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# Asymptotic Behavior of Solutions to the Semilinear Wave Equation with Time-dependent Damping

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**Abstract.** We consider the Cauchy problem for the semilinear wave equation with time-dependent damping

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = f(u)\,, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}^N \\ (u,u_t)(0,x) = (u_0,u_1)(x)\,, & x \in \mathbf{R}^N\,. \end{cases} \tag{$*$}$$

When $b(t) = (t+1)^{-\beta}$ with $0 \le \beta < 1$, the damping is effective and the solution $u$ to $(*)$ behaves as that to the corresponding parabolic problem. When $f(u) = O(|u|^\rho)$ as $u \to 0$ with $1 < \rho < \frac{N+2}{[N-2]_+}$(the Sobolev exponent), our main aim is to show the time-global existence of solutions for small data in the supercritical exponent $\rho > \rho_F(N) := 1 + 2/N$. We also obtain some blow-up results on the solution within a finite time, so that the smallness of the data is essential to get global existence in the supercritical exponent case.

## 1. Introduction

We consider the Cauchy problem for the semilinear wave equation with time-dependent damping

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = f(u)\,, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}^N \\ (u,u_t)(0,x) = (u_0,u_1)(x)\,, & x \in \mathbf{R}^N\,, \end{cases} \tag{1.1}$$

where the coefficient of damping

$$b(t) = b_0(t+1)^{-\beta}\,, \quad 0 \le \beta < 1 (b_0 : \text{positive constant}) \tag{1.2}$$

and the sourcing semilinear term

$$f(u) = \pm|u|^\rho \quad \text{or} \quad |u|^{\rho-1}u\,, \quad \rho > 1\,. \tag{1.3}$$

Throughout this paper we assume

$$1 < \rho < \infty (N = 1,2) \quad \text{and} \quad 1 < \rho < \frac{N+2}{N-2} (N \ge 3) \tag{1.4}$$

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(denote this simply by $1 < \rho < \frac{N+2}{[N-2]_+}$) and

$$(u_0, u_1) \in H^1 \times L^2, \quad \operatorname{supp}\{u_0, u_1\} \subset \{x; |x| \leq L\} =: B_L \tag{1.5}$$

for some positive constant $L$. Then there exists a unique weak solution $u$ to (1.1) in

$$X_T := C([0,T); H^1) \cap C^1([0,T); L^2) \quad \text{for some} \quad T > 0,$$

whose support

$$\operatorname{supp}\{u(t,\cdot)\} \subset B_{t+L} = \{x; |x| \leq t + L\} \tag{1.6}$$

(Strauss [20]). In these conditions our concern is with time-global existence of solutions and blow-up within a finite time.

When $f(u) \equiv 0$, Wirth has obtained in [23, 24] that, if $-1 < \beta < 1$, then "the damping term $+b(t)u_t$ is effective", and that the solution has "the diffusion phenomena" if $-1/3 < \beta < 1$, that is, the solution behaves as that of the corresponding diffusion equation

$$-\Delta\phi + b(t)\phi_t = 0 \tag{1.7}$$

as $t \to \infty$ (see also Yamazaki [25, 26]). Therefore, for (1.1) we can expect that there is some critical exponent and, in the supercritical exponent the time-global existence theorem of solutions for small data holds, while in the critical and subcritical exponents the blow-up phenomena within a finite time occurs.

The solution $\phi$ to (1.7) with $\phi(0,x) = \phi_0(x) \in L^q$ $(1 \leq q \leq \infty)$ is given by

$$\begin{aligned}\phi(t,x) &= \left[e^{B(t)\Delta}\phi_0\right](x), \quad B(t) = \int_0^t \frac{d\tau}{b(\tau)} \\ &= (4\pi B(t))^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4B(t)}} \phi_0(y)dy =: [G_B(t,\cdot) * \phi_0](x),\end{aligned} \tag{1.8}$$

so that for $1 \leq q \leq p \leq \infty$

$$\|\phi(t,\cdot)\|_{L^p} \leq Ct^{-\frac{(1+\beta)N}{2}(\frac{1}{q}-\frac{1}{p})}\|\phi_0\|_{L^q}. \tag{1.9}$$

If the solution $u$ to (1.1) behaves as $\phi$ with $q = 1$, then

$$\int_0^\infty \int_{\mathbf{R}^N} b(t)^{-1}|f(u)|(t,x)dxdt \leq C\int_0^\infty (t+1)^{\beta-\frac{(1+\beta)N}{2}(1-\frac{1}{\rho})\rho}dt < \infty$$

provided that $\beta - \frac{(1+\beta)N}{2}(1-\frac{1}{\rho})\rho < -1$ or $\rho > 1 + \frac{2}{N}$. Hence we expect the critical exponent to be the Fujita exponent

$$\rho_F(N) := 1 + \frac{2}{N}, \tag{1.10}$$

named after his pioneering work [1]. In fact, when $f(u) = -|u|^{\rho-1}u$, that is, the semilinear term works as absorbing, the critical exponent is believed to be $\rho_F(N)$ even in the time-dependent damping case (Nishihara and Zhai [18]). When $b(t) \equiv const. > 0$, there are many literatures [2]–[8], [10]–[12], [14]–[16], [19, 21, 27] etc. See also the references therein. In the case of space-dependent damping, see [9, 17].

Our main aim in this paper is to show the global existence theorem for small initial data in the case of supercritical exponent and some blow-up results.

