)$ the space of all (complex-valued) smooth functions on \mathbf{R}^N with compact support. We say that $u_n \rightarrow u$ in $L_{\text{loc}}^q(\mathbf{R}^N)$ if $u_n \rightarrow u$ in $L^q(K)$ for each compact subset K of \mathbf{R}^N .

The symbol 2^* stands for $2N/(N-2)$ if $N \geq 3$ and for ∞ if $N = 2$. For any sequence $\{u_n\}_n$ and an element u in a Banach space X , we write " $u_n \rightharpoonup u$ " if the sequence $\{u_n\}_n$ converges weakly to u in X as $n \rightarrow \infty$. The notation $X' \langle \cdot, \cdot \rangle_X$ stands for the coupling between the dual X' and X .

By abuse of notation, we write, e.g., $f(x, |u|^2)$ to indicate either the function $f(\cdot, |u|^2(\cdot))$ or the value $f(x, |u(x)|^2)$ if there is no fear of confusion.

For a magnetic vector potential $A = (A_1, \dots, A_N)$, we define the magnetic field $B = dA = (B_{jk})_{j,k=1}^N$, where $B_{jk} = \partial_j A_k - \partial_k A_j$. Define

$$|B(x)| = \left(\sum_{j < k} |B_{jk}(x)|^2 \right)^{1/2} \quad (1.2)$$

and

$$\gamma_B(x) = \left(\sum_{j < k} \left| \nabla \left(\frac{B_{jk}(x)}{|B(x)|} \right) \right|^2 \right)^{1/2}. \quad (1.3)$$

To formulate the main results, we make the following assumptions.

- (A.1) The magnetic vector potential A belongs to $C^2(\mathbf{R}^N, \mathbf{R}^N)$.
- (A.2) $\liminf_{|x| \rightarrow \infty} |B(x)| > 0$. In particular, $B^{-1}(0)$ is bounded.
- (A.3) $\limsup_{|x| \rightarrow \infty} \gamma_B(x) < \infty$.
- (A.4) The set $\Omega = \text{int}(B^{-1}(0))$ is non-empty and has smooth boundary.
- (A.5) $\Omega \subset A^{-1}(0)$ and $\text{Meas}(B^{-1}(0) \setminus \Omega) = 0$. Here, $\text{Meas}(\cdot)$ stands for the Lebesgue measure.

For the nonlinearity, we make the following assumptions.

- (f.1) The function f belongs to $C(\mathbf{R}^N \times [0, \infty), \mathbf{R})$.
- (f.2) There exists $p \in (2, 2^*)$ and $C > 0$ such that $\sup_{x \in \mathbf{R}^N} |f(x, t^2)| \leq C(1 + t^{p-2})$ for any $t > 0$. Moreover, $\lim_{t \rightarrow +0} \sup_{x \in \mathbf{R}^N} |f(x, t)| = 0$.
- (f.3) There exists $\theta > 2$ such that $0 < \theta F(x, t) \leq f(x, t)t$ for all $x \in \mathbf{R}^N$ and $t > 0$, where $F(x, t) = \frac{1}{2} \int_0^t f(x, \tau) d\tau$.
- (f.4) There exists $t_0 > 0$ such that $\inf_{x \in \mathbf{R}^N} F(x, t_0^2) > 0$.

(f.5) The function $f(x, \cdot)$ is strictly increasing for a.e. $x \in \mathbf{R}^N$.

(f.6) There exists $C > 0$ such that $\sup_{x \in \mathbf{R}^N} |f(x, |u+v|^2)(u+v) - f(x, |u|^2)u| \leq C(1 + |u|^{p-2} + |v|^{p-2})|v|$ holds for all $u, v \in \mathbf{C}$. Here, the constant p is as in (f.2).

Throughout this paper, we denote by $\nabla_{\lambda A}$ the distributional covariant derivative $\nabla - i\lambda A$. For any $\lambda > 0$, we introduce the Banach space

$$E_\lambda = \{u \in L^2(\mathbf{R}^N) \mid \nabla_{\lambda A} u \in L^2(\mathbf{R}^N, \mathbf{C}^N)\} \quad (1.4)$$

with norm $\|u\|_\lambda = \|\nabla_{\lambda A} u\|_{L^2(\mathbf{R}^N, \mathbf{C}^N)} = (\sum_{j=1}^N \|(\partial_j - i\lambda A_j)u\|_{L^2(\mathbf{R}^N)}^2)^{1/2}$. We define the functional

$$J_\lambda(u) = \frac{1}{2} \|\nabla - i\lambda A\|_{L^2(\mathbf{R}^N, \mathbf{C}^N)}^2 u - \int_{\mathbf{R}^N} F(x, |u|^2) dx \quad (1.5)$$

on E_λ . It is standard that any critical point u of J_λ is a weak solution to the equation (1.1), i.e., u satisfies $(\nabla_{\lambda A} u, \nabla_{\lambda A} v)_{L^2(\mathbf{R}^N, \mathbf{C}^N)} = \int_{\mathbf{R}^N} f(x, |u|^2) u \bar{v} dx$ for any $v \in E_\lambda$.

The main results of this paper are the following three theorems.

THEOREM 1.1. *Assume (A.1)–(A.5) and (f.1)–(f.4). For any integer $m \geq 1$, there exists $\Lambda_m > 0$ such that the equation (1.1) has at least m weak solutions $\{v_j^\lambda\}_{j=1}^m$ in E_λ for any $\lambda \geq \Lambda_m$.*

Moreover, for any $j = 1, 2, \dots, m$, there exist λ -independent constants a_j, b_j , and c_j such that $0 < a_j \leq \|v_j^\lambda\|_{L^p(\mathbf{R}^N)} \leq b_j < a_{j+1}$ and $\|v_j^\lambda\|_\lambda \leq c_j$ for all $\lambda \geq \Lambda_m$. Here, the constant p is as in (f.2).

Define $I_\Omega(u) = \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 - \int_\Omega F(x, |u|^2) dx$ for $u \in H_0^1(\Omega)$. Throughout this paper, we regard the space $H_0^1(\Omega)$ as a closed subspace of $H^1(\mathbf{R}^N)$ by zero extension.

THEOREM 1.2. *Assume (A.1)–(A.5) and (f.1)–(f.4). Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence such that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Assume that, for every n , the equation (1.1) with $\lambda = \lambda_n$ has a weak solution u_n in E_{λ_n} satisfying: $\sup_n \|u_n\|_{\lambda_n} < \infty$ and $\inf_n \|u_n\|_{L^p(\mathbf{R}^N)} > 0$. Then there exist a subsequence $\{u_{n(k)}\}_k$ of $\{u_n\}_n$ and $v \in H_0^1(\Omega)$ such that $\|u_{n(k)} - v\|_{\lambda_{n(k)}} \rightarrow 0$; $u_{n(k)} \rightarrow v$ in $L^q(\mathbf{R}^N)$ for every $q \in [2, 2^*)$; $J_{\lambda_{n(k)}}(u_{n(k)}) \rightarrow I_\Omega(v)$; and v is a nontrivial weak solution to*

$$-\Delta v = f(x, |v|^2)v. \quad (1.6)$$

Define $\mathcal{N}_\lambda = \{u \in E_\lambda \setminus \{0\} \mid E'_\lambda \langle J'_\lambda(u), u \rangle_{E_\lambda} = 0\}$ and $c_\lambda = \inf_{\mathcal{N}_\lambda} J_\lambda$. Define $\mathcal{N}_\Omega = \{u \in H_0^1(\Omega) \setminus \{0\} \mid H_0^1(\Omega)' \langle I'_\Omega(u), u \rangle_{H_0^1(\Omega)} = 0\}$ and $c(\Omega) = \inf_{\mathcal{N}_\Omega} I_\Omega$.

It is well-known that $c(\Omega)$ is positive and attained by some critical point of I_Ω .

THEOREM 1.3. *Assume (A.1)–(A.5) and (f.1)–(f.6). Then there exists $\Lambda > 0$ such that for any $\lambda \geq \Lambda$, the value c_λ is attained by some critical point $u_\lambda \in E_\lambda$ of J_λ . Moreover, $\lim_{\lambda \rightarrow \infty} c_\lambda = c(\Omega)$.*

REMARK 1.4. (i) We give examples of A which satisfy all the conditions (A.1)–(A.5) with $A^{-1}(0) = B^{-1}(0) = B_1(0)$. Fix $\chi \in C^\infty([0, \infty))$ satisfying: $0 \leq \chi \leq 1$ and $0 \leq \chi' \leq 1$ on $[0, \infty)$; $\chi(r) = 0$ if and only if $0 \leq r \leq 1$; $\chi(r) = 1$ if $r \geq 4$; χ is increasing. Then the examples are given by $A(x) = \chi(|x|^2)(-x_2, x_1)/2$, where $x = (x_1, x_2) \in \mathbf{R}^2$ for $N = 2$, and $A(x) = \chi(|x|^2)(-x_2, x_1, 0)/2 + (0, 0, \chi(x_3))$ for $N = 3$. Similar examples can be constructed in higher dimensions.

(ii) The power-type nonlinearity $f(t) = t^{(p-2)/2}$, where $2 < p < 2^*$, satisfies all the conditions (f.1)–(f.6).

(iii) Let $\{v_j^\lambda\}$ be the solutions obtained in Theorem 1.1. Let $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Then for each j , the sequence $\{u_n\}_n = \{v_j^{\lambda_n}\}_n$ satisfies the assumption as in Theorem 1.2.

(iv) All of the statements of the theorems above remain valid for the equation $-(\nabla - i\lambda A)^2 u + \mu u = f(x, |u|^2)u$ for any $\mu < \mu_1$, where μ_1 is the first eigenvalue of the Dirichlét Laplacian on Ω . Proofs are similar.

The organization of this paper is as follows. In Section 2, we formulate the Avron-Herbst-Simon type formulae for the quadratic form $\|u\|_\lambda^2$. In Section 3, we give preliminary results; some basic properties of the functional J_λ , a spectral concentration of $-(\nabla - i\lambda A)^2$ as $\lambda \rightarrow \infty$, and a variation of fountain theorem. In Sections 4, 5, and 6, we give proofs of Theorems 1.1, 1.2, and 1.3, respectively.

2. Lower estimates of quadratic forms

We assume (A.1)–(A.5) in the following. Let E_λ be as in (1.4). Let $H(\lambda)$ be the unique self-adjoint operator associated with the form $\|\nabla_{\lambda A} u\|_{L^2(\mathbf{R}^N, \mathbf{C}^N)}^2$ on $C_0^\infty(\mathbf{R}^N)$. Then $H(\lambda)$ is essentially self-adjoint on $C_0^\infty(\Omega)$, and E_λ is the form domain of $H(\lambda)$ (e.g., [Le-Si]).

