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Some Results on Walk Regular Graphs Which Are Cospectral to Its Complement

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Abstract. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where S_k denotes the neighborhood of the vertex k. We say that a graph G of order n is walk regular if and only if its vertex deleted subgraphs $G_i = G \setminus i$ are cospectral for $i = 1, 2, \ldots, n$. We here establish necessary and sufficient conditions under which a walk regular graph G which is cospectral to its complement $\overline{G}$ is strongly regular.

1. Introduction

Let G be a simple graph of order n. The spectrum of the simple graph G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix $A = A(G)$ and is denoted by $\sigma(G)$. The Seidel spectrum of G consists of the eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of its $(0, -1, 1)$ adjacency matrix $A^* = A^*(G)$ and is denoted by $\sigma^*(G)$. Let $P_G(\lambda) = |\lambda I - A|$ and $P_G^*(\lambda) = |\lambda I - A^*|$ denote the characteristic polynomial and the Seidel characteristic polynomial, respectively. Let $c = a + b\sqrt{m}$ and $\overline{c} = a - b\sqrt{m}$ where a and b are two nonzero integers and m is a positive integer such that m is not a perfect square. We say that $A^c = [c_{ij}]$ is the conjugate adjacency matrix of G if $c_{ij} = c$ for any two adjacent vertices i and j, $c_{ij} = \overline{c}$ for any two nonadjacent vertices i and j, and $c_{ij} = 0$ if $i = j$. The conjugate spectrum of G is the set of the eigenvalues $\lambda_1^c \geq \lambda_2^c \geq \cdots \geq \lambda_n^c$ of its conjugate adjacency matrix $A^c = A^c(G)$ and is denoted by $\sigma^c(G)$. Let $P_G^c(\lambda) = |\lambda I - A^c|$ denote the conjugate characteristic polynomial of G.

Further, we say that an eigenvalue μ of G is main if and only if $\langle \mathbf{j}, \mathbf{P}\mathbf{j} \rangle = n \cos^2 \alpha > 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The quantity $\beta = |\cos \alpha|$ is called the main angle of μ. Similarly, we say that $\mu^c \in \sigma^c(G)$ is the conjugate main eigenvalue if and only if $\langle \mathbf{j}, \mathbf{P}^c \mathbf{j} \rangle = n \cos^2 \gamma > 0$, where $\mathbf{P}^c$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_{A^c}(\mu^c)$. The quantity $\beta^c = |\cos \gamma|$ is called the conjugate main angle of μ^c.

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Let $\mathcal{M}(G)$ be the set of all main eigenvalues of G. Then $|\mathcal{M}(G)| = |\mathcal{M}(\overline{G})|$, where $\overline{G}$ denotes the complement of G. According to [3], we have $|\mathcal{M}(G)| = |\mathcal{M}^c(G)|$, where $\mathcal{M}^c(G)$ denotes the set of all conjugate main eigenvalues of G.

PROPOSITION 1 (Lepović [3]). *Let $\lambda \in \sigma^c(G)$ be a conjugate eigenvalue of the graph G with multiplicity $p \geq 1$ and let q be the multiplicity of the eigenvalue $\dfrac{\lambda + \overline{c}}{2b\sqrt{m}} \in \sigma(G)$. Then $p - 1 \leq q \leq p + 1$.*

Next, replacing λ with $x + y\sqrt{m}$ the conjugate characteristic polynomial $P^c_G(\lambda)$ can be transformed into the form

$$P^c_G(x + y\sqrt{m}) = Q_n(x, y) + \sqrt{m}\, R_n(x, y)\,, \tag{1}$$

where $Q_n(x, y)$ and $R_n(x, y)$ are two polynomials of order n in variables x and y, whose coefficients are integers. Besides, according to [3]

$$P^c_{\overline{G}}(x - y\sqrt{m}) = Q_n(x, y) - \sqrt{m}\, R_n(x, y)\,. \tag{2}$$

We note from (1) and (2) that $x_0 + y_0\sqrt{m} \in \sigma^c(G)$ and $x_0 - y_0\sqrt{m} \in \sigma^c(\overline{G})$ if and only if x_0 and y_0 is a solution of the following system of equations

$$Q_n(x, y) = 0 \quad \text{and} \quad R_n(x, y) = 0\,. \tag{3}$$

THEOREM 1 (Lepović [3]). *Let G and H be two graphs of order n. Then $P^c_G(\lambda) = P^c_H(\lambda)$ if and only if $P_G(\lambda) = P_H(\lambda)$ and $P_{\overline{G}}(\lambda) = P_{\overline{H}}(\lambda)$.*

PROPOSITION 2 (Lepović [6]). *Let G be a graph of order n. Then G is cospectral to its complement $\overline{G}$ if and only if $Q_n(-a, -\lambda) = Q_n(-a, \lambda)$ and $R_n(-a, -\lambda) = -R_n(-a, \lambda)$.*

PROPOSITION 3. *Let G be a graph of order n. Then G is cospectral to its complement $\overline{G}$ if and only if $Q_n(\lambda, 0) = P^c_G(\lambda)$ and $R_n(\lambda, 0) = 0$.*

PROOF. First, since $P^c_G(\lambda + 0 \cdot \sqrt{m}) = Q_n(\lambda, 0) + \sqrt{m}\, R_n(\lambda, 0)$ and $P^c_{\overline{G}}(\lambda - 0 \cdot \sqrt{m}) = Q_n(\lambda, 0) - \sqrt{m}\, R_n(\lambda, 0)$, using (1) and (2) we obtain that $P^c_G(\lambda) = P^c_{\overline{G}}(\lambda)$ if and only $Q_n(\lambda, 0) = P^c_G(\lambda)$ and $R_n(\lambda, 0) = 0$. Using Theorem 1 we obtain the proof. □