THEOREM 1.1 (Small data global existence). *Assume that the continuous function $f$ satisfies $|f(u)| = O(|u|^\rho)$ in the neighborhood of $u = 0$, and that*

$$I_0^2 := \int_{\mathbf{R}^N} e^{\frac{(1+\beta)|x|^2}{2(2+\delta)}} (|u_1|^2 + |\nabla u_0|^2 + |u_0|^{\rho+1}) dx \tag{1.11}$$

*is sufficiently small with some small $\delta > 0$ for the data (1.5). Then, when $\rho_F(N) < \rho < \frac{N+2}{[N-2]_+}$, there exists a unique global solution $u \in X_\infty = C([0,\infty);\ H^1) \cap C^1([0,\infty);\ L^2)$, which satisfies*

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\le C(\delta) I_0 (t+1)^{-\frac{(1+\beta)N}{4}+\frac{\varepsilon}{2}} \\ \|(u_t, \nabla u)(t,\cdot)\|_{L^2} &\le C(\delta) I_0 (t+1)^{-\frac{(1+\beta)(N+2)}{4}+\frac{\varepsilon}{2}} \end{aligned} \tag{1.12}$$

*for some small $\varepsilon = \varepsilon(\delta) > 0$ and large $C(\delta) > 0$ with $\varepsilon(\delta) \to 0$ and $C(\delta) \to \infty$ as $\delta \to 0$.*

Note that the decay rates (1.12) are almost same as those of $\phi$ in (1.8). So we believe that the asymptotic profile of the solution $u$ is

$$\theta_0 G_B(t,x) := \theta_0 (4\pi B(t))^{-\frac{N}{2}} e^{-\frac{|x|^2}{4B(t)}}$$

for some constant $\theta_0$. Concerning the blow-up we have the following two theorems.

THEOREM 1.2 (Blow-up for any small data). *Assume that*

$$f(u) = |u|^\rho \quad \textit{with} \quad 1 + \frac{2\beta}{N} \le \rho \le 1 + \frac{1+\beta}{N} \tag{1.13}$$

*($\rho > 1$ if $\beta = 0$), and that*

$$\int_{\mathbf{R}^N} u_i(x) dx \ge 0 (i = 0, 1) \quad \textit{with} \quad \int_{\mathbf{R}^N} (u_0 + u_1)(x) dx > 0\,. \tag{1.14}$$

*Then the solution $u \in X_T$ to (1.1) does not exist globally.*

THEOREM 1.3 (Blow-up for some data). *Assume that*

$$f(u) = |u|^{\rho-1} u \quad \textit{with} \quad 1 + \frac{4\beta}{N} \le \rho \le 1 + \frac{2(1+\beta)}{N} \tag{1.15}$$

*($\rho > 1$ if $\beta = 0$), and that*

$$\begin{aligned} &\frac{1}{2}\big(\|u_1\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) - \frac{1}{\rho+1}\|u_0\|_{L^{\rho+1}}^{\rho+1} \le 0\,, \quad \textit{and} \\ &\int_{\mathbf{R}^N}(u_0u_1)(x)dx \ge 0 \quad \textit{with} \quad \int_{\mathbf{R}^N}[u_0(u_0+u_1)](x)dx > 0\,. \end{aligned} \tag{1.16}$$

*Then the solution $u \in X_T$ to (1.1) does not exist globally.*

Theorems 1.2–1.3 are both based on the following blow-up lemma for the ordinary differential inequality.

LEMMA 1.1. *Let $\alpha > 0$, $0 \le \beta < 1$ and $\Psi(t)$ satisfy*

$$\begin{aligned} &(t+1)^\beta \Psi''(t) + \Psi'(t) \ge c_0(t+1)^{\beta-\gamma}|\Psi|^\alpha \Psi(t)\,, \\ &\Psi(0) \ge 0,\ \Psi'(0) \ge 0 \quad \textit{with} \quad \Psi(0) + \Psi'(0) > 0 \end{aligned} \tag{1.17}$$

*for some constant $c_0 > 0$. Then $\Psi(t)$ blows up within a finite time provided that*

$$2\beta \le \gamma \le \beta + 1\,. \tag{1.18}$$

*More precisely, if $\Psi(0) = \varepsilon > 0 (0 < \varepsilon \ll 1)$, then the life-span $T_\varepsilon$ of $\Psi(t)$ is estimated from above as*

$$T_\varepsilon \le \begin{cases} C\varepsilon^{-\frac{\alpha}{1+\beta-\gamma}} & \gamma < \beta + 1 \\ e^{C\varepsilon^{-\alpha}} & \gamma = \beta + 1\,, \end{cases} \tag{1.19}$$

*where $T_\varepsilon := \sup\{T;\ \Psi(t) < \infty,\ 0 < t < T\}$.*

REMARK 1.1. When $\beta = 0$, Lemma 1.1 was given by Li and Zhou [12]. See also Todorova and Yordanov [21], and Zhou [28]. See also Qi. Zhang [27] for the different method on the blow-up. Theorems 1.2 and 1.3 are, respectively, based on [21] and [28].

Let the data be $(\varepsilon u_0, \varepsilon u_1)$, $\varepsilon > 0$, instead of $(u_0, u_1)$ in (1.5). Then Theorem 1.3 does not hold for small $\varepsilon > 0$, because the assumption $(1.16)_1$ ( 1-st property of (1.16) ) breaks since

$$\frac{\varepsilon^2}{2}\big(\|u_1\|_{L^2}^2 + \|\nabla u_0\|_{L^2}\big) - \frac{\varepsilon^{\rho+1}}{\rho+1}\|u_0\|_{L^{\rho+1}}^{\rho+1} > 0 \quad \text{as} \quad \varepsilon \to 0+\ .$$

Note that some exponents $\rho$ in Theorem 1.3 are in the supercritical exponent interval $(\rho_F(N), \frac{N+2}{[N-2]_+})$ when $\beta > 0$. Therefore, the smallness condition on the data in Theorem 1.1 is essential.

On the other hand, Theorem 1.2 hold for any small $\varepsilon > 0$. We expect this kind of blow-up result for any $\rho \in (1, \rho_F(N)]$. But we could not show this. Our interval of exponents in (1.13) is not satisfactory. Also, we can obtain the estimate on the life-span $T_\varepsilon$ of $u$ as $\varepsilon \to 0$ in (3.11) later, but we are not sure whether it is optimal. In Theorem 1.3 the estimate on $T_\varepsilon$ has no meaning because $\varepsilon$ is not necessarily small.

Throughout this paper, by $C_i(a, b, \ldots)$ or $c_i(a, b, \ldots)$ denote several positive constants depending on $a, b, \ldots$. Without confusions, denote them only by $C$ or $c$ which are changed from line to line. By $\|\cdot\|_X$ we denote the norm in the Banach space $X$. By $\|\cdot\|$ we simply denote the $L^2$-norm in the Lebesgue space $L^2 = L^2(\mathbf{R}^N)$.

