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On Asymptotics of a Second Order Linear ODE with a Turning-regular Singular Point

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Abstract. The O.D.E. studied is of second order of the Schroedinger type constaining a small parameter, considered as the Planck constant, and it has a turning point and a regular singular point both situated at the origin. A case where the turning point and the singular point coincide has a very typical figure of the existence region (canonical domain) of the asymptotic solutions. Several cases of different order of the turning point are shown.

1. Introduction

1.1. The differential equation studied in this paper is

$$\varepsilon^2 \frac{d^2 y}{dx^2} - \left(x^m - \frac{\varepsilon}{x}\right) y = 0 \quad (x, y \in \mathbf{C};\ \ 0 < |x| \le x_0\,,\ 0 < \varepsilon \le \varepsilon_0;\ \ m \in \mathbf{N})\,, \tag{1.1}$$

where x_0 and ε_0 are constants. This differential equation has a turning point and a regular singular point, both of which are situated at the origin. We do not have a one step method to obtain an asymptotic approximation to the solution as $\varepsilon \to 0$ in the whole domain $0 < |x| \le x_0$, so we split (1.1) into two different types of the differential equation whose solutions are obtained separately (§2, 3) and then we connect them by what we call a matching matrix in a common domain as shown in §5. The differential equation (1.1) is represented in the matrix form:

$$\varepsilon \frac{dY}{dx} = \begin{bmatrix} 0 & 1 \\ x^m - \dfrac{\varepsilon}{x} & 0 \end{bmatrix} Y\,, \tag{1.2}$$

where Y is a 2-by-2 matrix. (1.2) has the first two terms of

$$\varepsilon \frac{dY}{dx} = \left\{ \begin{bmatrix} 0 & 1 \\ x^m & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 \\ -1/x & 0 \end{bmatrix} + O(\varepsilon^2) \right\} Y\,. \tag{1.3}$$

If $O(\varepsilon^2)$ is small for x and ε, then a solution of (1.3) is a regular perturbation of one of (1.2) with respect to a small ε. In this sense (1.2) is dominant to (1.3)

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1.2. Our aim is to get two types of the formal solution of (1.1) and match them as $\varepsilon \to 0$. In order to do it, analyzing Stokes curve configuration is important (§4). The case of $m = 1$ was studied in Nakano [6] but it is not complete because it is analyzed on one sheet of complex plane.

REMARK: In the last section, we correct a mistake in the former paper Nakano [8].

2. The outer equation

2.1. The differential equation (1.2) is written in the form

$$x^{(m+2)/2}(x^{-m-1}\varepsilon)\frac{dZ}{dx} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (x^{-m-1}\varepsilon)\begin{bmatrix} 0 & 0 \\ -1 & -\frac{m}{2}x^{m/2} \end{bmatrix}\right) Z\,, \tag{2.1}$$

where $Y := \operatorname{diag}\left[1,\ x^{m/2}\ \right] Z$. This differential equation is called an *outer equation* of (1.2) and it should be analyzed when $x^{-m-1}\varepsilon \to 0$, that is, for x in a sub-domain $K\varepsilon^{1/(m+1)} \leq |x| \leq x_0$ (K = large constant, cf. §3) of the whole domain $0 < |x| \leq x_0$. By the diagonalization transformation:

$$Z := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (x^{-m-1}\varepsilon)\begin{bmatrix} 0 & a^+ \\ a^- & 0 \end{bmatrix} + (x^{-m-1}\varepsilon)^2\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} + \cdots\right) W\,, \tag{2.2}$$

the differential equation (2.1) becomes

$$\begin{aligned} x^{(m+2)/2}(x^{-m-1}\varepsilon)\frac{dW}{dx} = \Bigg(&\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (x^{-m-1}\varepsilon)\begin{bmatrix} 2a^- & 0 \\ 0 & -2a^+ \end{bmatrix} \\ &+ \left(x^{-m-1}\varepsilon\right)^2\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} + \cdots\Bigg) W\,, \end{aligned} \tag{2.3}$$

which is of a diagonal form. The symbols $a^\pm$ are given by

$$a^\pm := \frac{1}{4}\left(-1 \pm \frac{m}{2}x^{m/2}\right) \tag{2.4}$$

and * denotes some constant. A leading term of the formal solution of (2.3) is given by

$$W = \frac{1}{x^{m/4}}\exp\left(\alpha\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)\left(\alpha := \frac{2}{m+2}\frac{1}{\varepsilon}x^{(m+2)/2} + \frac{1}{m}\frac{1}{x^{m/2}}\right). \tag{2.5}$$