PROPOSITION 4 (Lepović [3]). *Let G be a connected or disconnected graph of order n. Then*

$$P_G\left(\frac{\lambda - b\sqrt{m}}{2b\sqrt{m}}\right) = \frac{c P^c_G(\lambda - a) + (-1)^n \overline{c} P^c_{\overline{G}}(-\lambda - a)}{2^{n+1} a (b\sqrt{m}\,)^n}\,. \tag{4}$$

2. Some auxiliary results

PROPOSITION 5. *Let G be a connected or disconnected graph of order n. If n is an even number then*

$$P_G\left(\frac{x+a+(y-b)\sqrt{m}}{2b\sqrt{m}}\right) = \frac{a(Q_n(x,y)+Q_n(\overline{x},y))+mb(R_n(x,y)+R_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n} + \frac{b\sqrt{m}\,(Q_n(x,y)-Q_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n} + \frac{a\sqrt{m}\,(R_n(x,y)-R_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n},$$

where $\overline{x} = -x-2a$.

PROPOSITION 6. *Let G be a connected or disconnected graph of order n. If n is an odd number then*

$$P_G\left(\frac{x+a+(y-b)\sqrt{m}}{2b\sqrt{m}}\right) = \frac{a(Q_n(x,y)-Q_n(\overline{x},y))+mb(R_n(x,y)-R_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n} + \frac{b\sqrt{m}\,(Q_n(x,y)+Q_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n} + \frac{a\sqrt{m}\,(R_n(x,y)+R_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n},$$

where $\overline{x} = -x-2a$.

PROOF. Replacing $\lambda - a$ with $x + y\sqrt{m}$ in (4) and making use of (1) and (2) we easily obtain Propositions 5 and 6. □

PROPOSITION 7. *Let G be a connected or disconnected graph of order n. If n is an even number then*

$$P_{\overline{G}}\left(-\frac{x+a+(y+b)\sqrt{m}}{2b\sqrt{m}}\right) = \frac{a(Q_n(x,y)+Q_n(\overline{x},y))-mb(R_n(x,y)+R_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n} - \frac{b\sqrt{m}\,(Q_n(x,y)-Q_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n} + \frac{a\sqrt{m}\,(R_n(x,y)-R_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n},$$

where $\overline{x} = -x-2a$.

PROPOSITION 8. *Let G be a connected or disconnected graph of order n. If n is an odd number then*

$$P_{\overline{G}}\left(-\frac{x+a+(y+b)\sqrt{m}}{2b\sqrt{m}}\right) = \frac{-\,a(Q_n(x,y)-Q_n(\overline{x},y))+mb(R_n(x,y)-R_n(\overline{x},y))}{2^{n+1}a\,(b\sqrt{m})^n}$$

$$+ \frac{b\sqrt{m}\,(Q_n(x, y) + Q_n(\overline{x}, y))}{2^{n+1}a\,(b\sqrt{m})^n}$$
$$- \frac{a\sqrt{m}\,(R_n(x, y) + R_n(\overline{x}, y))}{2^{n+1}a\,(b\sqrt{m})^n},$$

where $\overline{x} = -x - 2a$.

PROOF. Applying (4) to its complement $\overline{G}$ and replacing $-\lambda - a$ with $x + y\sqrt{m}$, we easily obtain Propositions 7 and 8. □

PROPOSITION 9. *Let G be a connected or disconnected graph of order n. If n is an even number then*

$$P_G^*\Big(-\frac{x + a + y\sqrt{m}}{b\sqrt{m}}\Big) = \frac{Q_n(x, y) + Q_n(\overline{x}, y) + \sqrt{m}\,(R_n(x, y) - R_n(\overline{x}, y))}{2\,(b\sqrt{m})^n},$$

where $\overline{x} = -x - 2a$.

PROPOSITION 10. *Let G be a connected or disconnected graph of order n. If n is an odd number then*

$$P_G^*\Big(-\frac{x + a + y\sqrt{m}}{b\sqrt{m}}\Big) = -\frac{Q_n(x, y) - Q_n(\overline{x}, y) + \sqrt{m}\,(R_n(x, y) + R_n(\overline{x}, y))}{2\,(b\sqrt{m})^n},$$

where $\overline{x} = -x - 2a$.

PROOF. Using that $P_G^*(-2\lambda - 1) = 2^{n-1}(P_{\overline{G}}(-\lambda - 1) + (-1)^n P_G(\lambda))$ (see [2]), by an easy calculation we obtain the statements using Propositions 5, 6, 7 and 8. □

Further, let S be any subset of the vertex set $V(G)$. To switch G with respect to S means to remove all edges connecting S with $\overline{S} = V(G) \setminus S$, and to introduce edges between all nonadjacent vertices in G which connect S with $\overline{S}$. Two graphs G and H are switching (Seidel switching) equivalent if one of them is obtained from the other by switching. It is known that switching equivalent graphs have the same Seidel spectrum.