The proof of the next lemma is elementary.

LEMMA 2.1. *Let K be an open subset of \mathbf{R}^N and let $\zeta \in C_0^1(K)$. Then we have the following assertions for any $u \in E_\lambda$:*

(i) $\nabla_{\lambda A}(u|_K) = (\nabla_{\lambda A} u)|_K$ in the sense of distribution.

(ii) *If K is bounded, then ζu belongs to $H_0^1(K)$ and satisfies $\|\zeta u\|_{H^1(K)} \leq C_\lambda(\|u\|_\lambda + \|u\|_{L^2(\mathbf{R}^N)})$. Here, C_λ is independent of u .*

Define $\beta_{ij}(x) = B_{ij}(x)/|B(x)|$ for $x \in \mathbf{R}^N \setminus B^{-1}(0)$ and $1 \leq i, j \leq N$. We note that $\gamma_B(x) = \left(\sum_{i < j} |\nabla \beta_{ij}(x)|^2\right)^{1/2}$.

LEMMA 2.2 ([He-Na], Lemma B.3). *Let $\xi = (\xi_{ij})_{i,j}$ be a (constant) real anti-symmetric matrix with $\sum_{i < j} |\xi_{ij}|^2 = 1$. We have the inequality*

$$H(\lambda) \geq \sum_{l < m} \lambda \xi_{lm} B_{lm}$$

in the sense of the sesqui-linear form.

PROPOSITION 2.3. Assume that F is a closed subset of \mathbf{R}^N and $B^{-1}(0) \subset F$. Assume that $0 < s < \delta$ and $0 < s < 1/\sup_{(F_{\delta-s}^+)^c} \gamma_B$. Then

$$H(\lambda) \geq \lambda |B| \left(1 - s \left[\sup_{(F_{\delta-s}^+)^c} \gamma_B \right] \right) 1_{(F_{\delta+s}^+)^c} - c_0/s^2,$$

where

$$c_0 = \inf\{\|\nabla \mu\|_{L^2(\mathbf{R}^N, \mathbf{R}^N)}^2 / \|\mu\|_{L^2(\mathbf{R}^N)}^2 \mid \mu \in C_0^\infty(\text{int } B_1(0), \mathbf{R}) \setminus \{0\}\},$$

and 1_S stands for the characteristic function on S .

PROOF. The proof is essentially the same as that of Proposition B.1 in [He-Na]; however, we give a proof for the sake of completeness.

Take $\mu \in C_0^\infty(\text{int } B_1(0), \mathbf{R})$ with $\|\mu\|_{L^2} = 1$. For any $x_0 \in \mathbf{R}^N$ and $s > 0$, we define $\mu_{s,x_0}(x) = s^{-N/2} \mu((x - x_0)/s)$. Then $\text{supp}(\mu_{s,x_0}) \subset B_s(x_0)$, $\int_{\mathbf{R}^N} |\mu_{s,x_0}(x)|^2 dx = 1$ and $\int_{\mathbf{R}^N} |\nabla_x \mu_{s,x_0}(x)|^2 dx = s^{-2} \|\nabla \mu\|_{L^2(\mathbf{R}^N, \mathbf{R}^N)}^2$ for all $x \in \mathbf{R}^N$.

Let $0 < s < \delta$ and $x_0 \in (F_\delta^+)^c$. By Lemma 2.2 with $\xi_{ij} = \beta_{ij}(x_0)$, we have

$$\begin{aligned} H(\lambda) &\geq \sum_{l < m} \lambda \beta_{lm}(x_0) B_{lm}(x) \\ &= \lambda \left(|B(x)| + \sum_{l < m} (\beta_{lm}(x_0) - \beta_{lm}(x)) B_{lm}(x) \right) \\ &\geq \lambda |B(x)| \left(1 - \left[\sum_{l < m} |\beta_{lm}(x) - \beta_{lm}(x_0)|^2 \right]^{1/2} \right). \end{aligned} \quad (2.1)$$

By the mean-value theorem, for any $x \in B_s(x_0)$,

$$\begin{aligned} |\beta_{lm}(x) - \beta_{lm}(x_0)| &\leq |x - x_0| \int_0^1 |\nabla \beta_{lm}(tx + (1-t)x_0)| dt \\ &\leq |x - x_0| \left(\int_0^1 |\nabla \beta_{lm}(tx + (1-t)x_0)|^2 dt \right)^{1/2}. \end{aligned}$$

Therefore, it follows from the definition of γ_B that

$$\left[\sum_{l < m} |\beta_{lm}(x) - \beta_{lm}(x_0)|^2 \right]^{1/2} \leq s \sup_{B_s(x_0)} \gamma_B \quad (2.2)$$

for all $x \in B_s(x_0)$.

On the other hand, using the IMS type formula, which can be verified by direct computation, we have

$$\frac{1}{2}((\mu_{s,x_0})^2 H(\lambda) + H(\lambda)(\mu_{s,x_0})^2) = \mu_{s,x_0} H(\lambda) \mu_{s,x_0} - |\nabla_x \mu_{s,x_0}|^2$$

as a sesqui-linear form on $C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$. Integrating this with respect to x_0 over \mathbf{R}^N , we have, from (2.1) and (2.2),

$$\begin{aligned} H(\lambda) &= \int_{\mathbf{R}^N} \mu_{s,x_0} H(\lambda) \mu_{s,x_0} dx_0 - s^{-2} \|\nabla \mu\|_{L^2(\mathbf{R}^N, \mathbf{R}^N)}^2 \\ &\geq \int_{(F_\delta^+)^c} \mu_{s,x_0} H(\lambda) \mu_{s,x_0} dx_0 - s^{-2} \|\nabla \mu\|_{L^2(\mathbf{R}^N, \mathbf{R}^N)}^2 \\ &\geq \lambda |B| \int_{(F_\delta^+)^c} \left(1 - s \left[\sup_{B_s(x_0)} \gamma_B \right]\right) |\mu_{s,x_0}|^2 dx_0 - s^{-2} \|\nabla \mu\|_{L^2(\mathbf{R}^N, \mathbf{R}^N)}^2 \\ &\geq \lambda |B| \left(1 - s \left[\sup_{(F_{\delta-s}^+)^c} \gamma_B \right]\right) \int_{(F_\delta^+)^c} |\mu_{s,x_0}|^2 dx_0 - s^{-2} \|\nabla \mu\|_{L^2(\mathbf{R}^N, \mathbf{R}^N)}^2 \\ &= \lambda |B| \left(1 - s \left[\sup_{(F_{\delta-s}^+)^c} \gamma_B \right]\right) \left(1 - \int_{F_\delta^+} |\mu_{s,x_0}|^2 dx_0\right) - s^{-2} \|\nabla \mu\|_{L^2(\mathbf{R}^N, \mathbf{R}^N)}^2 \\ &\geq \lambda |B| \left(1 - s \left[\sup_{(F_{\delta-s}^+)^c} \gamma_B \right]\right) 1_{(F_{\delta+s}^+)^c} - s^{-2} \|\nabla \mu\|_{L^2(\mathbf{R}^N, \mathbf{R}^N)}^2. \end{aligned} \quad (2.3)$$

Here, the last inequality follows from the contraposition of the inclusion $(\text{supp}(\mu_{s,x_0}) \subset) B_s(x_0) \subset F_{\delta+s}^+$ for $x_0 \in F_\delta^+$.

Finally, taking the infimum over μ on both sides of (2.3), we have the conclusion. \blacksquare

LEMMA 2.4. *Let $R > 0$ satisfy $B^{-1}(0) \subset \text{int}(B_R(0))$. There exist $\Lambda_R > 0$ and $C_R > 0$ such that $\|u\|_{L^2(B_R(0))}^2 \leq C_R \|u\|_\lambda^2$ and $\lambda \|u\|_{L^2(B_R(0)^c)}^2 \leq C_R \|u\|_\lambda^2$ for any $u \in E_\lambda$ and any $\lambda \geq \Lambda_R$.*

In particular, there exist $\Lambda > 0$ and $C > 0$ such that $\|u\|_{L^2(\mathbf{R}^N)} \leq C \|u\|_\lambda$ for any $u \in E_\lambda$ and any $\lambda \geq \Lambda$.

PROOF. Let $r > 0$ satisfy that $B^{-1}(0) \subset \text{int}(B_r(0))$, and let $\varepsilon > 0$ small enough. Set $\delta = \varepsilon \text{dist}(B^{-1}(0), \partial B_r(0))$ and $s = \min\{\delta, 1/\sup_{B_{r-\delta}(0)^c} \gamma_B\}/3$, which are positive by (A.2) and (A.3). By Proposition 2.3 with $F = B_r(0)$, we have

$$\begin{aligned} \|u\|_\lambda^2 &\geq \lambda \left(\inf_{(F_{\delta+s}^+)^c} |B| \right) \left(1 - s \sup_{(F_{\delta-s}^+)^c} \gamma_B\right) \|u\|_{L^2((F_{\delta+s}^+)^c)}^2 - (c_0/s^2) \|u\|_{L^2(\mathbf{R}^N)}^2 \\ &=: \lambda C \|u\|_{L^2((F_{\delta+s}^+)^c)}^2 - c \|u\|_{L^2(\mathbf{R}^N)}^2 \end{aligned}$$

$$= (\lambda C - c) \|u\|_{L^2((F_{\delta+s}^+)^c)}^2 - c \|u\|_{L^2(F_{\delta+s}^+)}^2$$

for any $u \in E_\lambda$. Hence,

$$\|u\|_{L^2((F_{\delta+s}^+)^c)}^2 \leq (\lambda C - c)^{-1} (c \|u\|_{L^2(F_{\delta+s}^+)}^2 + \|u\|_\lambda^2). \quad (2.4)$$