2.2. Thus we can get the formal solution $\tilde{Y}_{out}$ of (1.2) which is called a *formal outer solution* of (1.2).

THEOREM 2.1. *The formal outer solution of* (1.2) *is given by*

$$\tilde{Y}_{out} := x^{m/4\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{\alpha\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \tag{2.6}$$

or

$$\tilde{Y}_{out} := \begin{bmatrix} x^{-m/4} & 0 \\ 0 & x^{m/4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{bmatrix}, \tag{2.7}$$

which is the leading term of an asymptotic expansion of a true outer solution of (1.2), *namely, there exists a true outer solution* Y_{out} *such that*

$$Y_{out} \sim \tilde{Y}_{out} \quad as \;\; x^{-m-1}\varepsilon \to 0 \tag{2.8}$$

in an outer domain, *i.e.*, *in a sector*

$$\mathsf{S}_m := \left\{ x \;\middle|\; K\varepsilon^{1/(m+1)} \le |x| \le x_0, \;\; -\frac{\pi}{m+2} < \arg x < \frac{3\pi}{m+2} \right\}. \tag{2.9}$$

The property (2.8) is a well known fact (cf. Wasow [12]). Notice that the arguments of x in the above sector S_m correspond to the arguments of the boundary of a canonical domain C_m^{∞} (cf. §3, 4). $\tilde{Y}_{out}$ is an outer *WKB approximation* to the solution of (1.2) of a matrix form.

3. The inner equation

3.1. We reduce (1.2) to another form in the complement ($0 < |x| < K\varepsilon^{1/(m+1)}$) of the sub-domain $K\varepsilon^{1/(m+1)} \le |x| \le x_0$. Let $x := \varepsilon^{1/(m+1)}t$ (a *stretching transform*) and $Y := \mathrm{diag}[1,\; \varepsilon^{m/2(m+1)}]U$, then (1.2) becomes a form such as

$$\varepsilon^{m/2(m+1)} \frac{dU}{dt} = \begin{bmatrix} 0 & 1 \\ p(t) & 0 \end{bmatrix} U \quad \left(p(t) := t^m - \frac{1}{t} \right), \tag{3.1}$$

which has a very similar form to (1.2) but lacks a term of ε and is called an *inner equation* of (1.2). The origin $t = 0$ is a regular singular point and zeros of $p(t)$ are turning points of (3.1), which are called *secondary turning points* of (1.2). Transforming (3.1) by

$$U := \left(\begin{bmatrix} 1 & -1 \\ \sqrt{p} & \sqrt{p} \end{bmatrix} + \frac{p'}{4p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \varepsilon^{m/2(m+1)} \right) V,$$

it becomes

$$\begin{aligned} \varepsilon^{m/2(m+1)} \frac{dV}{dt} = \Bigg(&\begin{bmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{bmatrix} - \frac{p'}{4p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \varepsilon^{m/2(m+1)} \\ &+ \frac{7p' - 4pp''}{32p^{5/2}} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \varepsilon^{1/(m+1)} + \cdots \Bigg) V. \end{aligned} \tag{3.2}$$

The formal solution of (3.2) is given by

$$V = \frac{1}{p^{1/4}} e^{\beta \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]} + \cdots \quad \left(\beta := \frac{1}{\varepsilon^{m/2(m+1)}} \int^t \sqrt{p}\, dt \right). \tag{3.3}$$

3.2. Thus we can get the formal solution $\tilde{Y}_{in}$ of (1.2) which is called a *formal inner solution* of (1.2).

THEOREM 3.1. *The formal inner solution of* (1.2) *is given by*

$$\tilde{Y}_{in} := \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{m/2(m+1)} \end{bmatrix} p^{1/4\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{\beta\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]} \tag{3.4}$$

or

$$\tilde{Y}_{in} := \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{m/2(m+1)} \end{bmatrix} \begin{bmatrix} p^{-1/4} & 0 \\ 0 & p^{1/4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{bmatrix}, \tag{3.5}$$

which is the leading term of the asymptotic expansion of a true inner solution of (1.2), *namely, there exists a true inner solution Y_{in} of* (1.2) *such that*

$$Y_{in} \sim \tilde{Y}_{in} \quad as \quad \begin{cases} \varepsilon \to 0 \\ t \to \infty \end{cases} \tag{3.6}$$

in a canonical domain

$$\mathsf{C}_m^{\infty} := \left\{ t \,\middle|\, 0 < |t| < \infty, -\frac{\pi}{m+2} < \arg t < \frac{3\pi}{m+2} \quad near \quad t = \infty \right\}. \tag{3.7}$$

The canonical domain C_m^{∞} comes from an inner domain $\mathsf{C}_m^K := \{t \,|\, 0 < |t| < K, \ -\pi/(m+2) < \arg t < 3\pi/(m+2) \ \ near \ \ t = \infty\}$ which is a complement of S_m in the x-plane. The inner equation (3.1) is to be originally analyzed in C_m^K, but it should be analyzed in an 'extended' domain C_m^{∞} in order to match Y_{out} and Y_{in} (cf. §5). C_1^{∞} and C_5^{∞} are shown in Figure 4-1 and 4-2, respectively, and an exact definition of C_m^{∞} and how to construct it are given in §4.

