TOKYO J. MATH.
VOL. 34, NO. 2, 2011

On the Existence of a Darling-Kac Set for the Renormalized Rauzy Map

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Abstract. It is well-known that the renormalized Rauzy map is conservative and ergodic. In this paper, we show that a Darling-Kac set exists for the renormalized Rauzy map. This implies the pointwise dual ergodicity of this map.

1. Introduction

Let d be a positive integer larger than 1. We denote by E the $d \times d$ identity matrix and by $I(i, j)$ the $d \times d$ matrix defined by

$$I(i, j)_{k,l} = \begin{cases} 1 & (k, l) = (i, j)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We also denote by π an irreducible permutation of $\{1, 2, \dots d\}$ where a permutation π is said to be irreducible if $\pi\{1, \dots, k\} = \{1, \dots, k\}$ implies $k = d$. We consider $\lambda = (\lambda_1, \dots, \lambda_d)$, $\lambda_i > 0$ for $1 \le i \le d$, $\sum_{i=1}^{d} \lambda_i = 1$ and put $\beta_0 = 0$ and $\beta_i = \beta_i(\lambda) = \sum_{j=1}^{i} \lambda_j$ for $1 \le i \le d$. We define $\lambda^\pi = (\lambda_1^\pi, \lambda_2^\pi, \dots, \lambda_d^\pi)$ by $\lambda_i^\pi = \lambda_{\pi^{-1}(i)}$ for $1 \le i \le d$. The interval exchange map T of $[0, 1)$ associated to (π, λ) is defined by $Tx = x - \beta_{i-1}(\lambda) + \beta_{\pi(i)-1}(\lambda^\pi)$ for $x \in [0, 1)$. Since π is irreducible, $T[0, \tau) = [0, \tau)$ implies $\tau = 1$. In the sequel, we assume T satisfies the infinite distinct orbit condition, i.e.

1. $\{T^n \beta_i\}_{n \in \mathbf{Z}} \cap \{T^n \beta_j\}_{n \in \mathbf{Z}} = \emptyset$ for any β_i and β_j, $1 \le i \ne j \le d-1$.
2. $\{T^n \beta_i\}_{n \in \mathbf{Z}}$ consists of infinitely many points for $1 \le i \le d-1$.

We consider a set of d-interval exchange maps and a map on this set. M. Keane [3] conjectured that almost every interval exchange map is uniquely ergodic. To discuss this problem, G. Rauzy [7] introduced a map on a set of interval exchange maps to itself which we call the Rauzy map or the Rauzy induction. Then W. Veech [10] showed that a.e. interval exchange map is uniquely ergodic if every "renormalized" Rauzy map is conservative with respect to Lebesgue measure. Indeed he showed that it is conservative ergodic (but its absolutely continuous invariant measure is not finite). A. Zorich [11] considered an induced map of a "renormalized" Rauzy map which is finite measure preserving and then A. Avila and A. Bufetov [2]

Received January 14, 2010; revised July 23, 2010
Partially supported by Grant-in Aid for Scientific research (No.21340027) by Japan Society for the Promotion of Science

showed that the correlation of Zorich's map is exponential decay. Moreover T. Morita [4], [5] discussed the central limit theorem and some other properties of the renormalized Rauzy-Veech-Zorich map. In this paper, we discuss the ergodic property of the renormalized Rauzy map as an infinite measure preserving transformation. We show that the renormalized Rauzy map is pointwise dual ergodic in the sense of J. Aaronson [1].

Let T be a measure preserving transformation on a measure space $(\mathbf{X}, \mathcal{B}, \mu)$ with $\mu(\mathbf{X}) = \infty$. T is said to be conservative if for any $A \in \mathcal{B}$, $\mu(A) > 0$, there exists a positive integer n such that $\mu(T^{-n}A \cap A) > 0$ and be ergodic if $T^{-1}A = A$, $A \in \mathcal{B}$ implies either $\mu(A) = 0$ or $\mu(A^c) = 0$, respectively. In the case $\mu(\mathbf{X}) = \infty$, we can not expect the strong law of large numbers such as the pointwise ergodic theorem even if T is conservative and ergodic. J. Aaronson introduced a notion of a strong law of large numbers in generalized sense ; a law of large numbers for T is a function $L : \prod^{\infty}\{0, 1\} \to [0, \infty]$ such that for any measurable subset A, for μ-a.e. $x \in \mathbf{X}$,

$$L(1_A(x),\ 1_A(Tx),\ 1_A(T^2x), \dots) = \mu(A)\,.$$

Then he gave some classes of transformations having this property. The notion of the pointwise dual ergodicity is one of sufficient conditions for T having a law of large numbers. A measure preserving transformation T on a measure space $(\mathbf{X}, \mathcal{B}, \mu)$ is said to be pointwise dual ergodic if there is a sequence of positive numbers (a_n) such that

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k f = \int_{\mathbf{X}} f d\mu \quad \mu\text{-a.e.}$$

for any integrable function f, where $\hat{T}$ is the pre-dual operator of T, i.e. $\int_{\mathbf{X}} \hat{T} f \cdot g\, d\mu = \int_{\mathbf{X}} f \cdot g \circ T d\mu$ for $f \in L^1(\mu)$, $g \in L^{\infty}(\mu)$, see [1].