Take and fix $\zeta_r \in C_0^\infty(\mathbf{R}^N)$ such that $0 \leq \zeta_r \leq 1$ on \mathbf{R}^N , $\zeta_r = 1$ on $B_{r+2\delta}$, and $\text{supp}(\zeta_r) \subset \text{int}[B_{2r}(0)]$. Since $|\zeta_r u| \in H_0^1(\text{int}[B_{2r}(0)])$ for any $u \in E_\lambda$ by Lemma 2.1 (ii), it follows from the Poincaré and the dia-magnetic inequalities that

$$\begin{aligned} \|u\|_{L^2(F_{\delta+s}^+)}^2 &\leq \|\zeta_r u\|_{L^2(B_{2r}(0))}^2 \leq C \|\nabla |\zeta_r u|\|_{L^2(B_{2r}(0), \mathbf{R}^N)}^2 \\ &\leq C \|\nabla_{\lambda A}(\zeta_r u)\|_{L^2(B_{2r}(0), \mathbf{C}^N)}^2 \leq C'' (\|u\|_{L^2(B_{2r}(0) \setminus F_{\delta+s}^+)}^2 + \|u\|_\lambda^2) \\ &\leq C'' (\|u\|_{L^2((F_{\delta+s}^+)^c)}^2 + \|u\|_\lambda^2) \\ &\leq C'' c (\lambda C - c)^{-1} \|u\|_{L^2(F_{\delta+s}^+)}^2 + C'' (1 + (\lambda C - c)^{-1}) \|u\|_\lambda^2, \end{aligned}$$

where we used (2.4) in the last inequality. Hence,

$$(\lambda C - c - C'' c) \|u\|_{L^2(F_{\delta+s}^+)}^2 \leq C'' (\lambda C - c + 1) \|u\|_\lambda^2. \quad (2.5)$$

Since r and ε are arbitrary under the restrictions described above and $F_{\delta+s}^+ = B_{r+\delta+s}(0)$, we deduce the lemma from (2.4) and (2.5). \blacksquare

COROLLARY 2.5. *Let $r > 0$ be fixed. Let $\{y_n\}_n$ be a sequence in \mathbf{R}^N such that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. There exist $\Lambda' > 0$, $N' > 0$, and $C' > 0$ such that*

$$\|u\|_\lambda^2 \geq \lambda C' \|u\|_{L^2(B_r(y_n))}^2$$

for any $\lambda \geq \Lambda'$, any $u \in E_\lambda$, and any $n \geq N'$.

COROLLARY 2.6. *Let $\delta > 0$ and $F = [B^{-1}(0)]_\delta^+$. There exists $C_\delta > 0$, independent of large λ , such that $\lambda \|u\|_{L^2(F^c)}^2 \leq C_\delta \|u\|_\lambda^2$ for any $u \in E_\lambda$.*

PROOF. Let $\delta > 0$ and $F = B^{-1}(0)$. Fix $R > 0$ so large that $[B^{-1}(0)]_\delta^+ \subset \text{int}(B_R(0))$. By (A.2), (A.3) and Proposition 2.3, for any small $s > 0$, we have

$$\begin{aligned} \|u\|_\lambda^2 &\geq \lambda \left(\inf_{(F_{\delta+s}^+)^c} |B| \right) (1 - s \sup_{(F_{\delta-s}^+)^c} \gamma_B) \|u\|_{L^2((F_{\delta+s}^+)^c)}^2 - (c_0/s^2) \|u\|_{L^2(\mathbf{R}^N)}^2 \\ &= (\lambda C - c) \|u\|_{L^2((F_{\delta+s}^+)^c)}^2 - c \|u\|_{L^2(F_{\delta+s}^+)}^2 \\ &\geq (\lambda C - c) \|u\|_{L^2((F_{\delta+s}^+)^c)}^2 - c \|u\|_{L^2(B_R(0))}^2 \\ &\geq (\lambda C - c) \|u\|_{L^2((F_{\delta+s}^+)^c)}^2 - c C_R \|u\|_\lambda^2 \end{aligned}$$

for any $u \in E_\lambda$, where we used Lemma 2.4 in the last inequality. The arbitrariness of δ (and small s) shows the lemma. \blacksquare

LEMMA 2.7. *For any $\lambda \geq \Lambda$, where Λ is as in Lemma 2.4, we have the following assertions:*

(i) *The Banach space E_λ is continuously embedded into $L^q(\mathbf{R}^N)$ for any $q \in [2, 2^*]$ if $N \geq 3$ and for any $q \geq 2$ if $N = 2$.*

(ii) *For any bounded measurable subset K of \mathbf{R}^N , the restriction map $E_\lambda \rightarrow L^q(K)$ is compact for any $q \in [2, 2^*]$ if $N \geq 3$ and for any $q \geq 2$ if $N = 2$.*

PROOF. Let q be as above. It follows from the Sobolev and the dia-magnetic inequalities, and Lemma 2.4 that

$$\begin{aligned} \|u\|_{L^q(\mathbf{R}^N)} &\leq C_{N,q} \|u\|_{H^1(\mathbf{R}^N)} \leq C_{N,q} (\|\nabla_{\lambda A} u\|_{L^2(\mathbf{R}^N, \mathbf{C}^N)}^2 + \|u\|_{L^2(\mathbf{R}^N)}^2)^{1/2} \\ &\leq C_{N,q} (1 + C^2)^{1/2} \|u\|_\lambda \end{aligned} \quad (2.6)$$

for any $u \in E_\lambda$ and any $\lambda \geq \Lambda$. Here, $C_{N,q}$ and C are independent of λ . This shows (i).

Next, we show (ii). Let K and q be as in (ii) above. Choose $R > 0$ such that $K \subset B_R(0)$, and take and fix $\zeta_K \in C_0^\infty(\mathbf{R}^N)$ such that $0 \leq \zeta_K \leq 1$ on \mathbf{R}^N , $\zeta_K = 1$ on K , and $\text{supp}(\zeta_K) \subset \text{int}[B_{2R}(0)]$.

The map $T_1 : E_\lambda \ni u \mapsto \zeta_R u \in H_0^1(\text{int}[B_{2R}(0)])$ is continuous by Lemmata 2.1 and 2.4, the embedding $T_2 : H_0^1(\text{int}[B_{2R}(0)]) \rightarrow L^q(B_{2R}(0))$ is compact by the Rellich-Kondrashov theorem, and the restriction $T_3 : L^q(B_{2R}(0)) \rightarrow L^q(K)$ is continuous. Thus, the composition $T_3 \circ T_2 \circ T_1 : E_\lambda \ni u \mapsto u|_K \in L^q(K)$ is compact for each $\lambda \geq \Lambda$. This completes the proof. \blacksquare

In the rest of this paper, we always assume the condition $\lambda \geq \Lambda$, where Λ be as in Lemma 2.4, in the place where E_λ is under consideration.

To apply the standard variational calculus, except for Subsection 3.2, we shall regard E_λ as a real Hilbert space equipped with the inner product $(u, v)_\lambda = \text{Re}(\nabla_{\lambda A} u, \nabla_{\lambda A} v)_{L^2(\mathbf{R}^N, \mathbf{C}^N)}$.

It follows from (A.5) that $\nabla_{\lambda A} u = \nabla u$ and $(u, v)_\lambda = \text{Re}(\nabla u, \nabla v)_{L^2(\Omega, \mathbf{C}^N)}$ hold for any $u, v \in H_0^1(\Omega)$. Therefore, we may regard the space $H_0^1(\Omega)$ as a closed subspace of the Hilbert space E_λ by zero extension.

3. Preliminaries

3.1. Variational structure of J_λ . In this subsection, we recall some (well-known) variational properties of J_λ . Assume (A.1)–(A.5) and (f.1)–(f.4). Let $p \in (2, 2^*)$ and $\theta > 2$ be as in (f.2) and (f.3).

LEMMA 3.1. (i) *There exist $C > 0$ and $C' > 0$ such that $\inf_{x \in \mathbf{R}^N} F(x, t^2) \geq Ct^\theta - C't^2$ for any $t \geq 0$.*

(ii) For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$0 < \theta F(x, t^2) \leq f(x, t^2)t^2 \leq \varepsilon t^2 + C_\varepsilon t^p$$

for any $t > 0$ and any $x \in \mathbf{R}^N$.

(iii) For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} \int_{\mathbf{R}^N} f(x, |u|^2)|v||w|dx &\leq \varepsilon \|v\|_{L^2(\mathbf{R}^N)} \|w\|_{L^2(\mathbf{R}^N)} \\ &\quad + C_\varepsilon \|u\|_{L^p(\mathbf{R}^N)}^{p-2} \|v\|_{L^p(\mathbf{R}^N)} \|w\|_{L^p(\mathbf{R}^N)}, \\ \int_{\mathbf{R}^N} f(x, |u|^2)|u|^2 dx + \int_{\mathbf{R}^N} F(x, |u|^2) dx &\leq \varepsilon \|u\|_{L^2(\mathbf{R}^N)}^2 + C_\varepsilon \|u\|_{L^p(\mathbf{R}^N)}^p, \\ \int_{\mathbf{R}^N} f(x, |u|^2)|u|^2 dx + \int_{\mathbf{R}^N} F(x, |u|^2) dx &\leq \varepsilon \|u\|_\lambda^2 + C_\varepsilon \|u\|_\lambda^p \end{aligned}$$

for any $u, v, w \in E_\lambda$. Here, C_ε is independent of large λ .

(iv) If u converges to v in $L^2(\mathbf{R}^N)$ and in $L^p(\mathbf{R}^N)$, then $\int_{\mathbf{R}^N} F(x, |u|^2) dx$ tends to $\int_{\mathbf{R}^N} F(x, |v|^2) dx$.

PROOF. First, we show (ii). By (f.2), for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $f(x, t^2)t^2 \leq \varepsilon t^2$ for all $t \in (0, t_\varepsilon]$ and all $x \in \mathbf{R}^N$, and $f(x, t^2)t^2 \leq C(t^2 + t^p) \leq C(t_\varepsilon^{-(p-2)} + 1)t^p =: C_\varepsilon t^p$ for all $t \geq t_\varepsilon$ and all $x \in \mathbf{R}^N$. The rest of the assertion follows from (f.3).

Second, we show (i). Let $t_0 > 0$ be as in (f.4). For $t \geq t_0$, we have $F(x, t) \geq Ct^{\theta/2}$ by integrating the inequality in (f.3). For $t < t_0$, the assertion follows from (ii).

Third, the assertion (iii) follows from (ii), the Hölder inequality, and (2.6).

Finally, we show (iv). For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $||u|^2 - |v|^2| \leq |u - v|(|u| + |v|) \leq C_\varepsilon^{-1}|u - v|^2 + \varepsilon(|u|^2 + |v|^2)$. Then it follows from the mean-value theorem and (ii) with $\varepsilon = 1$ that $2|F(x, |u|^2) - F(x, |v|^2)| \leq (|u|^2 - |v|^2) \int_0^1 f(t|u|^2 + (1-t)|v|^2) dt \leq C_p(C_\varepsilon|u - v|^2 + \varepsilon(|u|^2 + |v|^2))(1 + |u|^{p-2} + |v|^{p-2})$ for any $\varepsilon > 0$. Then the assertion follows from the Hölder inequality $\int |u|^2|v|^{p-2} \leq \|u\|_{L^p}^2 \|v\|_{L^p}^{p-2}$. \blacksquare

The next lemma is well-known (e.g., the proof of Theorem 1.22 in [Zo-Sc]).