The property (3.6) is called the *double asymptotic property* (Fedoryuk [2]). $\tilde{Y}_{in}$ is an inner *WKB approximation* to the solution of (1.2) of a matrix form.

4. A topology of Stokes curves and a canonical domain

4.1. Let us study a topology of Stokes curves for the equation (3.1). A *Stokes curve* for (3.1) is, by definition, a set of points t's given by

$$\{t \,|\, \Re\xi(a,t) = 0\}, \tag{4.1}$$

where

$$\xi(a,t) := \int_a^t \sqrt{p}\, dt \quad (p(a) = 0). \tag{4.2}$$

An *anti-Stokes curve* of (3.1) is defined by an equation

$$\Im\xi(a,t) = 0 \quad (p(a) = 0). \tag{4.3}$$

Those curves are particular level curves defined by $\Re\xi(a,t) = \text{const.}$ and $\Im\xi(a,t) = \text{const.}$, namely, they are the curves of level zero.

The general properties of Stokes curve configuration for a general rational function $p(t)$ are well known in Evgrafov-Fedoryuk [1], Fedoryuk [2] and Nakano [7]-[8]), and Hukuhara [5] and Paris-Wood [10] for local Stokes curves.

4.2. The outline of the Stokes curve configuration for (3.1) is as follows:

THEOREM 4.1. *The Stokes and anti-Stokes curves for* (3.1) *possess the following properties*:

(i) *The origin $t = 0$ is a regular singular point from which one Stokes curve and one anti-Stokes curve emerge.*

When m =odd, two lines $t < -1$, $0 < t < 1$ on the real axis are Stokes curves, and two lines $-1 < t < 0$, $1 < t$ are anti-Stokes curves.

When m =even, a line $0 < t < 1$ on the real axis is a Stokes curve and two lines $t < 0$, $1 < t$ on the real axis are anti-Stokes curves.

(ii) *The point at infinity $t = \infty$ is an irregular singular point and $m + 3$ Stokes curves emerge from* (*or tend to*) *$t = \infty$ at angles $\pm\frac{\pi}{m+2}$, $\pm\frac{3\pi}{m+2}$, $\pm\frac{5\pi}{m+2}, \dots$. Also, $m + 3$ anti-Stokes curves emerge from* (*or tend to*) *$t = \infty$ at middle angles between neighboring two Stokes curves.*

(iii) *All the zero $t = e^{2k\pi i/(m+1)}$ $(k = 0, 1, 2, 3, \dots)$ of $p(t)$ are situated on the unit circle $|t| = 1$ symmetrically with respect to the real axis and they are simple secondary turning points. From a turning point $t = e^{2k\pi i/(m+1)}$ three Stokes curves emerge at angles $\pm\frac{\pi}{3} + \frac{4k\pi}{3(m+1)}$, $\pi + \frac{4k\pi}{3(m+1)}$. Three anti-Stokes curves emerge from every zero at middle angles between neighboring two Stokes curves.*

(iv) *There is a Stokes curve connecting $\alpha := e^{2k\pi i/(m+1)}$ and $\alpha^* := e^{2\pi i - 2k\pi i/(m+1)}$. This Stokes curve crosses the anti-Stokes curve $-1 < t < 0$ and can not cross lines $t < -1$ and $0 < t < 1$.*

(v) *There is an anti-Stokes curve connecting $\alpha := e^{2k\pi i/(m+1)}$ and $\bar{\alpha} := e^{-2k\pi i/(m+1)}$. This anti-Stokes curve crosses only the Stokes curve $0 < t < 1$.*

(vi) *Any Stokes curve* (*resp., any anti-Stokes curve*) *can not cross other Stokes curves* (*resp., anti-Stokes curves*) *except for at turning points or at $t = \infty$.*

(vii) *A Stokes curve and an anti-Stokes curve emerging from a turning point tend to another turning point or to $t = \infty$.*

(viii) *Any Stokes curve and any anti-Stokes curve can not cross itself.*

(ix) *When a point $t = \alpha$ is a turning point or a simple pole, there are no* (*sums of*) *Stokes or anti-Stokes curves homotopic to a circle around α. Therefore there are no circle-like Stokes or anti-Stokes curves for* (3.1).

PROOF. (i) Near $t = 0$, we see that $p(t) \sim -1/t$. Then

$$\xi(0,t) := \int_0^t \sqrt{p(t)}\, dt \sim \int_0^t i t^{-1/2}\, dt = 2\sqrt{r} e^{i(\theta+\pi)/2} \quad (t := re^{i\theta})\,.$$

From the equation $\Re\xi(0,t)=0$, we get $\theta=0,\pm 2\pi,\pm 4\pi,\dots$. Therefore one Stokes curve emerges from $t=0$ at an angle 0. Similarly we get $\theta=\pm\pi$ from the equation $\Im\xi(0,t)=0$, so one anti-Stokes curve emerges from $t=0$ at an angle π.