In Section 2 we prove Theorem 1.1. In Section 3 we first show Lemma 1.1. Based on it, we prove both Theorem 1.2 and Theorem 1.3.

## 2. Global existence for small data in the supercritical exponent

Let $u \in X_T$ be a weak solution to

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = f(u)\,, & (t, x) \in \mathbf{R}^+ \times \mathbf{R}^N \\ (u, u_t)(0, x) = (u_0, u_1)(x)\,, & x \in \mathbf{R}^N\,, \end{cases} \tag{2.1}$$

whose support is in $B_{t+L} = \{x;\ |x| \le t + L\}$. Here, the semilinear term satisfies (1.3) and $b_0 = 1$ without loss of generality, that is,

$$b(t) = (t+1)^{-\beta}\,, \quad 0 \le \beta < 1\,. \tag{2.2}$$

For the proof of Theorem 1.1 it is enough for us to give a priori estimates. We apply the weighted energy method and introduce the weight

$$e^{2\psi}\,, \quad \psi(t, x) = \frac{a|x|^2}{(t+1)^{1+\beta}}\,, \tag{2.3}$$

originally in Todorova and Yordanov [21]. See also Nishihara and Zhai [18]. Then

$$\begin{aligned} \psi_t &= -(1+\beta)\frac{a|x|^2}{(t+1)^{2+\beta}} = -\frac{1+\beta}{t+1}\psi\,, \\ \nabla\psi &= \frac{2ax}{(t+1)^{1+\beta}}\,, \quad \Delta\psi = \frac{2aN}{(t+1)^{1+\beta}}\,. \end{aligned} \tag{2.4}$$

We choose the parameter $a$ as

$$a = \frac{1+\beta}{4(2+\delta)} \ \text{ for some small constant } \ \delta > 0\,, \tag{2.5}$$

then $-b(t)\psi_t = \frac{1+\beta}{4a}|\nabla\psi|^2 = (2+\delta)|\nabla\psi|^2$ and so

$$\frac{|\nabla\psi|^2}{-\psi_t} = \frac{b(t)}{2+\delta} \tag{2.6}$$

and

$$\Delta\psi = \frac{(1+\beta)N}{2(2+\delta)}\frac{b(t)}{t+1} =: \left(\frac{(1+\beta)N}{4} - \delta_1\right)\frac{b(t)}{t+1}\,. \tag{2.7}$$

Here and after, by $\delta_i$ denote the functions of $\delta$ satisfying

$$\delta_i = \delta_i(\delta) > 0\,(i = 1, 2, \ldots)\,, \quad \delta_i \to 0 \quad \text{as} \quad \delta \to 0\,. \tag{2.8}$$

Under these preparations, we show the following proposition, which yields Theorem 1.1.

PROPOSITION 2.1. *For* $0 \le t < T$ *and* $\varepsilon = 3\delta_1$, *define*

$$\begin{aligned} M(t) = \sup_{0<\tau<t} \bigg\{ &(\tau+1)^{\frac{(1+\beta)(N+2)}{2}-\varepsilon} \int_{\mathbf{R}^N} e^{2\psi}(u_t^2 + |\nabla u|^2)(\tau, x)dx \\ &+(\tau+1)^{\frac{(1+\beta)N}{2}-\varepsilon} \int_{\mathbf{R}^N} e^{2\psi} u^2(\tau, x)dx \bigg\}\,. \end{aligned} \tag{2.9}$$

*If* $M(t) \le \mu(\ll 1)$ *for some positive constant*, *then it holds*

$$\begin{aligned} M(t) + \int_0^t \bigg[ &(\tau+1)^{\frac{(1+\beta)N}{2}+1-\varepsilon} \int_{\mathbf{R}^N} e^{2\psi} u_t^2(\tau, x)dx \\ &+(\tau+1)^{\frac{(1+\beta)N}{2}+\beta-\varepsilon} \int_{\mathbf{R}^N} e^{2\psi} |\nabla u|^2(\tau, x)dx \\ &+(\tau+1)^{\frac{(1+\beta)N}{2}-1-\varepsilon} \int_{\mathbf{R}^N} e^{2\psi} u^2(\tau, x)dx \bigg] d\tau \\ \le C I_0^2\,. \end{aligned} \tag{2.10}$$

PROOF OF PROPOSITION 2.1. Multiplying (2.1) by $e^{2\psi}u_t$ and $e^{2\psi}u$, we have

$$\begin{aligned} &\frac{\partial}{\partial t}\bigg[\frac{e^{2\psi}}{2}(u_t^2 + |\nabla u|^2)\bigg] - \nabla\cdot(e^{2\psi}u_t\nabla u) \\ &\quad + e^{2\psi}\bigg(b(t) - \frac{|\nabla\psi|^2}{-\psi_t} - \psi_t\bigg)u_t^2 + \underbrace{\frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla u - u_t\nabla\psi|^2}_{(\#1)} \\ &= \frac{\partial}{\partial t}[e^{2\psi}F(u)] - 2e^{2\psi}\psi_t F(u) \ \ (F'(u) = f(u))\,, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\frac{\partial}{\partial t}\bigg[e^{2\psi}\bigg(uu_t + \frac{b(t)}{2}u^2\bigg)\bigg] - \nabla\cdot(e^{2\psi}u\nabla u) \\ &\quad + e^{2\psi}\bigg\{|\nabla u|^2 + \bigg(-\psi_t + \frac{\beta}{2(1+t)}\bigg)b(t)u^2 + \underbrace{2u\nabla\psi\cdot\nabla u}_{(\#2)} - 2\psi_t uu_t - u_t^2\bigg\} \\ &= e^{2\psi}uf(u)\,. \end{aligned} \tag{2.12}$$