PROPOSITION 11. *Let* $G^{(1)}$ *and* $G^{(2)}$ *be two switching equivalent graphs of order n. If n is an even number then*

$$Q_n^{(1)}(x, y) + Q_n^{(1)}(-x - 2a, y) = Q_n^{(2)}(x, y) + Q_n^{(2)}(-x - 2a, y)\,;$$
$$R_n^{(1)}(x, y) - R_n^{(1)}(-x - 2a, y) = R_n^{(2)}(x, y) - R_n^{(2)}(-x - 2a, y)\,,$$

where $Q_n^{(k)}(x, y)$ *and* $R_n^{(k)}(x, y)$ *are related to* $G^{(k)}$ *for* $k = 1, 2$.

PROPOSITION 12. *Let* $G^{(1)}$ *and* $G^{(2)}$ *be two switching equivalent graphs of order n. If n is an odd number then*

$$Q_n^{(1)}(x, y) - Q_n^{(1)}(-x - 2a, y) = Q_n^{(2)}(x, y) - Q_n^{(2)}(-x - 2a, y)\,;$$

$$R_n^{(1)}(x, y) + R_n^{(1)}(-x - 2a, y) = R_n^{(2)}(x, y) + R_n^{(2)}(-x - 2a, y)\,,$$

where $Q_n^{(k)}(x, y)$ *and* $R_n^{(k)}(x, y)$ *are related to* $G^{(k)}$ *for* $k = 1, 2$.

PROPOSITION 13. *Let G be a connected or disconnected graph G of order n. Then:*

$$Q_n(x, y) = \frac{1}{2}(Q_n(-a, \lambda^+) + Q_n(-a, \lambda^-)) + \frac{\sqrt{m}}{2}(R_n(-a, \lambda^+) - R_n(-a, \lambda^-))\,;$$

$$R_n(x, y) = \frac{1}{2}(R_n(-a, \lambda^+) + R_n(-a, \lambda^-)) + \frac{\sqrt{m}}{2m}(Q_n(-a, \lambda^+) - Q_n(-a, \lambda^-))\,,$$

where $\lambda^{\pm} = y \pm \dfrac{(x + a)\sqrt{m}}{m}$.

PROPOSITION 14. *Let G be a connected or disconnected graph G of order n. Then:*

$$Q_n(\overline{x}, y) = \frac{1}{2}(Q_n(-a, \lambda^+) + Q_n(-a, \lambda^-)) - \frac{\sqrt{m}}{2}(R_n(-a, \lambda^+) - R_n(-a, \lambda^-))\,;$$

$$R_n(\overline{x}, y) = \frac{1}{2}(R_n(-a, \lambda^+) + R_n(-a, \lambda^-)) - \frac{\sqrt{m}}{2m}(Q_n(-a, \lambda^+) - Q_n(-a, \lambda^-))\,,$$

where $\overline{x} = -x - 2a$.

PROOF. Replacing $x + y\sqrt{m}$ with $-a + \lambda\sqrt{m}$ and replacing $x - y\sqrt{m}$ with $-a - \lambda\sqrt{m}$ in relations (1) and (2) respectively, we easily obtain Propositions 13 and 14. □

3. Some preliminary results

Let i be a fixed vertex from the vertex set $V(G) = \{1, 2, \dots, n\}$ and let $G_i = G \setminus i$ be its corresponding vertex deleted subgraph. Let S_i denote the neighborhood of i, defined as the set of all vertices of G which are adjacent to i.

We say that a regular graph G of order n and degree $r \geq 1$ is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, understanding that G is not the complete graph K_n. We know that a regular connected graph is strongly regular if and only if it has exactly three distinct eigenvalues [1].

THEOREM 2 (Lepović [6]). *A regular graph G of order n and degree* $r \geq 1$ *is strongly regular if and only if its vertex deleted subgraphs* G_i *have exactly two main eigenvalues for* $i = 1, 2, \dots, n$.

DEFINITION 1. A graph G of order n is walk regular if the number of closed walks of length k starting and ending at vertex i is the same for any $i = 1, 2, \dots, n$.

We know that a graph G of order n is walk regular if and only if its vertex deleted subgraphs G_i are cospectral for $i = 1, 2, \dots, n$. Of course, if G is a walk regular graph then its complement $\overline{G}$ is also walk regular [6].

Further, let $G^\bullet = G \cup \bullet_x$ be the graph obtained from the graph G by adding a new isolated vertex x. We now have the following result [4].

PROPOSITION 15. *Let* $P^c_{G^\bullet}(x + y\sqrt{m}\,) = Q_{n+1}(x, y) + \sqrt{m}\, R_{n+1}(x, y)$. *Then*:

$$Q_{n+1}(x, y) = \frac{a^2 + mb^2}{2a}\left(Q_n(x, y) - (-1)^n\, Q_n(-x - 2a, y)\right) - mb\left(R_n(x, y) + (-1)^n\, R_n(-x - 2a, y)\right) + x\, Q_n(x, y) + my R_n(x, y)\,; \tag{5}$$

$$R_{n+1}(x, y) = \frac{a^2 + mb^2}{2a}\left(R_n(x, y) + (-1)^n\, R_n(-x - 2a, y)\right) - b\left(Q_n(x, y) - (-1)^n\, Q_n(-x - 2a, y)\right) + x\, R_n(x, y) + y\, Q_n(x, y)\,. \tag{6}$$

Let $H^{(i)}$ be switching equivalent to G with respect to $S_i \subseteq V(G)$ for $i = 1, 2, \dots, n$, understanding that S_i is the neighborhood of the vertex i. Then $H^{(i)} = H_i \cup \bullet_i$ where ' $\bullet_i$ ' is the isolated vertex denoted by ' i ' in G.