When $m=$ odd, we see that $p(t)<0$ for t satisfying $t<-1$ or $0<t<1$. Then the value of $\sqrt{p(t)}$ is pure imaginary, so two lines $t<-1$ and $0<t<1$ are Stokes curves. On the other hand, two lines $-1<t<0$ and $1<t$ are anti-Stokes curves because $p(t)>0$ and $\int^t \sqrt{p(t)}dt$ takes real values for t on these lines. In a similar way to this case, we can get the result for the even m's case.

(ii) For $t\sim\infty$, we can appreciate

$$\xi := \int^t \sqrt{p(t)}dt \sim \int^t t^{m/2}dt = \frac{2}{m+2}t^{(m+2)/2}\,.$$

Putting $t:=re^{i\theta}$, the equation $\Re\xi=0$ induces an equation $\cos(m+2)\theta/2=0$, from which we get $\theta=\pm\pi/(m+2),\pm 3\pi/(m+2),\pm 5\pi/(m+2),\dots$. Then Stokes curves emerge from (or go to) the infinity at these angles. One value of these θ's becomes π for the case of $m=$odd, then the line $t<-1$ is a Stokes curve (cf. (i)).

(iii) From the equation $t^{m+1}=1$ with $t:=re^{i\theta}$, we get

$$r=1,\ \theta=\frac{2k\pi}{m+1}\quad (k=0,1,2,\dots,m)\,.$$

Then we can get the $m+1$ zeros of $p(t)$ which are simple turning points of (3.1). At the zero $t:=e^{2k\pi i/(m+1)}$, holds the following approximation

$$p(t)\sim(m+1)\tau e^{-4k\pi i/(m+1)}\quad\text{as }\ \tau\to 0\,,$$

where $t:=\tau+e^{2k\pi i/(m+1)}$. Then,

$$\xi=\int^t\sqrt{p(t)}dt\sim\frac{2}{3}\sqrt{m+1}\ e^{-2k\pi i/(m+1)}\tau^{3/2}\quad\text{as }\ \tau\to 0\,.$$

By putting $\tau:=re^{i\phi}$, the equation $\Re(e^{-2k\pi i/(m+1)}\tau^{3/2})=0$ induces the equation

$$\cos(-2k\pi/(m+1)+3\phi/2)=0\,,$$

from which we get different three values of $\phi=\pm\pi/3+4k\pi/3(m+1),\ \pi+4k\pi/3(m+1)$. Thus three Stokes curves can emerge from the turning point $t:=e^{2k\pi i/(m+1)}$ at these three angles.

Similarly, the emerging angles of the anti-Stokes curves from the same turning point are given by the equation $\Im(e^{-2k\pi i/(m+1)}\tau^{3/2})=0$, i.e., $\sin(-2k\pi/(m+1)+3\phi/2)=0$.

(iv) Let us consider the complex plane on which there exist two points α and α^*. Define ξ such that

$$\xi(\alpha^*,\alpha):=\int_{\alpha^*}^{\alpha}\sqrt{p(t)}\,dt\,.$$

Since the integrand is an analytic function, we integrate it along the segments as follows:

$$\begin{aligned}\xi(\alpha^*,\alpha) &= \left(\int_{\alpha^*}^{0} + \int_{0}^{\alpha}\right)\sqrt{p(t)}\,dt \\ &= ie^{-k\pi i/(m+1)}\int_0^1 \sqrt{\frac{1-r^{m+1}}{r}}\,dr + i\,e^{k\pi i/(m+1)}\int_0^1 \sqrt{\frac{1-r^{m+1}}{r}}\,dr \\ &= i\,2\cos\frac{k\pi}{m+1}\int_0^1\sqrt{\frac{1-r^{m+1}}{r}}\,dr \in i\mathbf{R}\,.\end{aligned}$$

Therefore, we see that the equation $\Re\xi(\alpha,\alpha^*) = 0$ induces the result such that there exists a Stokes curve connecting two points α and α^*. This Stokes curve can not cross the line $0 < t < 1$ because this line is a Stokes curve (cf. (vi)).

Also,we can show that this Stokes curve can not cross a line $t < -1$ as follows: When m is odd, the line $t < -1$ is a Stokes curve then it can not cross this line.