When $b(t) \equiv 1$, the desired estimates were obtained by (2.11) and (2.12) ([21]), but in our problem we change (#1) and (#2) as follows:

$$
\begin{aligned}
(\#1) &= \frac{e^{2\psi}}{-\psi_t}\big(\psi_t^2|\nabla u|^2 - 2\psi_t u_t \nabla u \cdot \nabla\psi + u_t^2|\nabla\psi|^2\big) \\
&\geq \frac{e^{2\psi}}{-\psi_t}\left(\frac{1}{5}\psi_t^2|\nabla u|^2 - \frac{1}{4}u_t^2|\nabla\psi|^2\right) \\
&= e^{2\psi}\left\{\frac{1}{5}(-\psi_t)|\nabla u|^2 - \frac{b(t)}{4(2+\delta)}u_t^2\right\} \quad \text{by (2.6)}
\end{aligned}
\tag{2.13}
$$

and

$$
\begin{aligned}
(\#2) &= 4e^{2\psi}u\nabla u \cdot \nabla\psi - e^{2\psi}\nabla(u^2)\cdot\nabla\psi \\
&= 4e^{2\psi}u\nabla u \cdot \nabla\psi - \nabla\cdot(e^{2\psi}u^2\nabla\psi) + 2e^{2\psi}u^2|\nabla\psi|^2 + e^{2\psi}(\Delta\psi)u^2\,.
\end{aligned}
\tag{2.14}
$$

Hence, (2.11) and (2.12), respectively, change to

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2 + |\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}u_t\nabla u) \\
&\qquad + e^{2\psi}\left\{\left(\left(\frac{3}{4} - \frac{1}{2+\delta}\right)b(t) - \psi_t\right)u_t^2 + \frac{-\psi_t}{5}|\nabla u|^2\right\} \\
&\leq \frac{\partial}{\partial t}[e^{2\psi}F(u)] - 2e^{2\psi}\psi_t F(u)
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{b(t)}{2}u^2\right)\right] - \nabla\cdot\big\{e^{2\psi}(u\nabla u + u^2\nabla\psi)\big\} \\
&\qquad + e^{2\psi}\Bigg\{\underbrace{|\nabla u|^2 + 4u\nabla u\cdot\nabla\psi + (-\psi_t b(t) + 2|\nabla\psi|^2)u^2}_{(\#3)} \\
&\qquad\qquad + \left(\beta + \frac{(1+\beta)N}{2} - 2\delta_1\right)\frac{b(t)}{2(t+1)}u^2 - 2\psi_t uu_t - u_t^2\Bigg\} \\
&= e^{2\psi}uf(u)\,.
\end{aligned}
\tag{2.16}
$$

By (2.6),

$$
\begin{aligned}
(\#3) &= |\nabla u|^2 + 4u\nabla u\cdot\nabla\psi + (4+\delta)|\nabla\psi|^2u^2 \\
&= \left(1 - \frac{4}{4+\delta/2}\right)|\nabla u|^2 + \frac{\delta}{2}|\nabla\psi|^2u^2 + \Big|\frac{2}{\sqrt{4+\delta/2}}\nabla u + \sqrt{4+\delta/2}\,u\nabla\psi\Big|^2 \\
&\geq \delta_2(|\nabla u|^2 + b(t)(-\psi_t)u^2)\,.
\end{aligned}
\tag{2.17}
$$

To change the forms of (#2) and (#3) is a key point in the supercritical case, especially, to derive the last term $e^{2\psi}(\Delta\psi)u^2$ in (2.14). Hence, integrating (2.16) with (2.17) over $\mathbf{R}^N$, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^N} e^{2\psi}\left(uu_t+\frac{b(t)}{2}u^2\right)dx \\
&\quad+\int_{\mathbf{R}^N} e^{2\psi}\left\{\delta_2(|\nabla u|^2+b(t)(-\psi_t)u^2)+\left(\beta+\frac{(1+\beta)N}{2}-2\delta_1\right)\frac{b(t)}{2(t+1)}u^2\right. \\
&\qquad\qquad \left.-2\psi_t uu_t-u_t^2\right\}dx \\
&\le \int_{\mathbf{R}^N} e^{2\psi}uf(u)dx\,.
\end{aligned}
\tag{2.18}
$$

To cover the bad term $-u_t^2$ in the second term of (2.18), we integrate (2.15) over $\mathbf{R}^N$ and multiply it by $(t+t_0)^\beta$ $(t_0 \gg 1)$ to get

$$
\begin{aligned}
&\frac{d}{dt}\left[(t+t_0)^\beta\int_{\mathbf{R}^N}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dx\right]-\frac{\beta}{(t+t_0)^{1-\beta}}\int_{\mathbf{R}^N}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dx \\
&\quad+\int_{\mathbf{R}^N} e^{2\psi}\left\{\left(\frac{1}{4}+(-\psi_t)(t+t_0)^\beta\right)u_t^2+\frac{(-\psi_t)(t+t_0)^\beta}{5}|\nabla u|^2\right\}dx \\
&\le (t+t_0)^\beta\left[\frac{d}{dt}\int_{\mathbf{R}^N} e^{2\psi}F(u)dx-2\int_{\mathbf{R}^N}\psi_t F(u)dx\right]=:(NL)_1\,.
\end{aligned}
\tag{2.19}
$$

We now add (2.19) to $\nu\cdot$(2.18) $(0<\nu\ll 1)$:

$$
\begin{aligned}
&\frac{d}{dt}\hat{E}(t)+\hat{H}(t) \\
&\quad:=\frac{d}{dt}\int_{\mathbf{R}^N} e^{2\psi}\left\{\frac{(t+t_0)^\beta}{2}(u_t^2+|\nabla u|^2)+\underbrace{\nu uu_t}_{(\#4)}+\frac{\nu b(t)}{2}u^2\right\}dx \\
&\quad+\int_{\mathbf{R}^N} e^{2\psi}\left\{\left(\frac{1}{4}-\nu+(-\psi_t)(t+t_0)^\beta-\frac{\beta}{2(t+t_0)^{1-\beta}}\right)u_t^2\right. \\
&\qquad\qquad+\left(\frac{(-\psi_t)(t+t_0)^\beta}{5}+\nu\delta_2-\frac{\beta}{2(t+t_0)^{1-\beta}}\right)|\nabla u|^2 \\
&\qquad\qquad+\nu\delta_2 b(t)(-\psi_t)u^2+\left(\beta+\frac{(1+\beta)N}{2}-2\delta_1\right) \\
&\qquad\qquad\left.\times\frac{\nu b(t)}{2(t+1)}u^2\underbrace{-2\nu\psi_t uu_t}_{(\#5)}\right\}dx \\
&\quad\le (NL)_2\,,
\end{aligned}
\tag{2.20}
$$

where by the form $(NL)_1$ in (2.19)

$$(NL)_2 = \frac{d}{dt}\left[(t+t_0)^\beta \int_{\mathbf{R}^N} e^{2\psi} F(u)dx\right] + C\int_{\mathbf{R}^N} e^{2\psi}(1+(t+t_0)^\beta(-\psi_t))|u|^{\rho+1}dx\,. \quad (2.21)$$