A subset A of $\mathbf{X}$, $0 < \mu(A) < \infty$, is said to be a Darling-Kac set for a conservative ergodic transformation T if there is a sequence of positive numbers (a_n) such that

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A = \mu(A)$$

uniformly.

Suppose that $(X_i)_{i=1}^{\infty}$ is a sequence of random variables defined on a probability space $(\Omega, \mathcal{B}, \mathrm{Pr})$. (X_i) is said to be continued fraction mixing if the following holds; there exists $\psi(k)$, $k \geq 1$, such that for any positive integer k and n

$$|\mathrm{Pr}(A \cap B) - \mathrm{Pr}(A)\mathrm{Pr}(B)| < \mathrm{Pr}(A)\mathrm{Pr}(B)\psi(k)$$

for any measurable sets A generated by $X_1, \dots, X_n$ and B generated by X_{n+k+m}, $m \geq 0$ and $\psi(k) \to 0$ as $k \to \infty$ with $\psi(1) < \infty$.

Suppose that there exists a subset $D \subset \mathbf{X}$ of finite measure such that the first return map of T on D with a partition $\{B_i, i \geq 1\}$ of A induces a continued fraction mixing process, i.e.

1. $\{T^{-n}B_i, i \geq 1, n \geq 0\}$ generates the set of measurable subset of D
2. $\{X_n, n \geq 1\}$, $X_n(x) = i$ if $T_D^{n-1}(x) \in B_i$, is continued fraction mixing.

Then D is a Darling-Kac set and T is pointwise dual ergodic (see J. Aaronson [1], 3.7.5).

In this paper, we will show the following.

MAIN RESULT. *There exists a Darling-Kac set for the renormalized Rauzy map*, (see Theorem in Section 2).

To show this theorem, we show that there exists a "good" set, i.e. the first return map on this set satisfies conditions for a Kuzmin type theorem by F.Schweiger [8]. Because Schweiger's theorem implies that the map is continued fraction mixing (see H. Nakada and R. Natsui [6]), we see that the "good" set is a Darling-Kac set. We note the result by A. Avila and A. Bufetov's result [2] does not imply that Zorich's map is continued fraction mixing.

2. Definitions

First we define a set of irreducible permutations called Rauzy class. We define two maps a and b defined on the set of d-permutations by

$$a(\pi)(j) = \begin{cases} \pi(j) & \text{if} \quad j \leq \pi^{-1}(d)\,, \\ \pi(d) & \text{if} \quad j = \pi^{-1}(d) + 1\,, \\ \pi(d) + 1 & \text{if} \quad \pi(j) = d\,, \end{cases}$$

and

$$b(\pi)(j) = \begin{cases} \pi(j) & \text{if} \quad \pi(j) \leq \pi(d)\,, \\ \pi(j) + 1 & \text{if} \quad \pi(d) < \pi(j) < d\,, \\ \pi(d) + 1 & \text{if} \quad \pi(j) = d\,, \end{cases}$$

for an irreducible permutation π. The Rauzy class $\mathcal{R}(\pi_0)$ is the set of permutations which are mapped from π_0 by compositions of a and b. We define

$$||\lambda|| = \sum_{i=1}^{d} \lambda_i \qquad \text{for } \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix}$$

and

$$\Delta^{d-1} = \left\{ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix} : \lambda_i > 0, 1 \leq i \leq d, \sum_{i=1}^{d} \lambda_i = 1 \right\}.$$

Hereafter, we regard λ as a column vector. Put

$$\Delta_a^{d-1} = \{\lambda \in \Delta^{d-1},\ \lambda_d < \lambda_{\pi^{-1}(d)}\}\,,$$

$$\Delta_b^{d-1} = \{\lambda \in \Delta^{d-1},\ \lambda_d > \lambda_{\pi^{-1}(d)}\}\,,$$

and denote by ι the projection from the positive cone of $\mathbf{R}^d$ to the open regular simplex Δ^{d-1} i.e.

$$\iota(\lambda) = \frac{\lambda}{||\lambda||}$$

for a positive vector λ. We also define $n \times n$ matrices $A(\pi, a)$ and $A(\pi, b)$ by

$$\begin{cases} A(\pi, a) = (E + I(\pi^{-1}(d), d))P(\tau_{\pi^{-1}(d)})\,, \\ A(\pi, b) = E + I(d, \pi^{-1}(d))\,, \end{cases}$$

with permutations τ_k defined by

$$\tau_k(j) = \begin{cases} j & \text{if} \quad j \le k\,, \\ j+1 & \text{if} \quad k < j < d\,, \\ k+1 & \text{if} \quad j = d\,, \end{cases}$$

and the $d \times d$ matrix P defined by

$$P_{k,l}(\pi) = \begin{cases} 1 & \text{if} \quad l = \pi(k)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For $\lambda \in \Delta_\gamma^{d-1}$, $\gamma = a$ or b, we put

$$\lambda' = A^{-1}(\pi, \gamma)\lambda \tag{1}$$

and define

$$\mathcal{T}(\lambda, \pi) = (\iota(\lambda'), \gamma(\pi)) \quad \text{for } \lambda \in \Delta^{d-1}\,. \tag{2}$$