LEMMA 3.2. (i) If $u_n \rightharpoonup v$ in E_λ as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}^N} f(x, |u_n|^2) u_n \bar{w} dx = \int_{\mathbf{R}^N} f(x, |v|^2) v \bar{w} dx$$

for each $w \in E_\lambda$.

(ii) If $u_n \rightarrow v$ in E_λ as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \sup_{w \in E_\lambda \setminus \{0\}} \left| \int_{\mathbf{R}^N} f(x, |u_n|^2) u_n \bar{w} dx - \int_{\mathbf{R}^N} f(x, |v|^2) v \bar{w} dx \right| / \|w\|_\lambda = 0.$$

The next lemma follows from a simple calculation.

LEMMA 3.3. *Let $p > 2$, $a > 0$, and $b > 0$. The function $f(x) = ax^2/2 - bx^p/p$ on $[0, \infty)$ takes the maximum $f((a/b)^{1/(p-2)}) = \frac{p-2}{2p}(a^p/b^2)^{1/(p-2)}$. Moreover, $f(x) > 0$ if $0 < x < (a/b)^{1/(p-2)}$.*

The next lemma follows from the Hölder inequality.

LEMMA 3.4. *Let $1 < s < q < r < \infty$. Let K be a measurable subset of \mathbf{R}^N . We have $\|u\|_{L^q(K)} \leq \|u\|_{L^s(K)}^a \|u\|_{L^r(K)}^{1-a}$ for any $u \in L^s(K) \cap L^r(K)$, where $a = s(r-q)/[q(r-s)] \in (0, 1)$.*

Let J_λ be the functional on the real Banach space E_λ defined by (1.5). It follows from Lemma 3.2 that $J_\lambda \in C^1(E_\lambda, \mathbf{R})$ and the Fréchet derivative is given by the dual coupling

$$E'_\lambda \langle J'_\lambda(u), v \rangle_{E_\lambda} = \operatorname{Re} \left((u, v)_\lambda - \int_{\mathbf{R}^N} f(x, |u|^2) u \bar{v} dx \right)$$

for any $u, v \in E_\lambda$.

LEMMA 3.5. *There exist $r > 0$ and $b > 0$, both of which are independent of λ , such that $J_\lambda(u) \geq b$ if $\|u\|_\lambda = r$.*

PROOF. By Lemma 3.1 (iii) with $\varepsilon = 1/2$, we have

$$J_\lambda(u) = \frac{1}{2} \|u\|_\lambda^2 - \int_{\mathbf{R}^N} F(x, |u|^2) dx \geq \frac{1}{4} \|u\|_\lambda^2 - C_{1/2} \|u\|_\lambda^p \quad (3.1)$$

for any $u \in E_\lambda$. By Lemma 3.3, the rightmost of (3.1) takes the maximum $\frac{p-2}{2p} [(1/2)^p / (pC_{1/2})^2]^{1/(p-2)} =: b$ if $\|u\|_\lambda = [1/(2pC_{1/2})]^{1/(p-2)} =: r$. This proves the lemma. \blacksquare

For given $c \in \mathbf{R}$, we say that a sequence $\{u_n\}_n \subset E_\lambda$ is a $(\text{PS})_c$ -sequence for J_λ if $J_\lambda(u_n) \rightarrow c$ and $\|J'_\lambda(u_n)\|_{E'_\lambda} \rightarrow 0$ as $n \rightarrow \infty$. A functional is said to satisfy the $(\text{PS})_c$ -condition if any $(\text{PS})_c$ -sequence has a convergent subsequence.

LEMMA 3.6. *For any $(\text{PS})_c$ -sequence $\{u_n\}_n \subset E_\lambda$ for J_λ , we have:*

(i) $\limsup_{n \rightarrow \infty} \|u_n\|_\lambda^2 \leq [2\theta/(\theta - 2)]c$. In particular, $\{u_n\}_n$ is bounded in E_λ and $c \geq 0$. Here, θ is as in (f.3).

(ii) For any $q \in [2, 2^*]$ for $N \geq 3$ and for any $q \geq 2$ for $N = 2$, there exists $C_q > 0$ such that $\limsup_{n \rightarrow \infty} \|u_n\|_{L^q(\mathbf{R}^N)}^2 \leq C_q^2 [2\theta/(\theta - 2)]c$.

(iii) There exists $c_1 > 0$ such that if $c < c_1$ then $c = 0$.

(iv) There exists $\alpha > 0$ such that $\liminf_{n \rightarrow \infty} \|u_n\|_{L^p(\mathbf{R}^N)}^p \geq \alpha c$.

Here, all the constants C_q , c_1 , and α above are independent of λ , c , and of the choice of the $(\text{PS})_c$ -sequence.

PROOF. We give a proof of these standard results for the sake of completeness. Let $\{u_n\}_n$ be a $(PS)_c$ -sequence for J_λ . Then it follows from (f.3) that

$$\begin{aligned} c + o(1) + o(1)\|u_n\|_\lambda &= J_\lambda(u_n) - \frac{1}{\theta} E'_\lambda \langle J'_\lambda(u_n), u_n \rangle_{E_\lambda} \\ &= (1/2 - 1/\theta)\|u_n\|_\lambda^2 + \int_{\mathbf{R}^N} \left(\frac{1}{\theta} f(x, |u_n|^2) |u_n|^2 - F(x, |u_n|^2) \right) dx \\ &\geq [(\theta - 2)/(2\theta)]\|u_n\|_\lambda^2 \end{aligned}$$

as $n \rightarrow \infty$, which implies (i) because $o(1)\|u_n\|_\lambda \leq \varepsilon\|u_n\|_\lambda^2 + o(1)$ for any $\varepsilon > 0$. Then, applying (2.6) to (i) above, we have (ii). Note that the constant C_q from (2.6) depends only on N and q .

It follows from Lemma 3.1 (iii) with $\varepsilon = 1/2$ that

$$\begin{aligned} o(1)\|u_n\|_\lambda &= E'_\lambda \langle J'_\lambda(u_n), u_n \rangle_{E_\lambda} \geq \|u_n\|_\lambda^2 - \left(\frac{1}{2}\|u_n\|_\lambda^2 + C_{1/2}\|u_n\|_\lambda^p \right) \\ &= \frac{1}{4}\|u_n\|_\lambda^2 + \left(\frac{1}{4}\|u_n\|_\lambda^2 - C_{1/2}\|u_n\|_\lambda^p \right) \end{aligned} \quad (3.2)$$

as $n \rightarrow \infty$. Thus, by Lemma 3.3 with $a = 1/2$ and $b = pC_{1/2}$, we have

$$\|u_n\|_\lambda^2 \leq o(1)\|u_n\|_\lambda \quad (3.3)$$

if $\|u_n\|_\lambda \leq (2pC_{1/2})^{-1/(p-2)}$.

We now assume that $c < [(\theta - 2)/(2\theta)](2pC_{1/2})^{-2/(p-2)} =: c_1$. Then it follows from (i) that $\limsup_{n \rightarrow \infty} \|u_n\|_\lambda \leq ([2\theta/(\theta - 2)]c)^{1/2} < (2pC_{1/2})^{-1/(p-2)}$, and hence $\lim_{n \rightarrow \infty} \|u_n\|_\lambda = 0$ by (3.3). Then $J_\lambda(u_n) \rightarrow J_\lambda(0) = 0 = c$. This shows (iii).

Finally, we show (iv). By Lemma 3.1, (i), (ii), and (2.6), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, as $n \rightarrow \infty$,

$$\begin{aligned} c + o(1) + o(1)\|u_n\|_\lambda &= J_\lambda(u_n) - \frac{1}{2} E'_\lambda \langle J'_\lambda(u_n), u_n \rangle_{E_\lambda} \\ &= \int_{\mathbf{R}^N} \left(\frac{1}{2} f(x, |u_n|^2) |u_n|^2 - F(x, |u_n|^2) \right) dx \\ &\leq \varepsilon \|u_n\|_{L^2(\mathbf{R}^N)}^2 + C_\varepsilon \|u_n\|_{L^p(\mathbf{R}^N)}^p \\ &\leq \varepsilon C_2^2 \|u_n\|_\lambda^2 + C_\varepsilon \|u_n\|_{L^p(\mathbf{R}^N)}^p. \end{aligned} \quad (3.4)$$

Using (i), we deduce from (3.4) that

$$\liminf_{n \rightarrow \infty} \|u_n\|_{L^p(\mathbf{R}^N)}^p \geq C_\varepsilon^{-1} (1 - C_2^2 [2\theta/(\theta - 2)] \varepsilon) c.$$

Choose $\varepsilon > 0$ so that $C_2^2 [2\theta/(\theta - 2)] \varepsilon = 1/2$ and set $\alpha = C_\varepsilon^{-1}/2$. This completes the proof. \blacksquare

LEMMA 3.7. *Let $\{u_n\}_n$ be a $(PS)_c$ -sequence for J_λ . There exist a renamed subsequence $\{u_n\}_n$ and $v \in E_\lambda$ satisfying: (i) $u_n \rightharpoonup v$ in E_λ ; (ii) $u_n \rightarrow v$ in $L^q_{\text{loc}}(\mathbf{R}^N)$ for any $q \in [2, 2^*)$; (iii) $u_n(x) \rightarrow v(x)$ for a.e. $x \in \mathbf{R}^N$; (iv) $J'_\lambda(v) = 0$; (v) For any $R > 0$,*

$$\alpha c - \limsup_{n \rightarrow \infty} \|u_n\|_{L^p(B_R(0)^c)}^p \leq \|v\|_{L^p(\mathbf{R}^N)}^p \leq \{C_p^2[2\theta/(\theta-2)]c\}^{p/2}.$$

Here, the constants C_p and α are as in Lemma 3.6 (ii) and (iv).

PROOF. By a standard argument, we have (i)–(iii) since $\{u_n\}_n$ is bounded in E_λ . The assertion (iv) follows from (i) and Lemma 3.2.