When m is even, a line $t < 0$ is an anti-Stokes curve so it may cross the line $t < -1$, but it can not cross this line from a short calculation such as

$$\Re\xi(\alpha,-\beta) = -\sin\frac{k\pi}{m+1}\int_0^1\sqrt{\frac{1-r^{m+1}}{r}}\,dr - \int_0^{\beta}\sqrt{\frac{1+r^{m+1}}{r}}\,dr \neq 0 \quad (\beta > 1)\,.$$

We integrate $\xi(\alpha,-\beta)$ along the segments:

$$\xi(\alpha,-\beta) := \left(\int_{\alpha}^{0} + \int_{0}^{-\beta}\right)\sqrt{p}\,dt\,.$$

Thus $-\beta \notin \{t \mid \Re\xi(\alpha,t) = 0\}$ which means that there is no Stokes curve from α to $-\beta$. Similarly there exists no Stokes curve connecting two points α^* and $-\beta$. Therefor any Stokes curve does not exist connecting two secondary turning points α and α^* via a point on the line $t < -1$.

(v) Let us consider the complex plane on which there exist two points α and $\bar{\alpha}$. Define ξ such as

$$\xi(\bar{\alpha},\alpha) := \int_{\bar{\alpha}}^{\alpha}\sqrt{p(t)}\,dt\,.$$

Integrating the ξ along the segments we get

$$\begin{aligned}\xi(\bar{\alpha},\alpha) &= \left(\int_{\bar{\alpha}}^{0} + \int_{0}^{\alpha}\right)\sqrt{p(t)}\,dt \\ &= -i\,e^{-k\pi i/(m+1)}\int_0^1\sqrt{\frac{1-r^{m+1}}{r}}\,dr + i\,e^{k\pi i/(m+1)}\int_0^1\sqrt{\frac{1-r^{m+1}}{r}}\,dr \\ &= -2\sin\frac{k\pi}{m+1}\int_0^1\sqrt{\frac{1-r^{m+1}}{r}}\,dr \in \mathbf{R}\,.\end{aligned}$$

Therefore, we see that the equation $\Im\xi(\bar{\alpha}, \alpha) = 0$ induces the result such that there exists an anti-Stokes curve connecting two points α and $\bar{\alpha}$. This anti-Stokes curve can not cross the lines $-1 < t < 0$ and $0 < t$ because they are anti-Stokes curves (cf. (vi)).

The conditions (vi) to (ix) are well-known fact (Fedoryuk [2]). Q.E.D.

4.3. A *canonical domain* is, by definition, a simply connected domain bounded by Stokes curves on the t-plane (or the Riemann surface) which is mapped by $\xi = \xi(a, t)$ onto the whole ξ-plane except several slits. There are several canonical domains and we show one of them for a case of $m = 1$, i.e., C_1^{∞} (a shaded part of Figure 4-1). We should notice that arguments of the boundary, namely, $\arg(\lim_{\bar{l}_2 \ni t \to \infty} t) = -\pi/3$ and $\arg\left(\lim_{l_3 \ni t \to \infty} t\right) = \pi$, correspond to ones of the outer domain $\mathsf{S}_1 := \{x \mid K\varepsilon^{1/2} < |x| \le x_0,\ -\pi/3 < \arg x < \pi\}$ (cf. (2.9)).

The canonical domain C_1^{∞} is mapped by the function ξ defined by the integral $\xi = \xi(1, t)$ onto the whole ξ-plane, whose coordinate is $(\Re\,\xi, \Im\,\xi)$, except for two slits (Figure 4-1$'$). In Figure 4-1$'$, we use the same characters for the images, for instance, l_0 in Figure 4-1$'$ is an image of l_0 in Figure 4-1. The symbol l_0^* represents an image of l_0 on the second sheet.

In the canonical domain where a branch of $\sqrt{p}$ is determined in that manner, its real part takes positive values for $t > 1$. Branch cuts are situated along the Stokes curve $l_3 :\ \Re t < -1$ and the Stokes curve $l_0 :\ 0 < \Re t < 1$. Notice the asymptotic property such that $\Re\,\xi \to +\infty$ as $t \to \infty$ along the anti-Stokes curve L_0 and $\Re\,\xi \to -\infty$ as $t \to \infty$ along the anti-Stokes curve L_3 or L_1. The path P_+ is defined by a line along L_0, and the path P_- is defined by a line along L_3 or L_1. We can adopt $P_\pm$ as paths of integration of the integral (4.2) in order to show asymptoticity (3.6). These paths are also used when the matching matrix is calculated (cf. §5). Similarly, we can construct a canonical domain C_m^{∞} for other m case.

Next, we show two examples of the canonical domain: The first is C_5^{∞} and it is available for our matching (Figure 4-2, 2$'$). The second is the shaded part of Figure 4-3. It is however not available for the matching because the corresponding outer solution has not so large sector of asymptotics.