PROPOSITION 16 (Lepović [6]). *Let G be a walk regular graph of order $4n + 1$ and degree $2n$. If G is cospectral to its complement $\overline{G}$ then $P_{H_i}(\lambda) = P_{\overline{H}_i}(\lambda)$ for $i = 1, 2, \dots, 4n + 1$.*

THEOREM 3 (Lepović [6]). *Let G be a walk regular graph of order $4n+1$ and degree $2n$, which is cospectral to its complement $\overline{G}$. Then G is strongly regular if and only if G_i is cospectral to H_i for $i = 1, 2, \dots, 4n + 1$.*

PROPOSITION 17 (Lepović [7]). *Let G be a connected or disconnected regular graph of order n and degree r. Then*

$$P^c_{\overline{G}_i}(\lambda) = \frac{(-1)^{n-1}}{\lambda + \mu^c_1 + 2a}\left((\lambda - \overline{\mu}^c_1)\, P^c_{G_i}(-\lambda - 2a) - \frac{2a\, P^c_G(-\lambda - 2a)}{\lambda + \mu^c_1 + 2a}\right), \tag{7}$$

where $\mu^c_1 = (n - 1)a + (2r - (n - 1))b\sqrt{m}$ *and* $\overline{\mu}^c_1 = (n - 1)a - (2r - (n - 1))b\sqrt{m}$.

THEOREM 4 (Lepović [6]). *A graph G of order n has exactly k main eigenvalues if and only if $|\sigma^c_Q(G) \cap \sigma^c_R(G)| = n - k$, where $\sigma^c_Q(G) = \{x \mid Q_n(-a, x) = 0\}$ and $\sigma^c_R(G) = \{x \mid R_n(-a, x) = 0\}$.*

4. Main results

PROPOSITION 18. *Let G be a walk regular graph of order $4n+1$ and degree $2n$. If G is cospectral to its complement $\overline{G}$ then*

$$2mb^2 P^c_{H_i}(\lambda) = ((\lambda + a)^2 + 4na(\lambda + a) + mb^2) P^c_{G_i}(\lambda) \tag{8}$$

$$- ((\lambda + a)^2 - 4na(\lambda + a) - mb^2) P^c_{G_i}(\overline{\lambda})$$

$$2mb^2 P^c_{H_i}(\overline{\lambda}) = ((\lambda + a)^2 - 4na(\lambda + a) + mb^2) P^c_{G_i}(\overline{\lambda}) - ((\lambda + a)^2 + 4na(\lambda + a) - mb^2) P^c_{G_i}(\lambda)\,, \tag{9}$$

where $\overline{\lambda} = -\lambda - 2a$.

PROOF. Let $P^c_{G_i}(x + y\sqrt{m}) = Q^{(i)}_{4n}(x, y) + \sqrt{m}\, R^{(i)}_{4n}(x, y)$ for $i = 1, 2, \ldots, 4n + 1$. Since G_i and its complement $\overline{G}_i$ are cospectral (see [6]), we find that $Q^{(i)}_{4n}(\lambda, 0) = P^c_{G_i}(\lambda)$ and $R^{(i)}_{4n}(\lambda, 0) = 0$ (see Proposition 3). Let $P^c_{H_i}(x + y\sqrt{m}) = S^{(i)}_{4n}(x, y) + \sqrt{m}\, T^{(i)}_{4n}(x, y)$ for $i = 1, 2, \ldots, 4n + 1$. In view of Proposition 16 it turns out that $S^{(i)}_{4n}(\lambda, 0) = P^c_{H_i}(\lambda)$ and $T^{(i)}_{4n}(\lambda, 0) = 0$. Since G_i is switching equivalent to H_i with respect to $S_i \subseteq V(G_i)$, we obtain from Proposition 11,

$$Q^{(i)}_{4n}(\lambda, 0) + Q^{(i)}_{4n}(\overline{\lambda}, 0) = S^{(i)}_{4n}(\lambda, 0) + S^{(i)}_{4n}(\overline{\lambda}, 0)\,. \tag{10}$$

Further, since G is cospectral to its complement $\overline{G}$ we have $Q_{4n+1}(\lambda, 0) = P^c_G(\lambda)$ and $R_{4n+1}(\lambda, 0) = 0$. In view of this and using (7), we get

$$2a\,Q_{4n+1}(\lambda, 0) = (\lambda - 4na)((\lambda + 4na + 2a)Q^{(i)}_{4n}(\lambda, 0) - (\lambda - 4na)Q^{(i)}_{4n}(\overline{\lambda}, 0))\,,$$

which results in

$$a(Q_{4n+1}(\lambda, 0) - Q_{4n+1}(\overline{\lambda}, 0)) = ((\lambda + a) + (4n + 1)a)(\lambda + a)\, Q^{(i)}_{4n}(\lambda, 0) - ((\lambda + a) - (4n + 1)a)(\lambda + a)\, Q^{(i)}_{4n}(\overline{\lambda}, 0)\,. \tag{11}$$

Let $P^c_{H^{(i)}}(x + y\sqrt{m}) = S^{(i)}_{4n+1}(x, y) + \sqrt{m}\, T^{(i)}_{4n+1}(x, y)$ for $i = 1, 2, \ldots, 4n + 1$. Then using (5) we get

$$a(S^{(i)}_{4n+1}(\lambda, 0) - S^{(i)}_{4n+1}(\overline{\lambda}, 0)) = (a(\lambda + a) + mb^2)\, S^{(i)}_{4n}(\lambda, 0) + (a(\lambda + a) - mb^2)\, S^{(i)}_{4n}(\overline{\lambda}, 0)\,. \tag{12}$$