We show (v). Let $\{u_n\}_n$ be a renamed subsequence satisfying (i)–(iv). It follows from (i), Fatou's lemma, and Lemma 3.6 (i) that

$$\|v\|_{L^p(\mathbf{R}^N)}^2 \leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^p(\mathbf{R}^N)}^2 \leq C_p^2[2\theta/(\theta-2)]c.$$

On the other hand, it follows from (ii) and Lemma 3.6 (iv) that, for any $R > 0$,

$$\begin{aligned} \|v\|_{L^p(\mathbf{R}^N)}^p &\geq \|v\|_{L^p(B_R(0))}^p = \lim_{n \rightarrow \infty} \|u_n\|_{L^p(B_R(0))}^p \\ &\geq \liminf_{n \rightarrow \infty} \|u_n\|_{L^p(\mathbf{R}^N)}^p - \limsup_{n \rightarrow \infty} \|u_n\|_{L^p(B_R(0)^c)}^p \\ &\geq \alpha c - \limsup_{n \rightarrow \infty} \|u_n\|_{L^p(B_R(0)^c)}^p. \end{aligned}$$

These complete the proof. ■

3.2. Convergence of spectrum. In this subsection, we give a spectral property which follows from the norm resolvent convergence of the self-adjoint operator $H(\lambda)$ as $\lambda \rightarrow \infty$.

Let $\{\mu_j\}_{j=1}^\infty$ be the strictly increasing sequence of all *distinct* eigenvalues of the Dirichlet Laplacian $-\Delta_\Omega$ acting on the complex Hilbert space $L^2(\Omega)$ with $\text{Dom}(-\Delta_\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$. We denote by $\text{mult}(\mu_j)$ the multiplicity of the eigenvalue μ_j .

LEMMA 3.8. *For any integer $k \geq 1$ and any $\varepsilon > 0$, there exists $\Lambda_{k,\varepsilon} > 0$ such that, for any $\lambda \geq \Lambda_{k,\varepsilon}$,*

- (i) $H(\lambda)$ has at least $\sum_{j=1}^k \text{mult}(\mu_j)$ eigenvalues (counted multiplicities):

$$\mathcal{E}_k := \bigcup_{j=1}^k \bigcup_{l=1}^{\text{mult}(\mu_j)} \{\mu_{j,l}^\lambda\}.$$

- (ii) $\sum_{j=1}^k \sum_{l=1}^{\text{mult}(\mu_j)} |\mu_{j,l}^\lambda - \mu_j| < \varepsilon$.
 (iii) $\text{Spec}(H(\lambda)) \cap (-\infty, \mu_k + \varepsilon] = \mathcal{E}_k$.

PROOF. This is a consequence of the results obtained by [He-Na] (See also [He-He]).

In fact, since $\overline{\Omega}$ is compact by (A.2), it follows from Theorem 2.2 in [He-Na] that $H(\lambda)$ converges to a self-adjoint operator $K(\lambda)$ as $\lambda \rightarrow \infty$ in the norm resolvent sense. On the

other hand, since $A = 0$ on Ω by (A.5), it follows from Theorem 3.3 in [He-Na] that $K(\lambda)$ is unitarily equivalent to $-\Delta_\Omega$, therefore $\text{Spec}(K(\lambda)) = \text{Spec}(-\Delta_\Omega)$. Once we have the norm resolvent convergence, the spectral concentration follows from the standard argument as in the proof of Theorems VIII.23 and VIII.24 in [Re-Si] (or as in Appendix A in [He-Na]). \blacksquare

3.3. Fountain theorem. In this subsection, we formulate a variant of Theorem 3.5 of Willem's monograph [Wi]. The theorem will be used to produce $(\text{PS})_c$ -sequences for J_λ for certain c 's.

Let X be a *real* Banach space. Assume that X has a direct product decomposition $X = \overline{\bigoplus_{j \geq 1} X_j}$, where each X_j is a finite-dimensional subspace. For any $k \geq 1$, set $Y_k = \bigoplus_{1 \leq j \leq k} X_j$ and $Z_k = \overline{\bigoplus_{j \geq k} X_j}$. Clearly, $X = Y_k + Z_k$ and $Y_k \cap Z_k = X_k$.

Let G be a compact group which acts linearly and isometrically on X . Assume that every X_j are G -invariant. Assume further that for any bounded, G -invariant, open neighbourhood U of the origin of Y_{k+1} , every continuous G -equivariant map f from ∂U to Y_k has a zero for every $k \geq 1$.

Then we have the following version of a fountain theorem:

THEOREM 3.9. *Let $k \geq 1$. Assume that $\varphi \in C^1(X, \mathbf{R})$ and φ is G -invariant. Let $0 < r_k < \rho_k$ and set*

$$c_k = \inf_{\gamma \in \Gamma_k} \max_{u \in B_{\rho_k}(0) \cap Y_k} \varphi(\gamma(u)),$$

where $\Gamma_k = \{\gamma \in C(B_{\rho_k}(0) \cap Y_k, X) \mid \gamma \text{ is } G\text{-equivariant, } \gamma|_{\partial(B_{\rho_k}(0) \cap Y_k)} = \text{id}\}$. Assume that

$$b_k := \inf_{\partial B_{r_k}(0) \cap Z_k} \varphi > a_k := \max_{\partial B_{\rho_k}(0) \cap Y_k} \varphi. \quad (3.5)$$

Then we have the following assertions:

- (i) $c_k \geq b_k$, and
- (ii) For any $\varepsilon \in (0, (c_k - a_k)/2)$ and any $\delta > 0$, if $\gamma \in \Gamma_k$ satisfies $\sup_{u \in B_{\rho_k}(0) \cap Y_k} \varphi(\gamma(u)) \leq c_k + \varepsilon$, then there exists $u \in X$ such that $|\varphi(u) - c_k| \leq 2\varepsilon$; $\text{dist}(u, \gamma(B_{\rho_k}(0) \cap Y_k)) \leq 2\delta$; and $\|\varphi'(u)\|_{X'} \leq 8\varepsilon/\delta$.

In Theorem 3.5 of [Wi], it is assumed that $\dim X_j$ is independent of j . However, its proof applies to Theorem 3.9 above if we use the following intersection lemma instead of Lemma 3.4 in [Wi].

LEMMA 3.10. *Let $0 < r < \rho$. Assume that $\gamma \in C(B_\rho(0) \cap Y_k, X)$ is G -equivariant and $\gamma|_{\partial B_\rho(0) \cap Y_k} = \text{id}$. Then $\gamma(0) = 0$ and $\gamma(B_\rho(0) \cap Y_k) \cap \partial B_r(0) \cap Z_k \neq \emptyset$.*

PROOF. Let γ be as above. The equivariance of γ implies that $\gamma(0) \in Y_k^G := \{u \in Y_k \mid g(u) = u \text{ for all } g \in G\}$. Note that $g(0) = 0$ for all $g \in G$ because G acts linearly. Define a constant map $f : Y_{k+1} \rightarrow Y_k^G (\subset Y_k)$ by $f(u) = \gamma(0)$, and set $U = \text{int}[B_1(0)] \cap Y_{k+1}$. By the assumption on the action of G , $\partial U \cap f^{-1}(0) \neq \emptyset$. Hence, $\gamma(0) = 0$.

Set $U = \{u \in \text{int}[B_\rho(0)] \cap Y_k \mid \|\gamma(u)\|_X < r\}$, which is a bounded, G -invariant, open neighbourhood of the origin of Y_k . Let $P_{k-1} : X \rightarrow Y_{k-1}$ be the continuous projection and set $f = P_{k-1} \circ \gamma : \bar{U} \rightarrow Y_{k-1}$, which is continuous G -equivariant. By the assumption again, there exists $u \in \partial U \cap f^{-1}(0)$. Since $u \in \partial U$, either (i) $\|u\|_X = \rho$; or (ii) $\|\gamma(u)\|_X = r$. If (i) occurs, then $\gamma(u) = u$ and then $\rho = \|u\|_X = \|\gamma(u)\|_X \leq r$, which is a contradiction. Thus, $f(u) = 0$, $\|u\|_X < \rho$ and $\|\gamma(u)\|_X = r$, and then the element $\gamma(u)$ assures the non-emptiness. \blacksquare

In the next section we shall use the antipodal action of $G = \{\pm 1\}$ on \mathbf{R}^n to apply the fountain theorem. In this case, the assumption on the action of G is fulfilled because of the Borsuk-Ulam theorem:

LEMMA 3.11. *Let $N > n \geq 0$. Let U be a bounded, symmetric (i.e., invariant under the antipodal action of the group $G = \{\pm 1\}$), open neighbourhood of the origin of \mathbf{R}^N . Every continuous odd map f from ∂U to \mathbf{R}^n has a zero.*

PROOF. We may assume that $N > n + 1 \geq 2$ because the lemma is trivial if $n = 0$ and follows from the standard Borsuk-Ulam theorem (e.g., Theorem D.17 in [Wi]) if $N = n + 1$. Define the inclusion $\iota : \mathbf{R}^n \rightarrow \mathbf{R}^{N-1} = \mathbf{R}^n \times \mathbf{R}^{N-n-1}$ by $\iota(x) = (x; 0)$. Then the composition $\iota \circ f : \partial U \rightarrow \mathbf{R}^{N-1}$ is continuous and odd. By the Borsuk-Ulam theorem, there exists $x \in \partial U \cap (\iota \circ f)^{-1}(0)$. This implies that $x \in \partial U \cap f^{-1}(0)$. \blacksquare

4. Proof of Theorem 1.1

In this section, we show Theorem 1.1. The proof is an adaptation of that of Theorem 1.1 in [Ba-Pa-Wa].

Let E_λ be the (real) Hilbert space introduced at the end of Section 2, and let $-\Delta_\Omega$, $\{\mu_j\}_j$, $H(\lambda)$, and $\{\mu_{j,l}^\lambda\}_{j,l}$ be as in Subsection 3.2.

Fix $k \geq 1$ arbitrarily. Set

$$Y_k = \bigoplus_{j=1}^k \text{Ker}(-\Delta_\Omega - \mu_j) =: \bigoplus_{j=1}^k X_j$$

and

$$Y_k^\lambda = \bigoplus_{j=1}^k \bigoplus_{l=1}^{\text{mult}(\mu_j)} \text{Ker}(H(\lambda) - \mu_{j,l}^\lambda).$$

Set $(Y_k^\lambda)^\perp = \{u \in E_\lambda \mid (u, v)_\lambda = 0 \text{ for any } v \in Y_k^\lambda\}$. For notational convenience, we set $Y_0 = Y_0^\lambda = \{0\}$ and then $(Y_0^\lambda)^\perp = E_\lambda$. Set $Z_k^\lambda = E_\lambda \ominus Y_{k-1}$.

In what follows, we assume that $\lambda \geq \Lambda_{k+1,\varepsilon}$, where $\Lambda_{k+1,\varepsilon}$ is as in Lemma 3.8 and we set $\varepsilon = \min\{\mu_{j+1} - \mu_j\}_{j=1}^k / 100$.