The terms (#4) and (#5) are absorbed in the other good terms by choosing small parameters $\nu, \delta_2 (\nu \ll \delta_2)$ and large parameter $t_0$:

$$|(\#4)| \le \frac{\delta_2 \nu b(t)}{2} u^2 + \frac{\nu}{\delta_2}\frac{(t+t_0)^\beta}{2} u_t^2\,, \quad (2.22)$$

$$|(\#5)| \le \frac{1}{2}(-\psi_t)(t+t_0)^\beta u_t^2 + 2\nu^2 b(t)(-\psi_t)u^2\,. \quad (2.23)$$

In the results, denoting

$$E(t) = \int_{\mathbf{R}^N} e^{2\psi}(u_t^2 + |\nabla u|^2)(t,x)dx\,, \quad (2.24)$$

$$E_\psi(t) = \int_{\mathbf{R}^N} e^{2\psi}(-\psi_t)(u_t^2 + |\nabla u|^2)(t,x)dx \quad (2.25)$$

and, for $g(t,x) \ge 0$

$$J(t;g) = \int_{\mathbf{R}^N} e^{2\psi} g(t,x)dx\,, \quad (2.26)$$

$$J_\psi(t;g) = J(t;(-\psi_t)g) = \int_{\mathbf{R}^N} e^{2\psi}(-\psi_t)g(t,x)dx\,, \quad (2.27)$$

we have

$$\begin{aligned} &c_0(t+t_0)^\beta E(t) + (1-\delta_2)\frac{\nu b(t)}{2} J(t;u^2) \\ &\quad \le \hat{E}(t) \le C_0(t+t_0)^\beta E(t) + (1+\delta_2)\frac{\nu b(t)}{2} J(t;u^2) \end{aligned} \quad (2.28)$$

and

$$\begin{aligned} \hat{H}(t) \ge\ & c_1(\delta)(E(t) + (t+t_0)^\beta E_\psi(t) + b(t)J_\psi(t;u^2)) \\ & + \frac{\beta + \frac{(1+\beta)N}{2} - 2\delta_1}{t+1} \cdot \frac{\nu b(t)}{2} J(t;u^2)\,. \end{aligned} \quad (2.29)$$

The coefficient $(\beta + \frac{(1+\beta)N}{2} - 2\delta_1)/(t+1)$ of $\frac{\nu b(t)}{2} J(t;u^2)$ in (2.29) is important, for which the parameter $a$ was chosen in (2.5). For a moment we put

$$B = \beta + \frac{(1+\beta)N}{2} \quad (2.30)$$

and, by (2.28)–(2.29), multiply (2.20) by $(t+t_0)^{B-3\delta_1}$ to get

$$
\begin{aligned}
&\frac{d}{dt}\big[(t+t_0)^{B-3\delta_1}\hat{E}(t)\big] + (t+t_0)^{B-3\delta_1}\left(c_1(\delta) - \frac{C_0(B-3\delta_1)}{(t+t_0)^{1-\beta}}\right)E(t) \\
&\quad + (t+t_0)^{B+\beta-3\delta_1}\cdot c_1(\delta)E_\psi(t) + (t+t_0)^{B-\beta-3\delta_1}\cdot c_1(\delta)J_\psi(t;u^2) \\
&\quad + (t+t_0)^{B-3\delta_1}\left(\frac{B-2\delta_1}{t+1} - \frac{(B-3\delta_1)(1+\delta_2)}{t+t_0}\right)\frac{\nu b(t)}{2}J(t;u^2) \\
&\le (t+t_0)^{B-3\delta_1}(NL)_2\,,
\end{aligned}
\tag{2.31}
$$

and hence, by integrating (2.31) over $[0,t]$ and denoting $3\delta_1 = \varepsilon$,

$$
\begin{aligned}
&(t+1)^{B+\beta-\varepsilon}E(t) + (t+1)^{B-\beta-\varepsilon}J(t;u^2) \\
&\quad + \int_0^t \big[(\tau+1)^{B-\varepsilon}E(\tau) + (\tau+1)^{B+\beta-\varepsilon}E_\psi(\tau) \\
&\qquad + (\tau+1)^{B-1-\beta-\varepsilon}J(\tau;u^2) + (\tau+1)^{B-\beta-\varepsilon}J_\psi(\tau;u^2)\big]d\tau \\
&\le CI_0^2 + C\int_0^t (\tau+1)^{B-\varepsilon}(NL)_2 d\tau\,.
\end{aligned}
\tag{2.32}
$$

Considering $\int_0^t(\tau+1)^{B-\varepsilon}E(\tau)d\tau$ to be estimated, we multiply (2.19) by $(t+t_0)^{B-\beta+1-\varepsilon}$ and integrate it over $[0,t]$ to get

$$
\begin{aligned}
&(t+1)^{B+1-\varepsilon}E(t) - C(B-\beta+1-\varepsilon)\int_0^t (\tau+1)^{B-\varepsilon}E(\tau)d\tau \\
&\quad + c_2(\delta)\int_0^t \big[(\tau+1)^{B-\beta+1-\varepsilon}J(\tau;u_t^2) + (\tau+1)^{B+1-\varepsilon}E_\psi(\tau)\big]d\tau \\
&\le CI_0^2 + C\int_0^t (\tau+t_0)^{B-\beta+1-\varepsilon}(NL)_1 d\tau\,.
\end{aligned}
\tag{2.33}
$$