Since $Q_{4n+1}(\lambda, 0) - Q_{4n+1}(\overline{\lambda}, 0) = S^{(i)}_{4n+1}(\lambda, 0) - S^{(i)}_{4n+1}(\overline{\lambda}, 0)$ (see Proposition 12), by using (11), (12) and (10) we easily arrive at (8) and (9). □

Next, let $P^c_{G_i}(\lambda) = \sum_{k=0}^{4n} p_k \lambda^{4n-k}$ and let $P^c_{H_i}(\lambda) = \sum_{k=0}^{4n} q_k \lambda^{4n-k}$, understanding[1] that p_k and q_k are integers for $k = 0, 1, \ldots, 4n$. Let

$$P^c_{G_i}(\lambda - a) = \sum_{k=0}^{4n} x_k \lambda^{4n-k} \quad \text{and} \quad P^c_{H_i}(\lambda - a) = \sum_{k=0}^{4n} y_k \lambda^{4n-k}\,. \tag{13}$$

[1] We know that if $P^c_G(\lambda) = \sum_{k=0}^{n}(a_k + b_k\sqrt{m})\lambda^{n-k}$ then $P^c_{\overline{G}}(\lambda) = \sum_{k=0}^{n}(a_k - b_k\sqrt{m})\lambda^{n-k}$ where a_k and b_k are integral values for $k = 0, 1, \ldots, n$. In view of this it follows that $b_k = 0$ for $k = 0, 1, \ldots, n$ if and only if G is cospectral to its complement $\overline{G}$.

Besides, let $P^c_{G_i}(-\lambda-a)=\sum_{k=0}^{4n}\overline{x}_k\lambda^{4n-k}$ and let $P^c_{H_i}(-\lambda-a)=\sum_{k=0}^{4n}\overline{y}_k\lambda^{4n-k}$. With this notation we arrive at

PROPOSITION 19. *Let G be a walk regular graph of order $4n+1$ and degree $2n$. If G is cospectral to its complement $\overline{G}$ then $x_{2k}=y_{2k}$ for $k=0,1,\dots,2n$.*

PROOF. Replacing λ with $\lambda-a$ in (10) and keeping in mind that $Q^{(i)}_{4n}(\lambda,0)=P^c_{G_i}(\lambda)$ and $S^{(i)}_{4n}(\lambda,0)=P^c_{H_i}(\lambda)$ we have $P^c_{G_i}(\lambda-a)+P^c_{G_i}(-\lambda-a)=P^c_{H_i}(\lambda-a)+P^c_{H_i}(-\lambda-a)$. Then according to (13) we get $x_k+\overline{x}_k=y_k+\overline{y}_k$ for $k=0,1,\dots,4n$. It is not difficult to see that

$$x_k=\sum_{i=0}^{k}(-1)^{k+i}\binom{4n-i}{k-i}a^{k-i}p_i \quad\text{and}\quad \overline{x}_k=\sum_{i=0}^{k}(-1)^{i}\binom{4n-i}{k-i}a^{k-i}p_i\,, \tag{14}$$

for $k=0,1,\dots,4n$. Similarly, we have

$$y_k=\sum_{i=0}^{k}(-1)^{k+i}\binom{4n-i}{k-i}a^{k-i}q_i \quad\text{and}\quad \overline{y}_k=\sum_{i=0}^{k}(-1)^{i}\binom{4n-i}{k-i}a^{k-i}q_i\,, \tag{15}$$

for $k=0,1,\dots,4n$. Using (14) we easily obtain $x_{2k}+\overline{x}_{2k}=2x_{2k}$ for $k=0,1,\dots,2n$ and $x_{2k-1}+\overline{x}_{2k-1}=0$ for $k=1,2,\dots,2n$. Of course, we also have $y_{2k}+\overline{y}_{2k}=2y_{2k}$ for $k=0,1,\dots,2n$ and $y_{2k-1}+\overline{y}_{2k-1}=0$ for $k=1,2,\dots,2n$. Since $x_{2k}+\overline{x}_{2k}=y_{2k}+\overline{y}_{2k}$ we obtain the statement. □

PROPOSITION 20. *Let G be a walk regular graph of order $4n+1$ and degree $2n$. If G is cospectral to its complement $\overline{G}$ then*

$$mb^2y_{2k-1}=x_{2k+1}+4nax_{2k}\,, \tag{16}$$

for $k=1,2,\dots,2n$ understanding that $x_{4n+1}=0$.

PROOF. First, replacing λ with $\lambda-a$ in relations (8) and (9), by an easy calculation we obtain $mb^2(P^c_{H_i}(\lambda-a)-P^c_{H_i}(-\lambda-a))=(\lambda^2+4na\lambda)P^c_{G_i}(\lambda-a)-(\lambda^2-4na\lambda)P^c_{G_i}(-\lambda-a)$, which yields that

$$mb^2(y_k-\overline{y}_k)=x_{k+2}+4nax_{k+1}-(\overline{x}_{k+2}-4na\overline{x}_{k+1})\,,$$

for $k=0,1,\dots,4n$ understanding that $x_k=0$ and $y_k=0$ if $k\notin\{0,1,\dots,4n\}$. Since $y_{2k-1}+\overline{y}_{2k-1}=0$ and $x_{2k-1}+\overline{x}_{2k-1}=0$ for $k=1,2,\dots,2n$ and $x_{2k}=\overline{x}_{2k}$ for $k=0,1,\dots,2n$, we obtain the statement. □

PROPOSITION 21. *Let G be a walk regular graph of order $4n+1$ and degree $2n$. If G is cospectral to its complement $\overline{G}$ then*

$$((4n+1)-(2k+1))(4n+1)ax_{2k}+((4n+1)-2k)x_{2k+1}=0\,, \tag{17}$$

for $k=0,1,\dots,2n$.