Thus the map $\mathcal{T}$ is defined on $\Delta^{d-1} \times \mathcal{R}(\pi_0)$ and is a 2-1 map. Indeed, $A^{-1}(\pi, \gamma)\Delta_\gamma^{d-1}$ is a $(d-1)$-open simplex as well as Δ_γ^{d-1}, and

$$\mathcal{T}(\Delta_\gamma^{d-1} \times \{\pi\}) = \Delta^{d-1} \times \{\pi_1\}, \quad \pi_1 = \gamma(\pi) \tag{3}$$

for $\gamma = a$ or b. Moreover, $\mathcal{T}$ has an absolutely continuous invariant measure (not finite measure) and $\mathcal{T}$ is conservative ergodic with respect to this measure (see W. Veech [10]). We call $\mathcal{T}$ the renormalized Rauzy map. Note that $\mathcal{T}$ is defined for each Rauzy class $\mathcal{R}(\pi)$. For $(\lambda, \pi_0) \in \Delta^{d-1} \times \{\pi_0\}$, we define a sequence $\Gamma_1, \Gamma_2, \ldots$ by

$$\Gamma_k = \Gamma_k(\lambda, \pi_0) = \begin{cases} a & \text{if} \quad \lambda_d^{(k-1)} < \lambda^{(k-1)}_{(\pi_0^{(k-1)})^{-1}(d)} \\ b & \text{if} \quad \lambda_d^{(k-1)} > \lambda^{(k-1)}_{(\pi_0^{(k-1)})^{-1}(d)} \end{cases}$$

where $(\lambda^{(k)}, \pi_0^{(k)}) = \mathcal{T}^k(\lambda, \pi_0)$ and $\pi_0^{(k-1)} = \Gamma_{k-1} \cdots \Gamma_1(\pi_0)$. For a fixed finite sequence $(\gamma_1, \ldots, \gamma_n)$, $\gamma_i = a$ or b, $1 \le i \le n$, we denote by $C(\gamma_1, \ldots, \gamma_n)$ the cylinder set induced from $(\gamma_1, \ldots, \gamma_n)$ i.e.

$$C(\gamma_1, \ldots, \gamma_n) = \{(\lambda, \pi_0) : \lambda \in \Delta^{d-1}, (\Gamma_1, \ldots, \Gamma_n) = (\gamma_1, \ldots, \gamma_n)\}.$$

We rewrite the main result precisely as follows :

THEOREM. *If $\{\lambda \in \Delta^{d-1} : (\lambda, \pi_0) \in C(\gamma_1, \ldots, \gamma_n)\}$ is bounded away from $\partial\Delta^{d-1}$, then $C(\gamma_1, \ldots, \gamma_n)$ is a Darling-Kac set.*

We denote by η the Euclidean distance between $\{\lambda \in \Delta^{d-1} : (\lambda, \pi_0) \in C(\gamma_1, \ldots, \gamma_n)\}$ and $\partial\Delta^{d-1}$. We define

$$\hat{j}(\lambda, \pi_0) = \begin{cases} \min\{j \geq 1 : \mathcal{T}^j(\lambda, \pi_0) \in C(\gamma_1, \ldots, \gamma_n)\} \\ \qquad \text{if there exists } j \geq 1 \text{ such that } \mathcal{T}^j(\lambda, \pi_0) \in C(\gamma_1, \ldots, \gamma_n), \\ \infty \\ \qquad \text{if} \quad \mathcal{T}^j(\lambda, \pi_0) \notin C(\gamma_1, \ldots, \gamma_n) \text{ for all } j \geq 1, \end{cases}$$

for $(\lambda, \pi_0) \in C(\gamma_1, \ldots, \gamma_n)$. Because $\mathcal{T}$ is conservative, $\hat{j}(\lambda, \pi_0) < \infty$ for a.e. $(\lambda, \pi_0) \in C(\gamma_1, \ldots, \gamma_n)$ with respect to the volume measure of the cylinder set. For $(\lambda, \pi_0) \in C(\gamma_1, \ldots, \gamma_n)$ with $\hat{j}(\lambda, \pi_0) < \infty$, we put $\gamma_j = \Gamma_j(\lambda, \pi_0)$, for $n + 1 \leq j \leq \hat{j}(\lambda, \pi_0)$, where $\hat{j}(\lambda, \pi_0) > n$. Then we see that

$$(\lambda, \pi_0) \in C(\gamma_1, \ldots, \gamma_{\hat{j}(\lambda, \pi_0)}, \gamma_1, \ldots, \gamma_n).$$