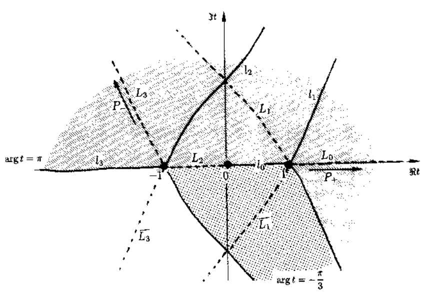

FIGURE 4-1. A canonical domain C_1^{∞}

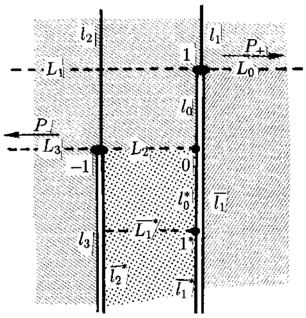

FIGURE 4-1$'$. An image of a canonical domain C_1^{∞}

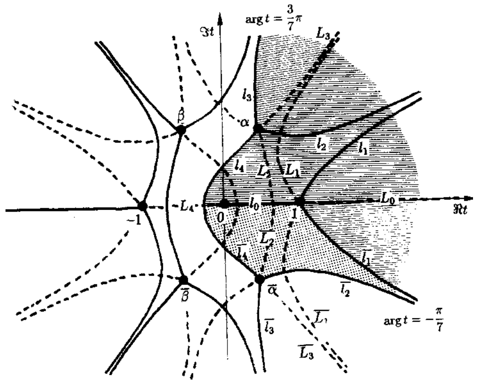

FIGURE 4-2. A canonical domain C_5^{∞}

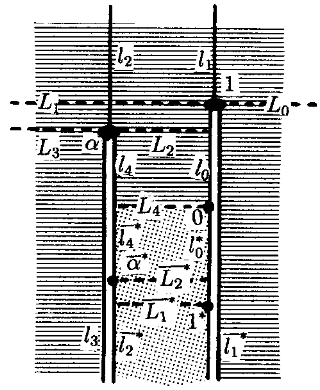

FIGURE 4-2′

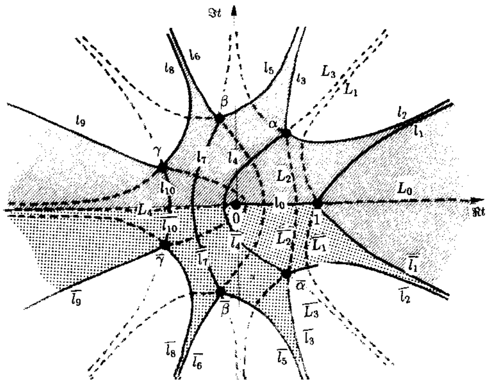

FIGURE 4-3. A canonical domain ($m = 6$)

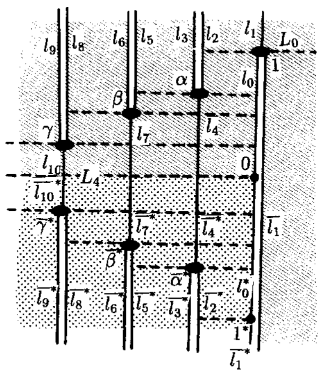

FIGURE 4-3′

5. A matching matrix and the main theorem

5.1. A matching matrix $\mathsf{M} := [\mathsf{m}_{ij}]$ between Y_{out} and Y_{in} is defined by the equality $Y_{out}\mathsf{M} = Y_{in}$, i.e.,

$$\tilde{Y}_{out}\mathsf{M} \sim \tilde{Y}_{in} \quad (\varepsilon \to 0)\,. \tag{5.1}$$

THEOREM 5.1. *The matching matrix defined by* (5.1) *is given by*

$$\mathsf{M} \sim \varepsilon^{m/4(m+1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\varepsilon \to 0)\,. \tag{5.2}$$

PROOF. From (5.1) we obtain

$$
\begin{aligned}
&\begin{bmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{bmatrix} \begin{bmatrix} \mathsf{m}_{11} & \mathsf{m}_{12} \\ \mathsf{m}_{21} & \mathsf{m}_{22} \end{bmatrix} \begin{bmatrix} e^{-\beta} & 0 \\ 0 & e^{\beta} \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x^{m/4} & 0 \\ 0 & x^{-m/4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{m/2(m+1)} \end{bmatrix} \begin{bmatrix} p^{-1/4} & 0 \\ 0 & p^{1/4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} .
\end{aligned} \tag{5.3}
$$

After a short calculation, asymptotics of the right hand side of (5.3) is given by

$$
\begin{aligned}
&\frac{1}{2} \begin{bmatrix} x^{m/4}p^{-1/4} + \varepsilon^{m/2(m+1)}x^{-m/4}p^{1/4} & -x^{m/4}p^{-1/4} + \varepsilon^{m/2(m+1)}x^{-m/4}p^{1/4} \\ -x^{m/4}p^{-1/4} + \varepsilon^{m/2(m+1)}x^{-m/4}p^{1/4} & x^{m/4}p^{-1/4} + \varepsilon^{m/2(m+1)}x^{-m/4}p^{1/4} \end{bmatrix} \\
&\qquad \sim \varepsilon^{m/4(m+1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\varepsilon \to 0) .
\end{aligned} \tag{5.4}
$$