PROOF. We note first that $\lambda_1 = 2n$ and $\lambda_i, -\lambda_i - 1$ for $i = 1, 2, \ldots, 2n$ are eigenvalues of G. Using Proposition 1 we find that $\lambda_1^c = 4na$ and $-a \pm (2\lambda_i + 1)b\sqrt{m}$ for $i = 1, 2, \ldots, 2n$ are the conjugate eigenvalues of G. Let $P_G^c(\lambda - a) = \sum_{k=0}^{4n+1} s_k \lambda^{(4n+1)-k}$ and let $P_{4n}(\lambda) = \sum_{k=0}^{4n} t_k \lambda^{4n-k}$ be a polynomial of degree $4n$ so that

$$P_G^c(\lambda - a) = (\lambda - (4n+1)a) \sum_{k=0}^{4n} t_k \lambda^{4n-k}. \tag{18}$$

Using the last relation we obtain $s_k = t_k - (4n+1)at_{k-1}$ for $k = 0, 1, \ldots, 4n+1$. Further, we note that $(4n+1)a$ and $\pm(2\lambda_i + 1)b\sqrt{m}$ are the roots of $P_G^c(\lambda - a)$. Of course, it means that $\pm(2\lambda_i + 1)b\sqrt{m}$ are the roots of $P_{4n}(\lambda)$. The roots of $P_{4n}(\lambda)$ are symmetric with respect to the zero point, which provides that $t_{2k-1} = 0$ for $k = 1, 2, \ldots, 2n$. So we obtain (i) $s_{2k} = t_{2k}$ and (ii) $s_{2k+1} = -(4n+1)at_{2k}$. Finally, since[2] $(P_G^c(\lambda - a))' = (4n+1) P_{G_i}^c(\lambda - a)$ we obtain $x_k = \frac{((4n+1)-k)s_k}{4n+1}$. In view of this and $(4n+1)as_{2k} + s_{2k+1} = 0$ (see (i) and (ii)) we obtain the statement. □

We shall now establish a better connection between the coefficients of polynomials $P_{G_i}^c(\lambda)$ and $P_{H_i}^c(\lambda)$ than that is given in relation (16), as follows. First, let $c_0, c_1, \ldots, c_{2n}$ be some real values so that

$$x_{2k} = c_k \left((-1)^k ((4n+1) - 2k) \binom{2n}{k} (4n+1)^{k-1} (mb^2)^k \right), \tag{19}$$

for $k = 0, 1, \ldots, 2n$. Since $x_0 = \binom{4n}{0} p_0$ and $x_2 = \binom{4n}{2} a^2 p_0 - \binom{4n-1}{1} ap_1 + \binom{4n-2}{0} p_2$ we[3] obtain $x_0 = 1$ and $x_2 = -2n(4n-1)mb^2$ (see (14)). Consequently, using (19) we find that $c_0 = 1$ and $c_1 = 1$. Further, using (17) and (19) we obtain

$$x_{2k+1} = c_k \left((-1)^{k+1} 4na \binom{2n-1}{k} (4n+1)^k (mb^2)^k \right), \tag{20}$$

for $k = 0, 1, \ldots, 2n$ understanding that $\binom{2n-1}{2n} = 0$. Using (19) and (20) we can easily see that (16) is transformed into

$$y_{2k-1} = c_k \big((-1)^k 4na (4n+1)^{k-1} (mb^2)^{k-1} \big) \Delta_{n,k},$$

where $\Delta_{n,k} = ((4n+1) - 2k)\binom{2n}{k} - (4n+1)\binom{2n-1}{k}$. Finally, since $\Delta_{n,k} = \binom{2n-1}{k-1}$ we easily arrive at

$$y_{2k+1} = c_{k+1} \left((-1)^{k+1} 4na \binom{2n-1}{k} (4n+1)^k (mb^2)^k \right), \tag{21}$$

[2]Let $P_{G_i}^c(x + y\sqrt{m}) = Q_{n-1}^{(i)}(x, y) + \sqrt{m} R_{n-1}^{(i)}(x, y)$ for $i = 1, 2, \ldots, n$. Since $(P_G^c(\lambda))' = \sum_{i=1}^n P_{G_i}^c(\lambda)$ we find that $\frac{\partial Q_n(x,y)}{\partial x} = \sum_{i=1}^n Q_{n-1}^{(i)}(x, y)$ and $\frac{\partial R_n(x,y)}{\partial x} = \sum_{i=1}^n R_{n-1}^{(i)}(x, y)$.

[3]We know that $a_0 + b_0\sqrt{m} = 1 + 0 \cdot \sqrt{m}$ and $a_1 + b_1\sqrt{m} = 0 + 0 \cdot \sqrt{m}$. Besides, we know that $a_2 = -\binom{n}{2}(a^2 + mb^2)$ and $b_2 = 2ab(\binom{n}{2} - 2e)$ where $e = e(G)$ is the number of edges of G (see [5]).

for $k = 0, 1, \ldots, 2n - 1$.