Moreover we have

$$\mathcal{T}^{\hat{j}(\lambda, \pi_0)} C(\gamma_1, \ldots, \gamma_{\hat{j}(\lambda, \pi_0)}, \gamma_1, \ldots, \gamma_n) = C(\gamma_1, \ldots, \gamma_n) \tag{4}$$

and

$$\mathcal{T}^j(\lambda, \pi_0) \notin C(\gamma_1, \ldots, \gamma_n)$$

for $1 \leq j < \hat{j}(\lambda, \pi_0)$. The identity (4) follows from (3) by induction. We note that the cardinality of such cylinder sets is countable. Thus we have a countable partition

$$\{C(\gamma_1, \ldots, \gamma_{\hat{j}}, \gamma_1, \ldots, \gamma_n)\}$$

of $C(\gamma_1, \ldots, \gamma_n)$ (mod. 0). We denote by $\{\xi_1, \xi_2, \ldots\}$, this countable partition of $C(\gamma_1, \ldots, \gamma_n)$. From the definition, we have

$$\mathcal{T}^{\hat{j}} C(\gamma_1, \ldots, \gamma_{\hat{j}}, \gamma_1, \ldots, \gamma_n) = C(\gamma_1, \ldots, \gamma_n)$$

for all such $C(\gamma_1, \ldots, \gamma_{\hat{j}}, \gamma_1, \ldots, \gamma_n)$'s, here, $\hat{j} = \hat{j}(\lambda, \pi_0)$ for some $(\lambda, \pi_0) \in C(\gamma_1, \ldots, \gamma_n)$. For each $C(\gamma_1, \ldots, \gamma_{\hat{j}}, \gamma_1, \ldots, \gamma_n)$, since $\mathcal{T}^{\hat{j}}$ is given by concatenations of linear maps of the form $A^{-1}(\pi, \gamma)$ and ι, $C(\gamma_1, \dots, \gamma_{\hat{j}}, \gamma_1, \dots, \gamma_n)$ is a $(d-1)$-simplex. We define

$$T(\lambda, \pi_0) = \mathcal{T}^{\hat{j}}(\lambda, \pi_0)$$

for $(\lambda, \pi_0) \in C(\gamma_1, \dots, \gamma_{\hat{j}}, \gamma_1, \dots, \gamma_n)$. From the definition of $\hat{j}$, T is the first return map of $\mathcal{T}$ to $C(\gamma_1, \dots, \gamma_n)$.

3. Proof of Theorem

In the sequel, the second coordinate π of (λ, π) is always π_0 when we concentrate our discussion only on $C(\gamma_1, \dots, \gamma_n)$. Thus we can identify $(\lambda, \pi_0) \in \Delta^{d-1} \times \{\pi_0\}$ with $\lambda \in \Delta^{d-1}$. So we regard all subsets or points in $\Delta^{d-1} \times \{\pi_0\}$ as sets or points in Δ^{d-1}, respectively. Now we will show that T satisfies the conditions (A), (B), (C), (D), (E) and (F) by F.Schweiger [8]. In our case, these can be rewritten as follows :
We put

$$< \xi_{i_1}, \dots, \xi_{i_u} >= \xi_{i_1} \cap T^{-1}\xi_{i_2} \cap \cdots \cap T^{-(u-1)}\xi_{i_u} .$$

(A)

$$\lim_{u \to \infty} \sup_{i_1, \dots, i_u} \text{diam } < \xi_{i_1}, \dots, \xi_{i_u} >= 0 ,$$

"diam" means the diameter by the Euclidean distance. Hereafter we consider the Euclidean distance.

(B), (D) and (G)

$$T^u < \xi_{i_1}, \dots, \xi_{i_u} >= C(\gamma_1, \dots, \gamma_n) \tag{5}$$

for any $i_1, \dots, i_u$, $u \geq 1$.

Although, the condition (B), (D) and (G) are different shapes in F. Schweiger [8] all of them follow from the identity (5), which is a consequence of (4).

Let $\omega(\xi_{i_1}, \dots, \xi_{i_u})$ is a function defined by

$$\int_E \omega(\xi_{i_1}, \dots, \xi_{i_u}) d\nu = \int_{T^{-1}E \cap < \xi_{i_1}, \dots, \xi_{i_u} >} d\nu$$

for any Borel subset of $C(\gamma_1, \dots, \gamma_n)$, where ν denotes the normalized volume measure of $C(\gamma_1, \dots, \gamma_n)$.

(C) There exists a constant $C_1 \geq 0$ (independent of $\xi_{i_1}, \dots, \xi_{i_u}$) such that

$$\text{ess} \sup_{\lambda \in C(\gamma_1, \dots, \gamma_n)} \omega(\xi_{i_1}, \dots, \xi_{i_u})(\lambda) \leq C_1 \text{ ess} \inf_{\lambda \in C(\gamma_1, \dots, \gamma_n)} \omega(\xi_{i_1}, \dots, \xi_{i_u})(\lambda) .$$

(E) For any $(\xi_{i_1}, \dots, \xi_{i_u})$, $u \geq 1$, there exists a Lipschitz continuous version of $\omega(\xi_{i_1}, \dots, \xi_{i_u})$ by the same Lipschitz constant R_1, i.e.

$$|\omega(\xi_{i_1}, \dots, \xi_{i_u})(\lambda) - \omega(\xi_{i_1}, \dots, \xi_{i_u})(\lambda')| \leq R_1 \cdot \nu(< \xi_{i_1}, \dots, \xi_{i_u} >) \cdot \mathrm{d}\,(\lambda, \lambda')$$

for any $\lambda, \lambda' \in C(\gamma_1, \dots, \gamma_n)$, where d $(\cdot, \cdot)$ denotes the Euclidean distance.