The components of the left hand side of (5.3) have the form

$$\mathsf{m}_{ij}\, e^{\pm(\alpha+\beta)} \quad \text{or} \quad \mathsf{m}_{ij}\, e^{\pm(\alpha-\beta)} .$$

Between α and β holds an equality:

$$\beta = \alpha - \frac{1}{4(3m+2)} \varepsilon^{-1/2} t^{-(3m+2)/2} + \cdots .$$

Then we see

$$\alpha + \beta \sim \eta^{(m+2)/2} \varepsilon^{-(3m+2)/4(m+1)} , \quad \alpha - \beta = O(\varepsilon^{1/4}) \quad (\varepsilon \to 0) , \tag{5.5}$$

where

$$x := \eta \varepsilon^{1/2(m+1)} , \quad t := \eta \varepsilon^{-1/2(m+1)} \quad (|\eta| = 1) . \tag{5.6}$$

The arguments x and t thus defined belong to a common part of the t-domain C_m^{∞} and the x-domain S_m, and we see that $|x| \to 0$ and $|t| \to \infty$ as $\varepsilon \to 0$. The new parameter η is defined soon later. Two relations (5.3) and (5.4) give four asymptotic relations:

$$\begin{cases} \mathsf{m}_{11}\, e^{\alpha-\beta} \sim \varepsilon^{m/4(m+1)} , & \mathsf{m}_{12}\, e^{\alpha+\beta} \sim 0 , \\ \mathsf{m}_{21}\, e^{-(\alpha+\beta)} \sim 0 , & \mathsf{m}_{22}\, e^{-(\alpha-\beta)} \sim \varepsilon^{m/4(m+1)} \end{cases} \quad (\varepsilon \to 0) .$$

From two diagonal relations we get

$$\mathsf{m}_{11} \sim \varepsilon^{m/4(m+1)} , \quad \mathsf{m}_{22} \sim \varepsilon^{m/4(m+1)} \quad (\varepsilon \to 0) ,$$

because we see that $\exp\{\pm(\alpha - \beta)\} \to 1$ as $\varepsilon \to 0$ due to (5.5).

Let the variable t (resp., x) vary in the canonical domain C_m^{∞} (resp., in the sector S_m). If we choose "η" of (5.6) as a unit vector on the path P_+ for a large t, then $\Re\,(\alpha + \beta) \to +\infty$ as $\varepsilon \to 0$. Thus we must choose $\mathsf{m}_{12} \sim 0$ $(\varepsilon \to 0)$. Similarly, if we choose η as a unit vector on the path P_- for a large t, then we see that $\Re\,(-(\alpha + \beta)) \to +\infty$ as $\varepsilon \to 0$. Thus we must choose $\mathsf{m}_{21} \sim 0$. There exist certainly two paths $P_\pm$ in C_m^{∞} because C_m^{∞} can be mapped onto the whole ξ-plane (except several vertical slits) by $\xi = \xi(a,t)$ $(p(a) = 0)$ and the inverse images of, say, the real and the imaginary axes contain routes for $P_{\pm}$. Q.E.D.

The above matching matrix M has been calculated in one of sectors, i.e., in the sector $-\pi/(m+2) < \arg x < 3\pi/(m+2)$ of the x-plane. If we want to match in other sector we must choose other canonical domain in the t-plane corresponding to the sector. Otherwise, we can use the Furry's rule (Fedoryuk [2], Furry [4]) around a secondary turning point, which is simple order, in order to continue a solution to another canonical domain.

5.2. Summing up the results so far, we can get the main theorem as follows:

THE MAIN THEOREM. *The differential equation* (1.1) (*or* (1.2)) *possesses a formal outer solution* (*an outer WKB approximation*) (2.6) (*or* (2.7)) *which is an asymptotic expansion of the true outer solution in a sector* (*an outer domain*) (2.9) *as* $x^{-m-1}\varepsilon \to 0$.

The differential equation (1.1) *possesses a formal inner solution* (*an inner WKB approximation*) (3.4) (*or* (3.5)) *which is an asymptotic expansion of the true inner solution in a canonical domain* (*an inner domain*) *as* $\varepsilon \to 0$ *or* $t \to \infty$.

Several concrete canonical domains are given in Figures 4-1 $\sim$ 3. *They are bounded by Stokes curves whose general abstract property for the inner differential equation* (3.1) *is given in Theorem* 4.1.

The arguments of the outer domain's boundary are $-\pi/(m+2)$ *and* $3\pi/(m+2)$, *and those of the inner domain's boundary are same for a large* t, *and two domains have a common part in which the outer and the inner solutions are related by the matching matrix* (5.2).