PROPOSITION 22. *Let G be a walk regular graph of order* $4n + 1$ *and degree* $2n$, *which is cospectral to its complement* $\overline{G}$. *Then G is strongly regular*[4] *if and only if* $c_k = 1$ *for* $k = 0, 1, \ldots, 2n$.

PROOF. Let us assume $c_k = 1$ for all values $k = 0, 1, \ldots, 2n$. Then using (20) and (21) we obtain $x_{2k+1} = y_{2k+1}$ for $k = 0, 1, \ldots, 2n-1$. In view of this and Proposition 19 it follows that $x_k = y_k$ for $k = 0, 1, \ldots, 4n$. Therefore, using (13) we find that $P^c_{G_i}(\lambda) = P^c_{H_i}(\lambda)$. Using Theorems 1 and 3 we obtain the proof. □

PROPOSITION 23. *Let G be a walk regular graph of order* $4n + 1$ *and degree* $2n$. *If G is cospectral to its complement* $\overline{G}$ *then*

$$P^c_G(\lambda - a) = (\lambda - (4n+1)a) \sum_{k=0}^{2n} c_k \left((-1)^k \binom{2n}{k} (4n+1)^k (mb^2)^k \right) \lambda^{4n-2k},$$

where $c_0 = 1$ *and* $c_1 = 1$.

PROOF. According to the proof of Proposition 21 we have $(4n + 1)x_k = ((4n + 1) - k)s_k$. Since $s_{2k} = t_{2k}$ and $t_{2k-1} = 0$ we obtain the statement using (18) and (19). □

PROPOSITION 24. *Let G be a walk regular graph of order* $4n+1$ *and degree* $2n$. *If G is cospectral to its complement* $\overline{G}$ *then*

$$Q^{(i)}_{4n}(-a, \lambda^+) = \sum_{k=0}^{2n} x_{2k} \lambda^{4n-2k} \quad \text{and} \quad S^{(i)}_{4n}(-a, \lambda^+) = \sum_{k=0}^{2n} y_{2k} \lambda^{4n-2k},$$

where $\lambda^+ = \frac{\lambda\sqrt{m}}{m}$.

PROOF. Since $P^c_{G_i}(\lambda - a) + P^c_{G_i}(-\lambda - a) = Q^{(i)}_{4n}(\lambda - a, 0) + Q^{(i)}_{4n}(-\lambda - a, 0)$ and $P^c_{H_i}(\lambda - a) + P^c_{H_i}(-\lambda - a) = S^{(i)}_{4n}(\lambda - a, 0) + S^{(i)}_{4n}(-\lambda - a, 0)$ we obtain the statement using Propositions 13 and 14. □

Since $x_{2k} = y_{2k}$ for $k = 0, 1, \ldots, 2n$ we have $Q^{(i)}_{4n}(-a, \lambda) = S^{(i)}_{4n}(-a, \lambda)$ for $i = 1, 2, \ldots, 4n + 1$. Since $Q^{(i)}_{4n}(-a, \lambda) = Q^{(j)}_{4n}(-a, \lambda)$ we also have $S^{(i)}_{4n}(-a, \lambda) = S^{(j)}_{4n}(-a, \lambda)$ for $i, j = 1, 2, \ldots, 4n + 1$. Besides, using Proposition 12 we find that $T^{(i)}_{4n+1}(-a, \lambda) = R_{4n+1}(-a, \lambda)$ for $i = 1, 2, \ldots, 4n + 1$.

PROPOSITION 25. *Let G be a walk regular graph of order* $4n+1$ *and degree* $2n$, *which is cospectral to its complement* $\overline{G}$. *Then* $P^c_{H_i}(\lambda) = P^c_{H_j}(\lambda)$ *for* $i, j = 1, 2, \ldots, 4n + 1$.

[4] In the meantime we have demonstrated that a walk regular graph G which is cospectral to its complement $\overline{G}$ is strongly regular if and only if $\Delta(G_i) = \Delta(H_i)$, where $\Delta(G)$ denotes the number of triangles of a graph G. In other words, it means that G is strongly regular if and only if $c_2 = 1$.

PROOF. It is sufficient to show $T_{4n}^{(i)}(-a,\lambda) = T_{4n}^{(j)}(-a,\lambda)$ for $i, j = 1, 2, \ldots, 4n+1$. Indeed, using (6) we get

$$T_{4n+1}^{(i)}(-a,\lambda) = \lambda\, S_{4n}^{(i)}(-a,\lambda) + \frac{mb^2}{a}\, T_{4n}^{(i)}(-a,\lambda)\,, \tag{22}$$

from which we obtain the statement. □

THEOREM 5. *Let G be a walk regular graph of order $4n+1$ and degree $2n$, which is cospectral to its complement $\overline{G}$. Then $\sigma^c(G_i) \setminus \mathcal{M}^c(G_i) = \sigma^c(H_i) \setminus \mathcal{M}^c(H_i)$ for $i = 1, 2, \ldots, 4n+1$.*

PROOF. First, replacing λ with $-a + \lambda\sqrt{m}$ in relation (7) we obtain the following system of equations[5]

$$Q_{4n+1}(-a,\lambda) = -(4n+1)^2\, a\, Q_{4n}^{(i)}(-a,\lambda) - (4n+1)\, m\lambda\, R_{4n}^{(i)}(-a,\lambda)\,; \tag{23}$$

$$R_{4n+1}(-a,\lambda) = (4n+1)\,\lambda\, Q_{4n}^{(i)}(-a,\lambda) + \frac{m\,\lambda^2}{a}\, R_{4n}^{(i)}(-a,\lambda)\,. \tag{24}$$

