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Existence of Standing Waves for Coupled Nonlinear Schrödinger Equations

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Abstract. In this paper we study the existence of standing waves for coupled nonlinear Schrödinger equations. The interaction between equations plays an important role in our study. When the interaction is strong, the least energy solution is a solution whose both components are positive. When the interaction is weak, the least energy solution is a semitrivial solution, namely a solution of a form $(u_1, 0)$ or $(0, u_2)$. Moreover, minimizing method on the Nehari type manifold with codimension 2 gives us a positive solution when the interaction is weak.

1. Introduction and main result

In this paper, we consider the existence of standing waves for the following coupled nonlinear Schrödinger equations:

$$\begin{cases} i\dfrac{\partial \psi_1}{\partial t} + \Delta_x \psi_1 + \lambda_1(x)\psi_1 + (\mu_1|\psi_1|^2 + \beta|\psi_2|^2)\psi_1 = 0 \quad \text{in} \quad (0,\infty)\times \mathbf{R}^N, \\ i\dfrac{\partial \psi_2}{\partial t} + \Delta_x \psi_2 + \lambda_2(x)\psi_2 + (\beta|\psi_1|^2 + \mu_2|\psi_2|^2)\psi_2 = 0 \quad \text{in} \quad (0,\infty)\times \mathbf{R}^N, \end{cases} \tag{$\tilde{\mathrm{E}}$}$$

where $\mu_1, \mu_2, \beta > 0$ are constants and the dimension N equals 2 or 3. The system ($\tilde{\mathrm{E}}$) appears in many physical problems, especially in the Hartree–Fock theory and nonlinear optics. We refer to [1, 2, 6, 9, 10, 14, 20, 22, 24] and references therein for more physical treatments.

To obtain standing waves, we substitute $\psi_j(t,x) = e^{i\tilde{\lambda}_j t}u_j(x)$ into ($\tilde{\mathrm{E}}$). Then $u_1(x), u_2(x)$ solve

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 \quad \text{in} \quad \mathbf{R}^N, \\ -\Delta u_2 + V_2(x)u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 \quad \text{in} \quad \mathbf{R}^N, \\ \qquad u_1, u_2 \in H^1(\mathbf{R}^N), \end{cases} \tag{E}$$

where $V_j(x) = \tilde{\lambda}_j - \lambda_j(x)$. In particular we are interested in a nontrivial positive solution of (E). Here, we say $u = (u_1, u_2)$ is a *nontrivial positive solution* of (E) if u solves (E) and both u_1, u_2 are positive in $\mathbf{R}^N$.

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Our aim of this paper is to study the existence of a nontrivial positive solution for the system with variable coefficients. Our work is motivated by Sirakov [20], and Ambrosetti–Colorado [2]. They consider (E) in constant coefficient case, which means that $V_j(x) \equiv \text{const} > 0$. Roughly speaking, they proved that there exist positive constants $\tilde{\beta}_1$ and $\tilde{\beta}_2$ such that if $0 \leq \beta < \tilde{\beta}_1$ or $\tilde{\beta}_2 < \beta$ holds, then (E) has a nontrivial positive solution. We remark that the existence problem becomes delicate when the coefficient depends on x. In Theorem 1.3 we give an example even if $V_j(x)$ is very close to constant, (E) does not have any nontrivial positive solutions.

In this paper, except for the nonexistence result (Theorem 1.3), we assume that $V_j(x)$ satisfies the following conditions:

(V1) $V_j(x) \in C^1(\mathbf{R}^N, \mathbf{R})$.

(V2) $0 < \inf_{x \in \mathbf{R}^N} V_j(x) \leq \sup_{x \in \mathbf{R}^N} V_j(x) \equiv V_{\infty,j} < \infty$.

(V3) $V_j(x) \to V_{\infty,j} \quad \text{as} \quad |x| \to \infty$.

Here we introduce some terminology. We call $u = (u_1, u_2)$ *nontrivial solution* if u solves (E) and $u_1, u_2 \not\equiv 0$. On the other hand, we call u *semitrivial solution* if u solves (E) and $u_1 \equiv 0$ or $u_2 \equiv 0$. We remark that if $V_j(x)$ satisfies (V1)–(V3), then (E) has a semitrivial solution. Indeed, the equation

$$\begin{cases} -\Delta u_1 + V_1(x) u_1 = \mu_1 u_1^3 \quad \text{in} \quad \mathbf{R}^N, \\ \qquad u_1 \in H^1(\mathbf{R}^N) \end{cases} \tag{E$_1$}$$

or

$$\begin{cases} -\Delta u_2 + V_2(x) u_2 = \mu_2 u_2^3 \quad \text{in} \quad \mathbf{R}^N, \\ \qquad u_2 \in H^1(\mathbf{R}^N) \end{cases} \tag{E$_2$}$$

has a nontrivial solution (for instance, see Willem [23]). Then $u = (u_1, 0)$ or $u = (0, u_2)$ is a semitrivial solution of (E).