6. Appendix: Correction to the former paper

In Nakano [7], we have studied the differential equation

$$\varepsilon^2 \frac{d^2y}{dx^2} - p(x)y = 0\,, \quad p(x) := \frac{(x-1)^2}{x^3}\,, \tag{6.1}$$

which is the simplest case having one turning point (at $x = 1$) and two irregular singular points (at $x = 0,\ \infty$). We suppose the x-plane's coordinate is (ν, μ). In that paper, we got a Stokes curve configuration (Figure 6.1). However, there was a small mistake in it. Now we will correct it.

Put

$$\xi(1,x) := \int_1^x p(x)dx = 2(x^{1/2} + x^{-1/2} - 2)\,. \tag{6.2}$$

In the polar coordinates: $x = r\,e^{i\theta}$, the real and imaginary parts of $\xi := \xi(1,x)$ are given by

$$\Re\,\xi = \frac{2}{\sqrt{r}}\left(r\,\cos\frac{\theta}{2} - 2\sqrt{r} + \cos\frac{\theta}{2}\right)\,, \quad \Im\,\xi = 2\left(\sqrt{r} - \frac{1}{\sqrt{r}}\right)\sin\frac{\theta}{2}\,.$$

The Stokes curves are determined by $\Re\xi = 0$ just like done in §4. We could not get the indefinite integral of $p(t)$ of (3.1), but (6.2) can be integrated easily and its real and imaginary parts are represented by elementary functions as above. We notice here that the differential equations with easily integrable coefficients are very rare. The Airy equation is one of them.

Since $\Re\xi = 0$ is a quadratic equation of $\sqrt{r}$, we can get

$$r = \frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}, \quad r = \frac{1-\sin\frac{\theta}{2}}{1+\sin\frac{\theta}{2}}, \tag{6.3}$$

from which we see that the Stokes curve configuration is symmetric with respect to the real axis, i.e., the ν-axis. From the first equation of (6.3) we can get the following derivatives

$$r' = \frac{\cos\frac{\theta}{2}}{\left(1-\sin\frac{\theta}{2}\right)^2}, \quad r'' = \frac{2+\sin\frac{\theta}{2}}{2\left(1-\sin\frac{\theta}{2}\right)^2}.$$

Then, we see that $r, r', r'' \to \infty \;\; (\theta \to \pi)$, and the r's curvature

$$\kappa := \frac{r^2+2r'^2-rr''}{(r^2+r'^2)^{3/2}} = \frac{\left(4-\sin\frac{\theta}{2}-2\sin^2\frac{\theta}{2}\right)\left(1-\sin\frac{\theta}{2}\right)^2}{2\cos\frac{\theta}{2}\left(2-\sin^2\frac{\theta}{2}\right)^{3/2}} \tag{6.4}$$

takes positive values for $\theta : -\pi < \theta < \pi$. Furthermore we see that the Stokes curve $\Re\xi = 0$

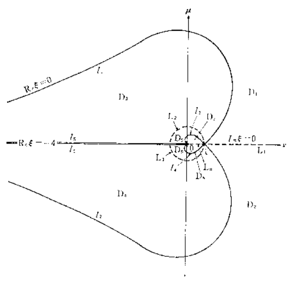

FIGURE 6-1

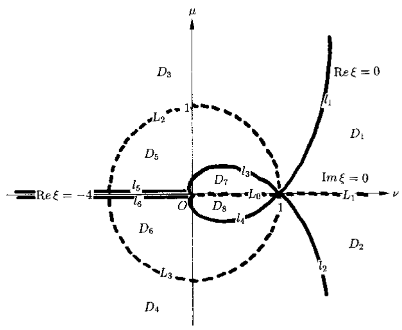

FIGURE 6-2. (Near $x = 0$)

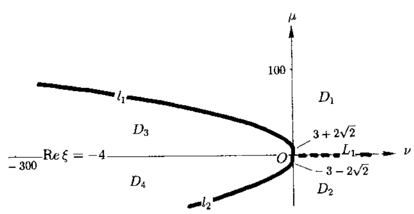

FIGURE 6-3. (Large $|x|$)

does not have any asymptote such as $\mu = a\ \nu + b\ \ (a, b = \text{const.})$ because a limit

$$\lim_{\theta \to \pm\pi} \left(r - \frac{b}{\sin\theta - a\cos\theta} \right)$$

does not exist. The Stokes curve l_1 intersects at $3 + 2\sqrt{2}$ on the μ-axis. Also, the Stokes curve l_2 intersects at $-3 - 2\sqrt{2}$ on the μ-axis. Thus we can get better figures like Figures 6-2 and 6-3. Figure 6-1 is in Nakano [7] which is exact near the unit circle but not good for $|\theta| > \pi/2$. The cause is very simple, i.e., neglecting (6.4).

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