Second, since $T_{4n+1}^{(i)}(-a,\lambda) = R_{4n+1}(-a,\lambda)$ and $S_{4n}^{(i)}(-a,\lambda) = R_{4n}^{(i)}(-a,\lambda)$ relation (22) is transformed into

$$R_{4n+1}^{(i)}(-a,\lambda) = \lambda\, Q_{4n}^{(i)}(-a,\lambda) + \frac{mb^2}{a}\, T_{4n}^{(i)}(-a,\lambda)\,. \tag{25}$$

We shall now demonstrate that G_i and H_i have the same number of main eigenvalues. Indeed, let $x \in \sigma_Q^c(G_i) \cap \sigma_R^c(G_i)$. Since $\sigma_Q^c(G_i) \cap \sigma_R^c(G_i) \subseteq \sigma_R^c(G)$ (see (24)) we get $R_{4n+1}(-a,x) = 0$. Using (25) we obtain $x \in \sigma_Q^c(H_i) \cap \sigma_R^c(H_i)$. Conversely, let $x \in \sigma_Q^c(H_i) \cap \sigma_R^c(H_i)$. Then Using (24) and (25) we get $R_{4n+1}(-a,x) = 0$ and $R_{4n}^{(i)}(-a,x) = 0$, which proves that $\sigma_Q^c(G_i) \cap \sigma_R^c(G_i) = \sigma_Q^c(H_i) \cap \sigma_R^c(H_i)$. In view of this and Theorem 4 we obtain $|\sigma^c(G_i) \setminus \mathcal{M}^c(G_i)| = |\sigma^c(H_i) \setminus \mathcal{M}^c(H_i)|$, which proves the assertion. According to [6], there exists a one-to-one correspondence between $\lambda^c \in \sigma^c(G) \setminus \mathcal{M}^c(G)$ and $x \in \sigma_Q^c(G) \cap \sigma_R^c(G)$, which completes the proof. □

THEOREM 6. *Let G be a walk regular graph of order $4n+1$ and degree $2n$, which is cospectral to its complement $\overline{G}$. Then G is strongly regular if and only if H_i has exactly two main eigenvalues for $i = 1, 2, \ldots, 4n+1$.*

PROOF. According to Theorem 5 the vertex deleted subgraphs G_i also have exactly two main eigenvalues for $i = 1, 2, \ldots, 4n+1$. Using Theorem 2 we obtain the statement. □

[5] Using (23) and (24) we easily obtain $\lambda Q_{4n+1}(-a,\lambda) + (4n+1)aR_{4n+1}(-a,\lambda) = 0$. The same relation could be obtained by using the equality $(\lambda + 4na + 2a)P^c_{\overline{G}}(\lambda) = (-1)^{4n+1}(\lambda - 4na)P^c_G(-\lambda - 2a)$.

Using Proposition 1 we obtain that $\sigma(G_i) \setminus \mathcal{M}(G_i) = \sigma(H_i) \setminus \mathcal{M}(H_i)$. Of course, we[6] also have $\sigma^*(G_i) \setminus \mathcal{M}^*(G_i) = \sigma^*(H_i) \setminus \mathcal{M}^*(H_i)$ for $i = 1, 2, \ldots, 4n + 1$, where $\mathcal{M}^*(G)$ is the set of all Seidel main eigenvalues of a graph G. Finally, since G_i and H_i are switching equivalent, we arrive at

PROPOSITION 26. *Let G be a walk regular graph of order $4n+1$ and degree $2n$, which is cospectral to its complement $\overline{G}$. Then $\mathcal{M}^*(G_i) = \mathcal{M}^*(H_i)$ for $i = 1, 2, \ldots, 4n + 1$.*

References

[1] R. J. ELZINGA,Strongly regular graphs: values of λ and μ for which there are only finitely many feasible (v, k, λ, μ), Electronic Journal of Linear Algebra ISSN 1081–3810, A publication of the International Linear Algebra Society, Volume 10, pp. 232–239, October 2003.

[2] M. LEPOVIĆ, On spectral complementary graphs, J. Math. (Novi Sad) **30**, No. 3 (2000), 83–91.

[3] M. LEPOVIĆ, On conjugate adjacency matrices of a graph, Discrete Math. **307** (2007), 730–738.

[4] M. LEPOVIĆ, The conjugate formal product of a graph, Applicable Analysis and Discrete Mathematics, **1** (2007), 427–437.

[5] M. LEPOVIĆ, On conjugate characteristic polynomial of a graph, J. Appl. Math. and Computing **28** (2008), 473–483. DOI:10.1007/s12190-008-0120-x

[6] M. LEPOVIĆ, Some results on walk regular and strongly regular graphs, New Zealand Journal of Mathematics, **38** (2008), 161–169.

[7] M. LEPOVIĆ, The conjugate characteristic polynomial of some compound graphs, International Journal of Computer Mathematics, in press

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[6]We know that if $\lambda \in \sigma(G) \setminus \mathcal{M}(G)$ then $-2\lambda - 1 \in \sigma^*(G) \setminus \mathcal{M}^*(G)$. In view of this it follows that $\sigma^*(G_i) \setminus \mathcal{M}^*(G_i) = \sigma^*(H_i) \setminus \mathcal{M}^*(H_i)$.