LEMMA 4.1. *There exist r and R_k , both of which are independent of λ , such that $0 < r < R_k$ and*

(i)

$$b_k^\lambda := \inf_{\partial B_r(0) \cap Z_k^\lambda} J_\lambda > 0 > -1 > a_k^\lambda := \max_{\partial B_{R_k}(0) \cap Y_k} J_\lambda.$$

(ii) $J_\lambda(u) \leq 0$ for any $u \in B_{R_k}(0)^c \cap Y_k$.

PROOF. First, the estimate of b_k^λ from below follows from Lemma 3.5 with the common r .

Next, we estimate a_k^λ from above. For any $u \in Y_k$, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2} \|u\|_\lambda^2 - \int_\Omega F(x, |u|^2) dx \leq \frac{1}{2} \|u\|_\lambda^2 - C \|u\|_{L^\theta(\Omega)}^\theta + C' \|u\|_{L^2(\Omega)}^2 \\ &\leq \left(\frac{1}{2} + C'' \right) \|u\|_\lambda^2 - C_k \|\nabla u\|_{L^2(\Omega, \mathbb{C}^N)}^\theta \\ &= \left(\frac{1}{2} + C'' \right) \|u\|_\lambda^2 - C_k \|u\|_\lambda^\theta, \end{aligned} \quad (4.1)$$

where we used Lemma 3.1 (i) in the first inequality and used Lemma 2.4 and the equivalence of norms on the λ -independent finite-dimensional space Y_k in the second inequality. Here, C'' and C_k are independent of λ . Since $\theta > 2$, we can take $R_k > 0$ so large that the rightmost of (4.1) is less than -1 if $\|u\|_\lambda \geq R_k$. This completes the proof. \blacksquare

Let R_k be the constant obtained in Lemma 4.1 above. Set

$$c_k^\lambda = \inf_{\gamma \in \Gamma_k^\lambda} \max_{u \in B_{R_k}(0) \cap Y_k} J_\lambda(\gamma(u)), \quad (4.2)$$

where $\Gamma_k^\lambda = \{\gamma \in C(B_{R_k}(0) \cap Y_k, E_\lambda) \mid \gamma \text{ is odd, } \gamma|_{\partial(B_{R_k}(0) \cap Y_k)} = \text{id}\}$.

LEMMA 4.2. *Let c_k^λ be as in (4.2). There exists a (PS) $_{c_k^\lambda}$ -sequence for J_λ for every large λ .*

PROOF. We take and fix an orthonormal basis $\{\tilde{f}_m\}_{m=k+1}^\infty$ of the subspace $Z_{k+1}^\lambda = E_\lambda \ominus Y_k$ of the Hilbert space E_λ .

We now apply Theorem 3.9 with $X = E_\lambda$; $G = \{\pm 1\}$ with the antipodal action; $\varphi = J_\lambda$; X_j is the space defined at the beginning of this section if $1 \leq j \leq k$, and set $X_j = \text{Span}_{\mathbb{C}}\{\tilde{f}_j\}$ if $j \geq k+1$; Y_k is the space defined above; $Z_k = Z_k^\lambda$; $r_k = r$; $\rho_k = R_k$; $c_k = c_k^\lambda$; $a_k = a_k^\lambda$; $b_k = b_k^\lambda$; and $\Gamma_k = \Gamma_k^\lambda$.

The assumption $b_k > a_k$ in Theorem 3.9 follows from Lemma 4.1. Hence, $c_k^\lambda \geq b_k^\lambda$, and for any $\varepsilon \in (0, (c_k^\lambda - a_k)/2)$ and any $\delta > 0$ there exists $u_\varepsilon \in E_\lambda$ such that $|J_\lambda(u_\varepsilon) - c_k^\lambda| \leq 2\varepsilon$ and $\|J'_\lambda(u_\varepsilon)\|_{E'_\lambda} \leq 8\varepsilon/\delta$. Moreover, it follows from Lemma 4.1 that $(c_k^\lambda - a_k)/2 \geq (b_k^\lambda -$

$a_k)/2 \geq 1/2$. Therefore, we can choose $\varepsilon = 1/(4n)$ and $\delta = 8$ so that $|J_\lambda(u_{1/n}) - c_k^\lambda| \leq 1/n$ and $\|J'_\lambda(u_{1/n})\|_{E'_\lambda} \leq 1/n$. Then we have a $(PS)_{c_k^\lambda}$ -sequence $\{u_{1/n}\}_n$. \blacksquare

LEMMA 4.3. *For any $(PS)_{c_k^\lambda}$ -sequence $\{u_n\}_n$ for J_λ , we have*

$$\limsup_{n \rightarrow \infty} \|u_n\|_\lambda^2 \leq [2\theta/(\theta - 2)]c_k^\lambda,$$

and $\limsup_{\lambda \rightarrow \infty} c_k^\lambda < \infty$.

PROOF. The first inequality follows from Lemma 3.6 (i). Since $\text{id} \in \Gamma_k^\lambda$, we have $c_k^\lambda \leq \max_{B_{R_k}(0) \cap Y_k} J_\lambda = \max_{B_{R_k}(0) \cap Y_k} I_\Omega$, which is finite and independent of λ . This shows the finiteness. \blacksquare

LEMMA 4.4. *Let c_k^λ be as in (4.2). Set*

$$\underline{c}_k^\lambda = \sup_{\rho > 0} \inf\{J_\lambda(u) \mid u \in (Y_{k-1}^\lambda)^\perp \cap \partial B_\rho(0)\}.$$

We have: $c_k^\lambda \geq \underline{c}_k^\lambda$ for any large λ .

PROOF. Let R_k be as in Lemma 4.1. If $\rho > 0$ satisfies $(Y_{k-1}^\lambda)^\perp \cap \partial B_\rho(0) \cap Y_k \cap [\text{int}(B_{R_k}(0))]^c \neq \emptyset$, then we deduce from Lemma 4.1 (ii) that

$$\inf\{J_\lambda(u) \mid u \in (Y_{k-1}^\lambda)^\perp \cap \partial B_\rho(0)\} \leq 0 \leq c_k^\lambda. \quad (4.3)$$

Next, we consider $\rho > 0$ satisfying $(Y_{k-1}^\lambda)^\perp \cap \partial B_\rho(0) \cap Y_k \cap [\text{int}(B_{R_k}(0))]^c = \emptyset$. For any $\gamma \in \Gamma_k^\lambda$, we set $\mathcal{O}_\gamma = \{u \in Y_k \cap \text{int}(B_{R_k}(0)) \mid \|\gamma(u)\|_\lambda < \rho\}$, which is a bounded, symmetric open neighborhood of $0 \in Y_k$. Let P_{k-1}^λ be the orthogonal projection from E_λ onto Y_{k-1}^λ for any $k \geq 1$. Since $P_{k-1}^\lambda \circ \gamma$ is a continuous odd map from $\overline{\mathcal{O}_\gamma}$ to Y_{k-1}^λ , by the Borsuk-Ulam theorem, there exists $v \in \partial \mathcal{O}_\gamma$ such that $P_{k-1}^\lambda(\gamma(v)) = 0$, i.e., $\gamma(v) \in (Y_{k-1}^\lambda)^\perp$.

Since $v \in \partial \mathcal{O}_\gamma$, either (i) $\|v\|_\lambda = R_k$; or (ii) $\|\gamma(v)\|_\lambda = \rho$. If (i) occurs, then $\gamma(v) = v$ since $\gamma \in \Gamma_k^\lambda$. Since $R_k = \|v\|_\lambda = \|\gamma(v)\|_\lambda \leq \rho$, the element $(\rho/R_k)v$ belongs to $(Y_{k-1}^\lambda)^\perp \cap \partial B_\rho(0) \cap Y_k \cap [\text{int}(B_{R_k}(0))]^c = \emptyset$. This contradiction shows that $\|v\|_\lambda < R_k$ and $\|\gamma(v)\|_\lambda = \rho$. Therefore, $v \in B_{R_k}(0) \cap Y_k$ and $\gamma(v) \in (Y_{k-1}^\lambda)^\perp \cap \partial B_\rho(0)$, and then $\inf\{J_\lambda(u) \mid u \in (Y_{k-1}^\lambda)^\perp \cap \partial B_\rho(0)\} \leq J_\lambda(\gamma(v)) \leq \max_{u \in B_{R_k}(0) \cap Y_k} J_\lambda(\gamma(u))$. This implies that

$$\inf\{J_\lambda(u) \mid u \in (Y_{k-1}^\lambda)^\perp \cap \partial B_\rho(0)\} \leq c_k^\lambda. \quad (4.4)$$

The lemma follows from (4.3) and (4.4). \blacksquare

LEMMA 4.5. *Let μ_k be as in Section 3. There exists $\Lambda'_k > 0$ such that*

$$\underline{c}_k^\lambda \geq C_0(\mu_k)^\delta$$

for any $\lambda \geq \Lambda'_k$. Here, both $C_0 > 0$ and $\delta > 0$ are independent of λ and k .

PROOF. By the definition of $(Y_{k-1}^\lambda)^\perp$ and Lemma 3.8, we have

$$\|u\|_\lambda^2 \geq \mu_k \|u\|_{L^2(\mathbf{R}^N)}^2 / 2 \quad (4.5)$$

for any $u \in (Y_{k-1}^\lambda)^\perp$ if λ is large enough. For $N \geq 3$, it follows from Lemma 3.4, (2.6) with $q = 2^*$, and (4.5) that

$$\begin{aligned} \|u\|_{L^p(\mathbf{R}^N)} &\leq \|u\|_{L^2(\mathbf{R}^N)}^a \|u\|_{L^{2^*}(\mathbf{R}^N)}^{1-a} \leq C_N \|u\|_{L^2(\mathbf{R}^N)}^a \|u\|_\lambda^{1-a} \\ &\leq C'_N \|u\|_\lambda / (\mu_k)^{a/2} \end{aligned} \quad (4.6)$$

for any $u \in (Y_{k-1}^\lambda)^\perp$. A similar argument replaced 2^* by $p+1$ yields (4.6) for $N = 2$. Here, the exponent a depends only on p and N . Then, for any $u \in (Y_{k-1}^\lambda)^\perp$ and for any $\varepsilon > 0$, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{2} \|u\|_\lambda^2 - \left(\varepsilon \|u\|_{L^2(\mathbf{R}^N)}^2 + C_\varepsilon \|u\|_{L^p(\mathbf{R}^N)}^p \right) \\ &\geq \frac{1}{2} (1 - 4\varepsilon/\mu_k) \|u\|_\lambda^2 - \frac{C_\varepsilon (C'_N)^p}{(\mu_k)^{ap/2}} \|u\|_\lambda^p, \end{aligned} \quad (4.7)$$

where we used Lemma 3.1, (4.5), and (4.6) in the second inequality.