We denote by $V(\xi_{i_1}, \dots, \xi_{i_u})$ the local inverse of $T|^u_{<\xi_{i_1},\dots,\xi_{i_u}>}$, i.e. $V(\xi_{i_1}, \dots, \xi_{i_u})$ is a map of $C(\gamma_1, \dots, \gamma_n)$ to $< \xi_{i_1}, \dots, \xi_{i_u} >$ such that $V(\xi_{i_1}, \dots, \xi_{i_u}) \circ T|^u_{<\xi_{i_1},\dots,\xi_{i_u}>}$ and $T|^u_{<\xi_{i_1},\dots,\xi_{i_u}>} \circ V(\xi_{i_1}, \dots, \xi_{i_u})$ are the identity maps of $< \xi_{i_1}, \dots, \xi_{i_u} >$ and $C(\gamma_1, \dots, \gamma_n)$, respectively.

(F) There exists a constant $R_2 \geq 0$ (independent of $\xi_{i_1}, \dots, \xi_{i_u}$) such that

$$\mathrm{d}\,(V(\xi_{i_1}, \dots, \xi_{i_u})(\lambda), V(\xi_{i_1}, \dots, \xi_{i_u})(\lambda')) \leq R_2 \cdot \mathrm{d}\,(\lambda, \lambda')$$

for any $\lambda, \lambda' \in C(\gamma_1, \dots, \gamma_n)$.

LEMMA 1. *There exists a positive constant* $\delta < 1$ *such that*

$$\mathrm{diam}\, C(\gamma_1, \dots \gamma_n, \gamma'_1, \dots \gamma'_l, \gamma_1, \dots, \gamma_n) < \delta \cdot \mathrm{diam}\, C(\gamma_1, \dots, \gamma_n)$$

for any sequence of $(\gamma'_1, \dots, \gamma'_l)$, *where*

$$\mathrm{diam}\, B = \sup\{\mathrm{d}(\lambda, \lambda') \, : \, \lambda, \lambda' \in B\} \qquad B \subset \mathbf{R}^d \, .$$

REMARK. $(\gamma'_1, \dots, \gamma'_l)$ can be empty and we always assume that

$$\gamma'_l \gamma'_{l-1} \cdots \gamma'_1 \gamma_n \cdots \gamma_1(\pi_0) = \pi_0 \, .$$

PROOF. Put

$$\Delta^{d-1}(\eta) = \{\lambda \in \Delta^{d-1} : \mathrm{d}\,(\lambda, \lambda') > \eta \text{ for any } \lambda' \in \partial \Delta^{d-1}\} \, .$$

Then $\Delta^{d-1}(\eta)$ is a $(d-1)$-open simplex. Moreover for each edge of Δ^{d-1}, there corresponds a parallel edge of $\Delta^{d-1}(\eta)$. Then, the distance of these edges is η. Because the diameter of a simplex is given by the length of its longest edge, we see

$$\mathrm{diam}\, \Delta^{d-1}(\eta) < \mathrm{diam}\, \Delta^{d-1} - 2\eta \, .$$

We put

$$\delta = \frac{\mathrm{diam}\,(\Delta^{d-1}) - 2\eta}{\mathrm{diam}\,(\Delta^{d-1})} \, .$$

For each $C(\gamma_1, \dots, \gamma_n, \gamma'_1, \dots, \gamma'_l, \gamma_1, \dots, \gamma_n)$, there exist a positive integer k and a $d \times d$ matrix M such that

$$\begin{aligned} T^k C(\gamma_1, \dots, \gamma_n, \gamma'_1, \dots, \gamma'_l, \gamma_1, \dots, \gamma_n) &= \iota(M C(\gamma_1, \dots, \gamma_n, \gamma'_1, \dots, \gamma'_l, \gamma_1, \dots, \gamma_n)) \\ &= C(\gamma_1, \dots, \gamma_n) \end{aligned} \tag{6}$$

and

$$\begin{aligned} T^k C(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l') &= \iota(MC(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l')) \\ &= \Delta^{d-1} . \end{aligned} \tag{7}$$

Indeed, M is given by products of matrices of the form $A^{-1}(\pi, \gamma)$. Here,

$$\operatorname{diam} (MC(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l', \gamma_1, \ldots, \gamma_n))$$

and

$$\operatorname{diam} (MC(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l'))$$

are given by the lengths of their longest edges, respectively. From (6) and (7), it turns out that the ratio of lengths of parallel edges is less than δ. By applying M^{-1}, the same holds for edges of

$$C(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l', \gamma_1, \ldots, \gamma_n)$$

and

$$C(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l') ,$$

that is, the ratio of lengths of the parallel edges. The ratio of

$$\operatorname{diam} C(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l', \gamma_1, \ldots, \gamma_n)$$

and

$$\operatorname{diam} C(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l')$$

is less than δ. Since

$$\operatorname{diam} C(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l', \gamma_1, \ldots, \gamma_n)$$