Adding (2.32) to $\mu\cdot$(2.33) $(0<\mu\ll 1)$, we have

$$
\begin{aligned}
&(t+1)^{\frac{(1+\beta)(N+2)}{2}-\varepsilon}E(t) + (t+1)^{\frac{(1+\beta)N}{2}-\varepsilon}J(t;u^2) \\
&\quad + \int_0^t \big[(\tau+1)^{\frac{(1+\beta)N}{2}+1-\varepsilon}J(\tau;u_t^2) + (\tau+1)^{\frac{(1+\beta)N}{2}+\beta-\varepsilon}J(\tau;|\nabla u|^2) \\
&\qquad + (\tau+1)^{\frac{(1+\beta)N}{2}-1-\varepsilon}J(\tau;u^2)\big]d\tau \\
&\le CI_0^2 + (NL)
\end{aligned}
\tag{2.34}
$$

(we dropped the terms about $E_\psi$, $J_\psi$ in the left-hand side) with

$$
\begin{aligned}
\text{(NL)} \qquad & \leq C(t+1)^{\frac{(1+\beta)(N+2)}{2}-\varepsilon} J(t;|u|^{\rho+1}) \\
& + C\int_0^t \Big[(\tau+1)^{\frac{(1+\beta)N}{2}+\beta-\varepsilon} J(\tau;|u|^{\rho+1}) \\
& \qquad + (\tau+1)^{\frac{(1+\beta)(N+2)}{2}-\beta-\varepsilon} J_\psi(\tau;|u|^{\rho+1})\Big] d\tau\,.
\end{aligned} \tag{2.35}
$$

We now go to estimate $(NL)$ by $M(t)$, and our goal is (2.40) below from (2.34).

LEMMA 2.1 (Gagliardo-Nirenberg). *Let $p, q, r$ $(1 \leq p, q, r, \leq \infty)$ and $\sigma \in [0,1]$ satisfy*

$$\frac{1}{p} = \sigma\left(\frac{1}{r} - \frac{1}{N}\right) + (1-\sigma)\frac{1}{q}$$

*except for $p=\infty$ or $r=N$ when $N \geq 2$. Then for some constant $C = C(p,q,r,N) > 0$ it holds*

$$\|g\|_{L^p} \leq C\|g\|_{L^q}^{1-\sigma}\|\nabla g\|_{L^r}^{\sigma}$$

*for any $g \in C_0^1(\mathbf{R}^N)$).*

Since

$$J(t;|u|^{\rho+1}) = \int_{\mathbf{R}^N} e^{2\psi}|u|^{\rho+1}dx = \int_{\mathbf{R}^N} |e^{\frac{2}{\rho+1}\psi}u|^{\rho+1}dx$$

and

$$\nabla(e^{\frac{2}{\rho+1}\psi}u) = \frac{2}{\rho+1}e^{\frac{2}{\rho+1}\psi}(\nabla\psi)u + e^{\frac{2}{\rho+1}\psi}\nabla u,$$

choosing $p=\rho+1$, $q=r=2$ and $\sigma = \frac{N(\rho-1)}{2(\rho+1)}$ $(<1)$, we have

$$
\begin{aligned}
& J(t;|u|^{\rho+1})^{\frac{1}{\rho+1}} \\
& \quad \leq C\left(\int_{\mathbf{R}^N} e^{\frac{4}{\rho+1}\psi}u^2dx\right)^{\frac{1-\sigma}{2}}\left(\int_{\mathbf{R}^N} e^{\frac{4}{\rho+1}\psi}|\nabla\psi|^2u^2dx + \int_{\mathbf{R}^N} e^{\frac{4}{\rho+1}\psi}|\nabla u|^2dx\right)^{\frac{\sigma}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
& J(t;|u|^{\rho+1}) \\
& \quad \leq C(t+1)^{-\frac{(1+\beta)(\rho+1)\sigma}{2}} J(t;u^2)^{\frac{\rho+1}{2}} + CJ(t;u^2)^{\frac{(\rho+1)(1-\sigma)}{2}} J(t;|\nabla u|^2)^{\frac{(\rho+1)\sigma}{2}}.
\end{aligned} \tag{2.36}
$$

Hence, by simple calculations of the exponent of $t+1$,

$$
\begin{aligned}
(t+1)^{\frac{(1+\beta)(N+2)}{2}-\varepsilon} J(t;|u|^{\rho+1}) &\le C(t+1)^{-\frac{(1+\beta)N}{2}(\rho-1-\frac{2}{N})+\frac{\rho-1}{2}\varepsilon} M(t)^{\frac{\rho+1}{2}} \\
&\le CM(t)^{\frac{\rho+1}{2}}
\end{aligned}
\tag{2.37}
$$

since $\rho > \rho_F(N)$ and $\varepsilon = 3\delta_1$ is small. Similarly,

$$
\begin{aligned}
&\int_0^t (\tau+1)^{\frac{(1+\beta)N}{2}+\beta-\varepsilon} J(\tau;|u|^{\rho+1})d\tau \\
&\qquad \le C\int_0^t (\tau+1)^{-1-\frac{(1+\beta)N}{2}(\rho-1-\frac{2}{N})+\frac{\rho-1}{2}\varepsilon} M(\tau)d\tau\,.
\end{aligned}
\tag{2.38}
$$

Since $-\psi_t = \frac{1+\beta}{t+1}\psi$, for small $\mu > 0$

$$
J_\psi(t;|u|^{\rho+1}) \le C(t+1)^{-1}\int_{\mathbf{R}^N} e^{2\psi}\psi|u|^{\rho+1}dx \le C(t+1)^{-1}\int_{\mathbf{R}^N} e^{(2+\mu)\psi}|u|^{\rho+1}dx.
$$

Hence, similar to (2.36) and (2.38),

$$
\begin{aligned}
&\int_0^t (\tau+1)^{\frac{(1+\beta)(N+2)}{2}-\beta-\varepsilon} J_\psi(\tau;|u|^{\rho+1})d\tau \\
&\qquad \le C\int_0^t (\tau+1)^{-1-\beta-\frac{(1+\beta)N}{2}(\rho-1-\frac{2}{N})+\frac{\rho-1}{2}\varepsilon} M(\tau)d\tau\,.
\end{aligned}
\tag{2.39}
$$

Combining (2.9), (2.34)–(2.35) with (2.36)–(2.39), we obtain

$$
\begin{aligned}
&(1-CM(t)^{\frac{\rho-1}{2}})M(t) \\
&\quad + \int_0^t \Big[(\tau+1)^{\frac{(1+\beta)N}{2}+1-\varepsilon} J(t;u_t^2) + (\tau+1)^{\frac{(1+\beta)N}{2}-\varepsilon} J(\tau;|\nabla u|^2) \\
&\qquad\qquad + (\tau+1)^{\frac{(1+\beta)N}{2}-1-\varepsilon} J(\tau;u^2)\Big]d\tau \\
&\le CI_0^2 + C\int_0^t (\tau+1)^{-1-\frac{1+\beta}{2}(\rho-1-\frac{2}{N})+\frac{\rho-1}{2}\varepsilon} M(\tau)d\tau\,.
\end{aligned}
\tag{2.40}
$$

Therefore, when $M(t) \le (1/2C)^{\frac{2}{\rho-1}}$, using the Gronwall inequality we have the desired estimate (2.10).