Setting $\varepsilon = \mu_1/8$, we have $1 - 4\varepsilon/\mu_k = 1 - \mu_1/(2\mu_k) \geq 1/2$. We deduce from (4.7) and Lemma 3.3 that

$$\begin{aligned} \underline{c}_k^\lambda &= \sup_{\rho>0} \inf \{ J_\lambda(u) \mid u \in (Y_{k-1}^\lambda)^\perp \cap \partial B_\rho(0) \} \\ &\geq \sup_{\rho>0} \left(\frac{1}{4} \rho^2 - \frac{1}{p} \frac{p C_\varepsilon (C'_N)^p}{(\mu_k)^{ap/2}} \rho^p \right) \\ &= \frac{p-2}{2p} \left(2^{-p} / [p C_\varepsilon (C'_N)^p / (\mu_k)^{ap/2}]^2 \right)^{1/(p-2)} \\ &=: C_0(\mu_k)^\delta. \end{aligned}$$

This proves the lemma. ■

LEMMA 4.6. *Assume that $B^{-1}(0) \subset B_{R-1}(0)$. Then there exist $C_{k,R} > 0$ and $a > 0$, both of which are independent of λ , such that any $(PS)_{c_k^\lambda}$ -sequence $\{u_n\}_n$ for J_λ satisfies $\limsup_{n \rightarrow \infty} \|u_n\|_{L^p(B_R(0)^c)} \leq C_{k,R} \lambda^{-a}$.*

PROOF. By Lemmata 2.4 and 4.3, there exists a constant $C_k > 0$ such that

$$\lambda C_R \|u_n\|_{L^2(B_R(0)^c)}^2 \leq \|u_n\|_\lambda^2 \leq C_k \quad (4.8)$$

for all n and λ large enough. For $N \geq 3$, it follows from Lemma 3.4, (2.6), and (4.8) that

$$\|u_n\|_{L^p(B_R(0)^c)} \leq \|u_n\|_{L^2(B_R(0)^c)}^a \|u_n\|_{L^{2^*}(B_R(0)^c)}^{1-a}$$

$$\begin{aligned}
&\leq C_N^{1-a} \|u_n\|_{L^2(B_R(0)^c)}^a \|u_n\|_\lambda^{1-a} \\
&\leq C_N^{1-a} (C_k)^{1/2} / (C_R \lambda)^{a/2}, \tag{4.9}
\end{aligned}$$

where a is from Lemma 3.4. This completes the proof for $N \geq 3$. A similar argument yields (4.9) for $N = 2$. \blacksquare

By Lemmata 4.2 and 3.7, we obtain a critical point $v_k^\lambda \in E_\lambda$ of J_λ . Fix $R > 0$ as in Lemma 4.6. It follows from Lemma 3.7 (v) with $v = v_k^\lambda$ and Lemma 4.6 that

$$\alpha c_k^\lambda - C_{k,R} \lambda^{-a} \leq \|v_k^\lambda\|_{L^p(\mathbf{R}^N)}^p \leq C_p^p C_k^{p/2} =: \beta_k.$$

Here, the constants α , C_p , and C_k are as in Lemma 3.6 (i), (ii), and (4.8), respectively. Thus, by Lemmata 4.5 and 4.4, for each $k \geq 1$,

$$\alpha_k := \alpha C_0 (\mu_k)^\delta / 2 \leq \alpha c_k^\lambda / 2 \leq \|v_k^\lambda\|_{L^p(\mathbf{R}^N)}^p \leq \beta_k$$

for any $\lambda \geq [2C_{k,R}/C_0(\mu_k)^\delta]^{1/a} + \Lambda'_k =: \Lambda(k)$. Here, Λ'_k is as in Lemma 4.5.

Since $0 < \mu_1 < \dots < \mu_k < \dots \rightarrow \infty$ as $k \rightarrow \infty$, it follows from Lemma 4.5 that there exists a sequence $\{k_j\}_{j=1}^\infty$ such that $2\beta_{k_j} < \alpha_{k_{j+1}}$ for any $\lambda \geq \Lambda(k_{j+1})$. (Clearly, we may assume that $\{\Lambda(k)\}_k$ is increasing.)

Therefore, for each $m \geq 1$, the weak solutions $\{v_{k_j}^\lambda\}_{j=1}^m$ satisfy

$$(0 < \alpha_1 <) \alpha_{k_j} \leq \|v_{k_j}^\lambda\|_{L^p(\mathbf{R}^N)}^p \leq \beta_{k_j} < \alpha_{k_{j+1}} \leq \|v_{k_{j+1}}^\lambda\|_{L^p(\mathbf{R}^N)}^p$$

for any $\lambda \geq \Lambda(k_m)$ and for any $j = 1, 2, \dots, m-1$.

We now complete the proof of Theorem 1.1 by setting $b_j = (\beta_{k_j})^{1/p}$, $a_j = (\alpha_{k_j})^{1/p}$, and $c_j = \sup_{\lambda \geq \Lambda(k_j)} \|v_{k_j}^\lambda\|_\lambda$, which is finite by Lemma 4.3 and the weak lower-semicontinuity of norms.

5. Proof of Theorem 1.2

In this section, we show Theorem 1.2. Our proof is essentially the same as that of Theorem 1.2 in [Ba-Pa-Wa].

Before proceeding to the proof, we recall a standard result from the theory of Sobolev spaces. The next result follows immediately from Corollary 5.3.5 in [Ca] for real-valued functions.

LEMMA 5.1. *Let Ω is a bounded, smooth open subset of \mathbf{R}^N . Assume that $v \in H^1(\Omega)$, $w \in H_0^1(\Omega)$, and $|v| \leq |w|$ a.e. Then $v \in H_0^1(\Omega)$.*

Let $\{\lambda_n\}_{n=1}^\infty$ be a diverging sequence and let $\{u_n\}_n$ be a set of critical points of J_{λ_n} satisfying: $u_n \in E_{\lambda_n}$, $\sup_n \|u_n\|_{\lambda_n} < \infty$, and $\inf_n \|u_n\|_{L^p(\mathbf{R}^N)} > 0$.

We remark that all the statements in Lemma 5.2 below remain valid if the condition “ $J'_{\lambda_n}(u_n) = 0$ ” is replaced by “ $\|J'_{\lambda_n}(u_n)\|_{E'_{\lambda_n}} \rightarrow 0$ as $n \rightarrow \infty$ ”.

LEMMA 5.2. *There exist $v \in H_0^1(\Omega)$ and a subsequence of $\{u_n\}_n$, still denoted by $\{u_n\}_n$, such that $u_n|_\Omega \rightharpoonup v$ in $H^1(\Omega)$, moreover,*

- (i) $u_n \rightarrow v$ in $L_{\text{loc}}^q(\mathbf{R}^N)$ for any $q \in [2, 2^*)$, and $u_n \rightarrow v$ in $L^2(\mathbf{R}^N)$;
- (ii) $u_n(x) \rightarrow v(x)$ for a.e. $x \in \mathbf{R}^N$;
- (iii) v is a weak solution to $-\Delta v = f(x, |v|^2)v$ on Ω .

PROOF. By (2.6), we have $\| |u_n| \|_{H^1(\mathbf{R}^N)} \leq C \sup_n \|u_n\|_{\lambda_n}$, where C is independent of n . Thus, $\{|u_n|\}_n$ is bounded in $H^1(\mathbf{R}^N)$, therefore we find a (renamed) subsequence such that $|u_n| \rightharpoonup w \in H^1(\mathbf{R}^N)$; $|u_n| \rightarrow w$ in $L_{\text{loc}}^q(\mathbf{R}^N)$ for any $q \in [2, 2^*)$; $|u_n(x)| \rightarrow w(x)$ for a.e. $x \in \mathbf{R}^N$ as $n \rightarrow \infty$.

Let $\delta > 0$. It follows from Corollary 2.6 that $\lambda_n \|u_n\|_{L^2(\mathbf{R}^N \setminus [B^{-1}(0)]_\delta^+)}^2 \leq C_\delta \sup_n \|u_n\|_{\lambda_n}^2$, where C_δ is independent of n . Thus,

$$\lim_{n \rightarrow \infty} \|u_n\|_{L^2(\mathbf{R}^N \setminus [B^{-1}(0)]_\delta^+)}^2 = 0 \quad (5.1)$$

since $\lambda_n \rightarrow \infty$. Using a countable covering argument, we choose a (renamed) subsequence such that $u_n(x) \rightarrow 0$ for a.e. $x \in [B^{-1}(0)]^c$. Therefore, $w(x) = 0$ for a.e. $x \in \Omega^c$ because of (A.5). Since $\partial\Omega$ has zero measure, we conclude that $w \in H_0^1(\Omega)$ (e.g., [Br], Theorem 9.18).

On the other hand, $\|\nabla u_n\|_{L^2(\Omega, \mathbf{C}^N)} \leq \|\nabla_{\lambda_n} u_n\|_{L^2(\mathbf{R}^N, \mathbf{C}^N)} \leq \sup_n \|u_n\|_{\lambda_n}$ by (A.5). Thus, $\{u_n|_\Omega\}_n$ is bounded in $H^1(\Omega)$, and then we find a subsequence, still denoted by $\{u_n\}_n$, such that $u_n|_\Omega \rightharpoonup v \in H^1(\Omega)$; $u_n|_\Omega \rightarrow v$ in $L^q(\Omega)$ for any $q \in [2, 2^*)$; $u_n(x) \rightarrow v(x)$ for a.e. $x \in \Omega$. Hence, $|v(x)| = w(x)$ for a.e. $x \in \Omega$, and then $v \in H_0^1(\Omega)$ by Lemma 5.1.

Let $q \in [2, 2^*)$. For any bounded measurable set K , we have $\|u_n - v\|_{L^q(K)}^q = \| |u_n| \|_{L^q(K \setminus \Omega)}^q + \|u_n|_\Omega - v\|_{L^q(K \cap \Omega)}^q \rightarrow 0$ as $n \rightarrow \infty$ by the L_{loc}^q -convergences. Moreover, by (5.1) and the L_{loc}^2 -convergence, we have

$$\|u_n - v\|_{L^2(\mathbf{R}^N)}^2 = \|u_n\|_{L^2(\mathbf{R}^N \setminus [B^{-1}(0)]_\delta^+)}^2 + \|u_n - v\|_{L^2([B^{-1}(0)]_\delta^+)}^2 \rightarrow 0$$

as $n \rightarrow \infty$. These imply (i) and (ii).