is given by the length of its longest edge and

$$C(\gamma_1, \ldots, \gamma_n, \gamma_1', \ldots, \gamma_l') \subset C(\gamma_1, \ldots, \gamma_n) \subset \Delta^{d-1}(\eta) ,$$

we have the assertion of the lemma. □

For a given sequence $\gamma_1', \ldots, \gamma_l'$, $\gamma_i = a$ or b for $1 \le i \le l$, we define

$$r_1(\gamma_1', \ldots, \gamma_l') = \begin{cases} \min \{r \ge 1 \,:\, \gamma_r' \cdots \gamma_{r+n-1}' = \gamma_1 \cdots \gamma_n, r+n-1 \le l\} & \text{if } r \text{ exists,} \\ \infty & \text{otherwise.} \end{cases}$$

$$\vdots$$

$$r_{j+1}(\gamma'_1, \dots, \gamma'_l) = \begin{cases} \min\{r \geq r_j + n \, : \, \gamma'_r \cdots \gamma'_{r+n-1} = \gamma_1 \cdots \gamma_n, r + n - 1 \leq l\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } r_j < \infty \text{ and } r \text{ exists,} \\ \infty \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

and put

$$L = \begin{cases} \max\{j \, : \, r_j < \infty\} & \text{if } r_1 < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2. *For a given* $\gamma'_1, \dots, \gamma'_l$,

$$\operatorname{diam} C(\gamma_1, \dots, \gamma_n, \gamma'_1, \dots, \gamma'_l, \gamma_1, \dots, \gamma_n) < \delta^{L+1} \cdot \operatorname{diam} C(\gamma_1, \dots, \gamma_n)\,.$$

PROOF. This follows by induction with Lemma 1. □

Let q be the number of k's, $2 \leq k \leq n$, such that

$$(\gamma_1, \dots, \gamma_n) = (\gamma_k, \dots, \gamma_{n+k-1})$$

and

$$\pi_0^{(k-1)} = \gamma_{k-1} \cdots \gamma_1(\pi_0) = \pi_0\,.$$

We put

$$\sigma(u) = \sup_{\xi_{i_1}, \dots, \xi_{i_u}} \operatorname{diam}\,(\xi_{i_1} \cap T^{-1}\xi_{i_2} \cap \cdots \cap T^{-(u-1)}\xi_{i_u})$$

for $u \geq 1$. The following Lemma implies that the condition (A) holds.

LEMMA 3. *There exists a constant* $C_0 > 0$ *such that*

$$\sigma(u) < C_0 \cdot (\delta^{\frac{1}{q}})^u\,.$$

PROOF. We estimate

$$\operatorname{diam}\,(\xi_{i_1} \cap T^{-1}\xi_{i_2} \cap \cdots \cap T^{-(u-1)}\xi_{i_u})$$

for any choice of $(\xi_{i_1}, \dots, \xi_{i_u})$. From the definition of the partition $\{\xi_i\}$, there exists $\gamma'_1, \dots, \gamma'_l$ such that

$$\xi_{i_1} \cap T^{-1}\xi_{i_2} \cap \cdots \cap T^{-(u-1)}\xi_{i_u} = C(\gamma_1, \dots, \gamma_n, \gamma'_1, \dots, \gamma'_l, \gamma_1, \dots, \gamma_n)$$

whenever $u \geq q + 1$. Then we see that

$$u \leq (L + 1)q\,.$$

From Lemma 2, we have

$$\frac{\operatorname{diam}\,(\xi_{i_1} \cap T^{-1}\xi_{i_2} \cap \cdots \cap T^{-(u-1)}\xi_{i_u})}{\operatorname{diam} C(\gamma_1, \dots, \gamma_n)} < \delta^{L+1}$$

$$< (\delta^{\frac{1}{q}})^u .$$

This shows the assertion of this lemma. □

Now we show that T satisfies (C), (E), and (F) of F. Schweiger [8]. For a given finite sequence $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_u}$ of elements of the partition $\{\xi_i\}$, there exists a positive integer k such that $T^u = \mathcal{T}^k$ on $< \xi_{i_1}, \ldots, \xi_{i_u} >$. Then $\mathcal{T}^k$ is bijective on $< \xi_{i_1}, \ldots, \xi_{i_u} >$ to $C(\gamma_1, \ldots, \gamma_n)$. We denote by $V(\xi_{i_1}, \ldots, \xi_{i_u})$ its local inverse. According to (1) and (2), We have matrices $A_1, \ldots, A_k$ of the form $A(\pi, a)$ or $A(\pi, b)$ and

$$V(\xi_{i_1}, \ldots, \xi_{i_u})(\lambda) = \iota(M\lambda)$$

for $\lambda \in C(\gamma_1, \ldots, \gamma_n)$ with

$$M = (m_{i\,j})_{1 \le i \le d,\, 1 \le j \le d} = A_1 A_2 \cdots A_k .$$

We note M is a non-negative matrix. Moreover, the Jacobian $J(M, \lambda)$ of $V(\xi_{i_1}, \ldots, \xi_{i_u})$ is given by $\dfrac{1}{||M\lambda||^d}$ (see W. Veech [9], 5.2).
From the definition of η, there exists $\eta_0 > 0$, such that $\eta_0 \le \lambda_i$, $1 \le i \le d$, for any $\lambda \in C(\gamma_1, \ldots, \gamma_n)$. Then we see the following lemma, which shows the condition (C).