## 3. Blow-up properties

We first prove Lemma 1.1, for which we need the following comparison lemma.

LEMMA 3.1 (Comparison lemma). *Suppose that the functions $k(t)$ and $h(t)$ satisfy*

$$
\begin{aligned}
a_1(t)k''(t) + k'(t) &\ge a_2(t)|k|^\alpha k(t) \\
a_1(t)h''(t) + h'(t) &\le a_2(t)|h|^\alpha h(t)
\end{aligned}
\tag{3.1}
$$

*for any* $t \geq 0$, *where* $\alpha \geq 0$ *and* $a_i(t) > 0 (t \geq 0),\ i = 1, 2$. *If*

$$k(0) > h(0)\,, \quad k'(0) \geq h'(0) \tag{3.2}$$

*or*

$$k(0) \geq h(0)\,, \quad k'(0) > h'(0)\,, \tag*{(3.2)$'$}$$

*then it holds*

$$k'(t) > h'(t) \quad \textit{and} \quad k(t) > h(t) \quad \textit{for any} \quad t > 0\,. \tag{3.3}$$

The proof is given in [12] and omitted.

REMARK 3.1. If $h(t) \equiv 0$, then $k(t) \geq 0$ provided that $k(0) \geq 0,\ k'(0) \geq 0$ with $k(0) + k'(0) > 0$. Hence the absolute value mark can be deleted in $(3.1)_1$.

PROOF OF LEMMA 1.1. Let $\Phi(t) \geq 0$ be the solution to

$$\Phi'(t) = \delta_0 c_0 (t+1)^{\beta-\gamma} \Phi(t)^{1+\frac{\alpha}{2}}, \quad \Phi(0) = \Psi(0) \geq 0 \tag{3.4}$$

with

$$\delta_0 = c_1 \Phi(0)^{\frac{\alpha}{2}} \quad (0 < c_1 \ll 1)\,. \tag{3.5}$$

Then, since $-\frac{2}{\alpha}\frac{d}{dt}\Phi(t)^{-\frac{\alpha}{2}} = \delta_0 c_0 (t+1)^{\beta-\gamma}$,

$$\Phi(t) = \begin{cases} \left[\Phi(0)^{-\frac{\alpha}{2}} - \dfrac{\alpha\delta_0 c_0}{2(1+\beta-\gamma)}\left((t+1)^{1+\beta-\gamma} - 1\right)\right]^{-\frac{2}{\alpha}} & (\beta - \gamma > -1) \\ \left[\Phi(0)^{-\frac{\alpha}{2}} - \dfrac{\alpha\delta_0 c_0}{2}\log(t+1)\right]^{-\frac{2}{\alpha}} & (\beta - \gamma = -1) \end{cases}$$

and hence $\Phi(t) \nearrow \infty$ as $t \to T_0 - 0$, where

$$T_0 = \begin{cases} \left(\dfrac{2(1+\beta-\gamma)}{\alpha\delta_0 c_0}\Phi(0)^{-\frac{\alpha}{2}} + 1\right)^{\frac{1}{1+\beta-\gamma}} - 1 & (\beta - \gamma > -1) \\ e^{\frac{2}{\alpha\delta_0 c_0}\Phi(0)^{-\frac{\alpha}{2}}} - 1 & (\beta - \gamma = -1)\,. \end{cases}$$

If $\Phi(0) = \varepsilon$, then by (3.5), $T_0$ is estimated as

$$T_0 \leq \begin{cases} C\varepsilon^{-\frac{\alpha}{1+\beta-\gamma}} & (\beta - \gamma > -1) \\ e^{C\varepsilon^{-\alpha}} & (\beta - \gamma = -1)\,. \end{cases} \tag{3.6}$$

We now show $\Psi(t) \geq \Phi(t)$ using Lemma 3.1. Since $\beta - \gamma \leq -\beta \leq 0$ by (1.18),

$$\Phi''(t) = \frac{(\delta_0 c_0)^2 (2+\alpha)}{2}(t+1)^{2(\beta-\gamma)}\Phi(t)^{1+\alpha} + \delta c_0 (\beta - \gamma)(t+1)^{\beta-\gamma-1}\Phi(t)^{1+\frac{\alpha}{2}}$$

$$\leq \frac{(\delta_0 c_0)^2(2+\alpha)}{2}(t+1)^{2(\beta-\gamma)}\Phi(t)^{1+\alpha}$$

and

$$(t+1)^\beta \Phi''(t) + \Phi'(t) \leq c_0(t+1)^{\beta-\gamma}\Phi(t)^{1+\alpha} \cdot \delta_0\left(\frac{(2+\alpha)\delta_0 c_0}{2}(t+1)^{2\beta-\gamma} + \Phi(0)^{-\frac{\alpha}{2}}\right).$$

Hence, the choice of $\delta$ in (3.5) means

$$\delta_0\left(\frac{(2+\alpha)\delta_0 c_0}{2}(t+1)^{2\beta-\gamma} + \Phi(0)^{-\frac{\alpha}{2}}\right) \leq c_1 \cdot \left(\frac{(2+\alpha)c_0 c_1}{2}\Phi(0)^\alpha + 1\right) \leq 1$$

when $\Phi(0) \leq C$, and, by (1.18)

$$(t+1)^\beta \Phi''(t) + \Phi'(t) \leq c_0(t+1)^{\beta-\gamma}\Phi(t)^{1+\alpha}. \tag{3.7}$$