Finally, We show (iii). Every u_n satisfies $E'_{\lambda_n} \langle J'_{\lambda_n}(u_n), \varphi \rangle_{E_{\lambda_n}} = 0$ for any $\varphi \in H_0^1(\Omega)$. Recall that $\nabla(u_n|_\Omega) = (\nabla u_n)|_\Omega$; $u_n|_\Omega \rightharpoonup v$ in $H^1(\Omega)$; and $u_n|_\Omega \rightarrow v$ in $L_{\text{loc}}^q(\Omega)$. Then we can deduce from Lemma 3.2 that v is a weak solution to (1.6). \blacksquare

We recall a lemma due to Lions (Lemma I.1 in [Li]):

LEMMA 5.3. *Let $r > 0$ and $q' \in [2, 2^*)$. Assume that $\{g_n\}_n$ is bounded in $H^1(\mathbf{R}^N)$ and satisfies $\lim_{n \rightarrow \infty} \sup_{y \in \mathbf{R}^N} \int_{B_r(y)} |g_n(x)|^{q'} dx = 0$. Then $\{g_n\}_n$ converges to 0 in $L^q(\mathbf{R}^N)$ for any $q \in (2, 2^*)$.*

LEMMA 5.4. *Let $\{u_n\}_n$ and v be as in lemma 5.2. A renamed subsequence satisfies: $u_n \rightarrow v$ as $n \rightarrow \infty$ in $L^q(\mathbf{R}^N)$ for any $q \in (2, 2^*)$. Moreover, $v \neq 0$.*

PROOF. Set $g_n = u_n - v \in E_{\lambda_n}$. It follows from (2.6) that $\| |g_n| \|_{H^1(\mathbf{R}^N)} \leq C \|g_n\|_{\lambda_n} \leq C(\sup_n \|u_n\|_{\lambda_n} + \|v\|_{H^1(\Omega)}) < \infty$. Thus, $\{|g_n|\}_n$ is bounded in $H^1(\mathbf{R}^N)$ and $g_n(x) \rightarrow 0$ for a.e. $x \in \mathbf{R}^N$ by Lemma 5.2 (ii).

We show that $|g_n| \rightarrow 0$ in $L^q(\mathbf{R}^N)$ for any $q \in (2, 2^*)$ by contradiction. Assume that $\{|g_n|\}_n$ does not converges to 0 in $L^q(\mathbf{R}^N)$ for some $q \in (2, 2^*)$. Then, by Lemma 5.3 with $r = 1$, $q' = 2$, and $g_n = |g_n|$, there exist $\delta > 0$, a sequence $\{y_n\}_n \subset \mathbf{R}^N$, and a (renamed) subsequence $\{|g_n|\}_n$ such that

$$\inf_n \|g_n\|_{L^2(B_1(y_n))}^2 \geq \delta. \quad (5.2)$$

Moreover, we have $|y_n| \rightarrow \infty$ since $g_n \rightarrow 0$ in $L_{\text{loc}}^2(\mathbf{R}^N)$ by Lemma 5.2 (i), and we have $g_n = u_n - v = u_n$ on $B_1(y_n)$ for any large n since $v \in H_0^1(\Omega)$. Then it follows from (5.2) and Corollary 2.5 that

$$\begin{aligned} \lambda_n \delta &\leq \lambda_n \|g_n\|_{L^2(B_1(y_n))}^2 = \lambda_n \|u_n\|_{L^2(B_1(y_n))}^2 \\ &\leq C' \|u_n\|_{\lambda_n}^2 \leq C' \sup_n \|u_n\|_{\lambda_n}^2 < \infty, \end{aligned}$$

where C' is independent of n . Since $\lambda_n \rightarrow \infty$, this is a contradiction.

Finally, we have $\|v\|_{L^p(\mathbf{R}^N)} \geq \inf_n \|u_n\|_{L^p(\mathbf{R}^N)} - \limsup_{n \rightarrow \infty} \|u_n - v\|_{L^p(\mathbf{R}^N)} = \inf_n \|u_n\|_{L^p(\mathbf{R}^N)} > 0$ by assumption. This shows that $v \neq 0$. \blacksquare

LEMMA 5.5. *Let $\{u_n\}_n$ and v be as in lemma 5.4. A renamed subsequence satisfies: (i) $\|u_n - v\|_{\lambda_n} \rightarrow 0$, and (ii) $J_{\lambda_n}(u_n) \rightarrow I_\Omega(v)$, as $\lambda \rightarrow \infty$.*

PROOF. We show (i). Since $v \in H_0^1(\Omega)$, it follows from (A.5) that

$$\begin{aligned} E'_{\lambda_n} \langle J'_{\lambda_n}(v), u_n - v \rangle_{E_{\lambda_n}} \\ = \operatorname{Re} \left(\int_{\Omega} \nabla v \overline{\nabla(u_n - v)} dx + \int_{\Omega} f(x, |v|^2) v \overline{(u_n - v)} dx \right). \end{aligned} \quad (5.3)$$

The first term on the right-hand side of (5.3) tends to 0 as $n \rightarrow \infty$ since $u_n|_{\Omega} \rightharpoonup v$ in $H^1(\Omega)$ by Lemma 5.2. By the L_{loc}^q -convergence of $\{u_n\}_n$ and Lemma 3.1 (iii), the second integral on the right-hand side of (5.3) tends to 0 as $n \rightarrow \infty$. Thus, we have $\lim_{n \rightarrow \infty} \langle J'_{\lambda_n}(v), u_n - v \rangle = 0$. Since $J'_{\lambda_n}(u_n) = 0$, it follows that, as $n \rightarrow \infty$,

$$\begin{aligned} o(1) &= E_{\lambda_n} \langle J'_{\lambda_n}(u_n) - J'_{\lambda_n}(v), u_n - v \rangle_{E_{\lambda_n}} \\ &= \|u_n - v\|_{\lambda_n}^2 \\ &\quad - \operatorname{Re} \int_{\mathbf{R}^N} f(|u_n|^2) u_n \overline{(u_n - v)} dx + \operatorname{Re} \int_{\Omega} f(|v|^2) v \overline{(u_n - v)} dx. \end{aligned}$$

From this, using Lemma 3.1 (iii) with $\varepsilon = 1$ and the L^q_{loc} -convergence, we have

$$\begin{aligned}
& \|u_n - v\|_{\lambda_n}^2 \\
& \leq (\|v\|_{L^2(\Omega)} \|u_n - v\|_{L^2(\Omega)} + C_1 \|v\|_{L^p(\Omega)}^{p-1} \|u_n - v\|_{L^2(\Omega)}) + o(1) \\
& = (\|u_n\|_{L^2(\mathbf{R}^N)} \|u_n - v\|_{L^2(\mathbf{R}^N)} + C_1 \|u_n\|_{L^p(\mathbf{R}^N)}^{p-1} \|u_n - v\|_{L^p(\mathbf{R}^N)}) + o(1) \\
& \leq C_p (\sup_n \|u_n\|_{\lambda_n} + \sup_n \|u_n\|_{\lambda_n}^{p-1}) (\|u_n - v\|_{L^2(\mathbf{R}^N)} + \|u_n - v\|_{L^p(\mathbf{R}^N)}) + o(1) \\
& \leq C'_p (\|u_n - v\|_{L^2(\mathbf{R}^N)} + \|u_n - v\|_{L^p(\mathbf{R}^N)}) + o(1)
\end{aligned}$$

as $n \rightarrow \infty$, where we used (2.6) in the first inequality and used the finiteness assumption. Then the assertion (i) follows from Lemmata 5.4 and 5.2 (i).

The assertion (ii) follows from (i) and the $L^p(\mathbf{R}^N)$ -convergence of u_n to v from Lemma 5.4. Indeed, since $v \in H_0^1(\Omega)$, we have

$$J_{\lambda_n}(u_n) - I_{\Omega}(v) = \frac{1}{2} (\|u_n\|_{\lambda_n}^2 - \|v\|_{\lambda_n}^2) - \int_{\mathbf{R}^N} (F(x, |u_n|^2) - F(x, |v|^2)) dx.$$

The first term on the right-hand side tends to zero as $n \rightarrow \infty$ by (i). The second term tends to zero by Lemma 3.1 (iv). \blacksquare

We now complete the proof of Theorem 1.2.

6. Proof of Theorem 1.3

In this section, we show Theorem 1.3. Assume (A.1)–(A.5) and (f.1)–(f.6). We may assume that λ is large enough.

Under the assumptions (f.1)–(f.5), we can show the positivity of the infimum c_λ over the Nehari manifold and the existence of a $(\text{PS})_{c_\lambda}$ -sequence for J_λ ; this can be shown as in the proof of Theorem 4.2 in [Wi]. Moreover, we have $\liminf_{\lambda \rightarrow \infty} c_\lambda \geq b > 0$, where b is as in Lemma 3.5, because of the mountain pass type characterization of c_λ obtained as in Section 4 in [Wi].

In addition, under (f.6), we can show that J_λ satisfies the $(\text{PS})_{c_\lambda}$ -condition for any large λ by using the results in Subsection 3.1 and the argument in the proof of Lemma 4.6; the proof is similar to that of Proposition 2.1 in [Ba-Wa2]. Therefore, the value c_λ is attained by some critical point $u_\lambda \in E_\lambda$ for J_λ (e.g., [Wi]). Note that it follows from the definition of the Nehari manifolds that $(0 < b \leq) c_\lambda \leq c(\Omega)$ since $H_0^1(\Omega) \subset E_\lambda$.

Finally, we show that $c_\lambda \rightarrow c(\Omega)$ as $\lambda \rightarrow \infty$ by contradiction. Assume that there exists a diverging sequence $\{\lambda_n\}_n$ such that $c_{\lambda_n} = J_{\lambda_n}(u_{\lambda_n}) \rightarrow A \in (0, c(\Omega))$ as $n \rightarrow \infty$. Then the sequence $\{u_{\lambda_n}\}_n$ satisfies the conditions: $J'_{\lambda_n}(u_{\lambda_n}) = 0$, $\sup_n \|u_{\lambda_n}\|_{\lambda_n} < \infty$, and $\inf_n \|u_{\lambda_n}\|_{L^p(\mathbf{R}^N)} > 0$ by Lemma 3.6 (i), (iv). In particular, all the assumptions of Theorem 1.2 with $u_n = u_{\lambda_n}$ are fulfilled. Hence, we find a (renamed) subsequence such that $J_{\lambda_n}(u_{\lambda_n}) \rightarrow$

$I_\Omega(v)$ for some nonzero critical point v of I_Ω . Therefore, $c(\Omega) \leq I_\Omega(v) = A < c(\Omega)$. This contradiction finishes the proof.

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