LEMMA 4. $$\sup_{\lambda^{(u)} \in C(\gamma_1, \ldots, \gamma_n)} J(M, \lambda^{(u)}) \le \frac{1}{\eta_0^d} \inf_{\lambda^{(u)} \in C(\gamma_1, \ldots, \gamma_n)} J(M, \lambda^{(u)})$$

Next, we show the following lemmas. The first one is used in the proof of the later.

LEMMA 5. *There exists a constant* $C_2 > 0$ *such that*

$$C_2 \cdot \nu(< \xi_{i_1}, \ldots, \xi_{i_u} >) \ge \sup_{\lambda^{(u)} \in C(\gamma_1, \ldots, \gamma_n)} J(M, \lambda^{(u)}) .$$

PROOF. From Lemma 4, we have

$$\begin{aligned} \nu(< \xi_{i_1}, \ldots, \xi_{i_u} >) &= \int_{C(\gamma_1, \ldots, \gamma_n)} J(M, \lambda) d\nu(\lambda) \\ &\ge \inf_{\lambda \in C(\gamma_1, \ldots, \gamma_n)} J(M, \lambda) \cdot \int_{C(\gamma_1, \ldots, \gamma_n)} d\nu(\lambda) \\ &\ge \eta_0^d \sup_{\lambda \in C(\gamma_1, \ldots, \gamma_n)} J(M, \lambda) \cdot \nu(C(\gamma_1, \ldots, \gamma_n)) . \end{aligned}$$

This implies the assertion of this lemma. Here we recall that ν is the normalized volume measure of $C(\gamma_1, \ldots, \gamma_n)$. □

LEMMA 6. *There exists a constant* $C_3 \ge 0$ *such that*

$$|J(M, \lambda) - J(M, \lambda')| \le C_3 \cdot \nu(< \xi_{i_1}, \ldots, \xi_{i_u} >) ||\lambda - \lambda'||$$

for $\lambda, \lambda' \in C(\gamma_1, \ldots, \gamma_n)$ *and any* $(i_1, \ldots, i_u)$.

PROOF. We put $\alpha = ||M\lambda|| = \sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}\lambda_j$, $\beta = ||M\lambda'|| = \sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}\lambda'_j$, and have

$$\begin{aligned}|J(M,\lambda) - J(M,\lambda')| &= \left|\frac{1}{\alpha^d} - \frac{1}{\beta^d}\right| \\ &= \left|\frac{(\beta-\alpha)(\beta^{d-1} + \beta^{d-2}\alpha + \beta^{d-3}\alpha^2 + \cdots \beta\alpha^{d-2} + \alpha^{d-1})}{\alpha^d\beta^d}\right| .\end{aligned}$$

Here we see

$$\begin{aligned}|\beta - \alpha| &= \left|\sum_{i=1}^{d}\left[\sum_{j=1}^{d} m_{i\,j}(\lambda'_j - \lambda_j)\right]\right| \\ &\le \sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}|\lambda'_j - \lambda_j| \\ &\le \sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}\cdot ||\lambda - \lambda'||\end{aligned}$$

and $0 < \alpha \le \sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}$, $0 < \beta \le \sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}$. We put $P = \sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}$. Then we have

$$\left|\frac{(\beta-\alpha)(\beta^{d-1} + \beta^{d-2}\alpha + \beta^{d-3}\alpha^2 + \cdots \beta\alpha^{d-2} + \alpha^{d-1})}{\alpha^d\beta^d}\right| \le \frac{P\cdot d\cdot P^{d-1}\cdot ||\lambda - \lambda'||}{\alpha^d\beta^d} .$$

We note $\lambda'_j > \eta_0$ for $1 \le j \le d$ and then

$$\sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}\lambda'_j \ge \eta_0 \cdot P .$$

Consequently, we see

$$\begin{aligned}|J(M,\lambda) - J(M,\lambda')| &\le \frac{P\cdot d\cdot P^{d-1}\|\lambda - \lambda'\|}{\alpha^d\cdot\eta_0^d\cdot P^d} \\ &\le \frac{d}{\eta_0^d}\cdot\frac{1}{\alpha^d}\cdot\|\lambda - \lambda'\| \\ &\le \frac{d}{\eta_0^d}\sup_{\lambda\in C(\gamma_1,\ldots,\gamma_n)} J(M,\lambda)\|\lambda - \lambda'\| .\end{aligned}$$

Finally, by Lemma 5, the righthand side of this inequality is bounded by

$$\frac{d}{\eta_0^d} C_2 \cdot \nu(< \xi_{i_1}, \ldots, \xi_{i_u} >) \cdot \|\lambda - \lambda'\|\,,$$

which shows the assertion of Lemma 6. □

Since

$$\mathrm{d}(\lambda, \lambda') \leq \|\lambda - \lambda'\| \leq \sqrt{d+1}\,\mathrm{d}\,(\lambda, \lambda')\,,$$

the condition (E) is an easy consequence of Lemma 6. The following Lemma 7 shows the final condition (F) by the above inequality.