By Remark 3.1, we can take $k(t) = \Psi(t)$, $h(t) = \Phi(t)$ in Lemma 3.1 and conclude $\Psi(t) \geq \Phi(t) \geq 0$ and the blow-up of $\Psi(t)$ within a finite time including the estimate (1.19) on the life-span. □

REMARK 3.2. If $L > 1$ and $\beta - \gamma < 0$, then $(t+L)^{\beta-\gamma} \geq L^{\beta-\gamma}(t+1)^{\beta-\gamma}$. Hence, even if $\Psi(t)$ satisfies

$$\begin{aligned} &(t+1)^\beta \Psi''(t) + \Psi'(t) \geq c_0(t+L)^{\beta-\gamma}\Psi(t)^{1+\alpha} \\ &\Psi(0) \geq 0, \quad \Psi'(0) \geq 0 \quad \text{with} \quad \Psi(0) + \Psi'(0) > 0, \end{aligned} \tag{3.8}$$

instead of (1.17), then Lemma 1.1 holds.

We now prove Theorems 1.2 and 1.3.

PROOF OF THEOREM 1.2. Let $u \in X_T$ be a local solution to (1.1) with $f(u) = |u|^\rho$ satisfying $\operatorname{supp}\{u(t,\cdot)\} \subset B_{t+L}$. We define

$$\Psi(t) = \int_{\mathbf{R}^N} u(t,x)dx = \int_{B_{t+L}} u(t,x)dx, \tag{3.9}$$

then, by $1/\rho + (\rho-1)/\rho = 1$

$$|\Psi(t)| \leq C\left(\int_{B_{t+L}} dx\right)^{\frac{\rho-1}{\rho}} \left(\int_{\mathbf{R}^N} |u|^\rho dx\right)^{\frac{1}{\rho}} \leq C(t+L)^{\frac{N(\rho-1)}{\rho}}\|u(t)\|_{L^\rho}.$$

Hence, by integrating (1.1) over $\mathbf{R}^N$,

$$\Psi''(t) + b(t)\Psi'(t) = \|u(t)\|_{L^\rho}^\rho \geq c(t+L)^{-N(\rho-1)}|\Psi(t)|^\rho$$

and

$$(t+1)^\beta \Psi''(t) + \Psi'(t) \geq c(t+L)^{\beta-N(\rho-1)}|\Psi(t)|^\rho. \tag{3.10}$$

By (1.14), $(1.17)_2$ follows and $\Psi(t) \geq 0$. Hence Lemma 1.1 and Remark 3.2 yield the blow-up of $\Psi(t)$, and the estimate of life-span $T_\varepsilon$

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-\frac{\rho-1}{1+\beta-N(\rho-1)}} & \rho < 1 + \dfrac{1+\beta}{N} \\ e^{C\varepsilon^{-(\rho-1)}} & \rho = 1 + \dfrac{1+\beta}{N}\,, \end{cases} \tag{3.11}$$

which completes the proof of Theorem 1.2. □

PROOF OF THEOREM 1.3. Let $u \in X_T$ be a local solution to (1.1) with $f(u) = |u|^{\rho-1}u$ whose support is in $B_{t+L}$. We define

$$\Psi(t) = \frac{1}{2}\int_{\mathbf{R}^N} u^2(t,x)dx = \frac{1}{2}\int_{B_{t+L}} u^2(t,x)dx\,. \tag{3.12}$$

Since $\Psi'(t) = \int_{\mathbf{R}^N}(uu_t)(t,x)dx$, $(1.17)_2$ follows from $(1.16)_2$.

Integrating $(1.1) \times u_t$ over $\mathbf{R}^N$, we have

$$\begin{aligned} \frac{d}{dt}E_0(t) := \frac{d}{dt}\left[\frac{1}{2}(\|u_t(t)\|^2 + \|\nabla u\|^2) - \frac{1}{\rho+1}\int_{\mathbf{R}^N}|u|^{\rho+1}dx\right] \\ = -(t+1)^{-\beta}\|u_t(t)\|^2 \leq 0 \end{aligned}$$

and hence $E_0(t) \leq E_0(0) \leq 0$ by $(1.16)_1$. Therefore,

$$\begin{aligned} \Psi''(t) &= \|u_t(t)\|^2 + \int_{\mathbf{R}^N}(uu_{tt})(t,x)dx \\ &= \|u_t(t)\|^2 + \int_{\mathbf{R}^N} u(\Delta u - (t+1)^{-\beta}u_t + |u|^{\rho-1}u)dx \\ &= \|u_t(t)\|^2 - \|\nabla u\|^2 - (t+1)^{-\beta}\int_{\mathbf{R}^N}(uu_t)(t,x)dx + \int_{\mathbf{R}^N}|u|^{\rho+1}(t,x)dx \\ &= -(t+1)^{-\beta}\Psi'(t) + (-E_0(t)) + 2\|u_t(t)\|^2 + \frac{\rho-1}{\rho+1}\int_{\mathbf{R}^N}|u|^{\rho+1}dx\,. \end{aligned} \tag{3.13}$$

Using $\frac{1}{(\rho+1)/2} + \frac{1}{(\rho+1)/(\rho-1)} = 1$, we have

$$\Psi(t) \leq C(t+L)^{\frac{N(\rho-1)}{\rho+1}}\left(\int_{\mathbf{R}^N}|u|^{\rho+1}(t,x)dx\right)^{\frac{2}{\rho+1}}$$

or

$$\int_{\mathbf{R}^N}|u|^{\rho+1}(t,x)dx \geq c(t+L)^{-\frac{N(\rho-1)}{2}}\Psi(t)^{1+\frac{\rho-1}{2}}\,. \tag{3.14}$$

From (3.13)–(3.14) it follows that

$$(t+1)^\beta\Psi''(t) + \Psi'(t) \geq c(t+L)^{\beta-\frac{N(\rho-1)}{2}}\Psi(t)^{1+\frac{\rho-1}{2}}\,. \tag{3.15}$$

Thus, when (1.15) or $2\beta \leq \frac{N(\rho-1)}{2} \leq \beta + 1$, the blow-up of $\Psi(t)$ occurs. □

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