LEMMA 7. *There exists a constant $C_4 > 0$ such that for any $i_1, \ldots, i_u$, we have*

$$||V(\xi_{i_1}, \ldots, \xi_{i_u})(\lambda) - V(\xi_{i_1}, \ldots, \xi_{i_u})(\lambda')|| \leq C_4 ||\lambda - \lambda'||$$

for $\lambda, \lambda' \in C(\gamma_1, \ldots, \gamma_n)$.

PROOF. We put again $\alpha = \sum_{i=1}^d \sum_{j=1}^d m_{i\,j}\lambda_j$ and $\beta = \sum_{i=1}^d \sum_{j=1}^d m_{i\,j}\lambda'_j$. The assertion of this lemma follows by the following calculations.

$$\begin{aligned}
&||V(\xi_{i_1}, \ldots, \xi_{i_u})(\lambda) - V(\xi_{i_1}, \ldots, \xi_{i_u})(\lambda')|| \\
&= \sum_{i=1}^d \left| \frac{\sum_{j=1}^d m_{i\,j}\lambda_j}{\alpha} - \frac{\sum_{j=1}^d m_{i\,j}\lambda'_j}{\beta} \right| \\
&= \sum_{i=1}^d \frac{\left| \beta \sum_{j=1}^d m_{i\,j}\lambda_j - \alpha \sum_{j=1}^d m_{i\,j}\lambda'_j \right|}{\alpha\,\beta} \\
&= \frac{\sum_{i=1}^d \left| \beta \sum_{j=1}^d m_{i\,j}\lambda_j - \beta \sum_{j=1}^d m_{i\,j}\lambda'_j + \beta \sum_{j=1}^d m_{i\,j}\lambda'_j - \alpha \sum_{j=1}^d m_{i\,j}\lambda'_j \right|}{\alpha\,\beta} \\
&\leq \frac{\sum_{i=1}^d \left| \sum_{j=1}^d m_{i\,j}(\lambda_j - \lambda'_j) \right|}{\alpha} + \frac{\left| \sum_{i=1}^d \sum_{j=1}^d m_{i\,j}(\lambda'_j - \lambda_j) \right| \sum_{i=1}^d \left(\sum_{j=1}^d m_{i\,j}\lambda'_j \right)}{\alpha\,\beta}
\end{aligned}$$

$$\leq \frac{\|\lambda - \lambda'\|}{\eta_0} + \frac{\left(\sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}\right)\|\lambda' - \lambda\| \sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}}{\eta_0 \cdot \left(\sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}\right) \eta_0 \cdot \left(\sum_{i=1}^{d}\sum_{j=1}^{d} m_{i\,j}\right)}$$

$$\leq \frac{\|\lambda - \lambda'\|}{\eta_0} + \frac{1}{\eta_0^2}\|\lambda' - \lambda\| .$$

□

It is easy to see that there exists an invariant probability measure μ for T equivalent to the volume measure of $C(\gamma_1, \ldots, \gamma_n)$. We consider the pre-dual operator $\hat{T}$ of T :

$$\int_{T^{-1}E} f d\nu = \int_E (\hat{T} f) d\nu$$

for any $f \in \mathcal{L}_1(\nu)$ and any measurable subset E of $C(\gamma_1, \ldots, \gamma_n)$. From lemmas 3 - 7, we have shown that T satisfies all conditions given by F.Schweiger [8]. Thus there exists an invariant probability measure μ for T equivalent to ν. We denote by h its density function $\frac{d\mu}{d\nu}$. This means that there exists a constant ρ, $0 < \rho < 1$, such that

$$\hat{T}^n f = \left(\int_{C(\gamma_1,\ldots,\gamma_n)} f d\nu\right) \cdot h + O(\alpha^{\sqrt{n}}), \qquad n \geq 1$$

whenever a real valued function f of $C(\gamma_1, \ldots, \gamma_n)$ satisfies

$$0 < m_0 \leq f \leq M_0$$

where m_0 and M_0 depend on f, and

$$|f(x) - f(y)| \leq K \cdot ||x - y|| \ \text{ for any } x, y \in C(\gamma_1, \ldots, \gamma_n)\,,$$

where K depends on f. We refer F. Schweiger [8] for the detail. As a consequence (see [6]), we see that T is continued fraction mixing with the partition $\{\xi_i\}$. This means that if we put $X_n(x) = i$ if $T^{n-1}(x) \in \xi_i$ for $x \in C(\gamma_1, \ldots, \gamma_n)$, then there exists a constant K_0 such that

$$|\mu(A \cap T^{-(n+k)}B) - \mu(A)\mu(B)| < K_0 \cdot \mu(A)\mu(B)\alpha^{\sqrt{k}}$$

for any measurable subset A generated by $X_1, \ldots, X_n$ and any measurable subset B of $C(\gamma_1, \ldots, \gamma_n)$. Thus, we see that T is continued fraction mixing and then $C(\gamma_1, \ldots, \gamma_n)$ is a Darling-Kac set of $\mathcal{T}$.

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