Hereafter, we fix $\mu_1, \mu_2 > 0$, $V_1(x)$, $V_2(x)$ and consider the range of $\beta > 0$ for which (E) has a nontrivial positive solution. Here we state the main theorem in this paper.

THEOREM 1.1. *Let $V_j(x)$ satisfy* (V1)–(V3). *Then there exist $\beta_1 > 0$ and $\beta_2 > \beta_1$ such that*

(i) *If $0 < \beta < \beta_1$, then* (E) *has a nontrivial positive solution.*

(ii) *If $\beta_2 < \beta$, then* (E) *has a nontrivial positive solution.*

Next, we consider whether the solutions obtained in the above theorem is the least energy solution or not. We say a solution $u = (u_1, u_2)$ of (E) is *the least energy solution* if

$$I(u_1, u_2) = \inf\{I(v_1, v_2) | (v_1, v_2) \not\equiv (0, 0) \text{ solves (E)}\} .$$

Here, we use notation: for $v = (v_1, v_2) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$,

$$I(v) = \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla v_1|^2 + V_1(x)v_1^2 + |\nabla v_2|^2 + V_2(x)v_2^2)dx - \frac{1}{4}\int_{\mathbf{R}^N} (\mu_1 v_1^4 + 2\beta v_1^2 v_2^2 + \mu_2 v_2^4)dx\,.$$

THEOREM 1.2. (i) *There exists a* $\beta_3 \in (0, \beta_2]$ *such that if* $\beta \in [0, \beta_3)$, *then the nontrivial positive solution obtained in Theorem* 1.1 (i) *is not the least energy solution.*

(ii) *If* $\beta > \beta_2$, *then the least energy solution of* (E) *is nontrivial. Here* β_2 *is given in Theorem* 1.1.

REMARK 1.1. Ambrosetti–Colorado [2] obtained a nontrivial positive solution of (E) in the constant coefficient case with the mountain pass argument on the Nehari manifold. When $\beta > 0$ is small, they showed that the nontrivial positive solution of (E) has a higher energy than the semitrivial positive solutions.

Next, we give the nonexistence result. We assume that $V_j(x)$ satisfies the following conditions:

(V1′) $V_j(x) \in C^1(\mathbf{R}^N, \mathbf{R})$, $\dfrac{\partial V_j}{\partial x_i} \in L^\infty(\mathbf{R}^N)$ for $1 \le i \le N$, $j = 1, 2$.

(V2′) $0 < \inf_{x\in\mathbf{R}^N} V_j(x) \le \sup_{x\in\mathbf{R}^N} V_j(x) \equiv V_{\infty,j} < \infty$.

(V3′) $\exists \nu \in \mathbf{R}^N\backslash\{0\}$ s.t. $\dfrac{\partial V_j}{\partial \nu}(x) = \sum_{i=1}^{N} \dfrac{\partial V_j}{\partial x_i}(x)\nu_i \ge 0$.

(V4′) $\exists j_0 \in \{1, 2\}$ s.t. $\dfrac{\partial V_{j_0}}{\partial \nu} \not\equiv 0$.

Here we state the nonexistence result.

THEOREM 1.3. *Let* $V_j(x)$ *satisfies* (V1′)–(V4′). *Then* (E) *has no nontrivial positive solution for any* $\beta > 0$.

REMARK 1.2. There is a function which is close to a constant and satisfies (V1′)–(V4′). For instance, setting $V_j(x) = \varepsilon \arctan(x_1) + \pi$, then $V_j(x)$ satisfies (V1′)–(V4′) and (E) has no nontrivial positive solution for any $\varepsilon > 0$. This fact implies that the existence of nontrivial positive solution is a delicate problem and we need the behavior of $V_j(x)$ at infinity for the existence.

We prove Theorem 1.1 by variational methods. To obtain a nontrivial solution of (E), we introduce the Nehari manifold $\mathcal{N}$ and the Nehari type manifold $\mathcal{M}$:

$$\mathcal{N} := \{u \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) | u \not\equiv (0,0),\ I'(u)u = 0\}\,,$$
$$\mathcal{M} := \{u \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) | u_1, u_2 \not\equiv 0,\ I'(u)(u_1, 0) = I'(u)(0, u_2) = 0\}\,.$$

When $\beta > 0$ is large, which implies the setting of Theorem 1.1(ii), a nontrivial solution will be obtained as a minimizer of I on $\mathcal{N}$ (see section 5).

When $\beta > 0$ is small, which is dealt in Theorem 1.1(i), our argument is straight forward and we will observe that $\inf_{\mathcal{N}} I$ is also attained. However the minimizer turns out to be a semitrivial function and the Nehari type manifold $\mathcal{M}$ plays a role to find a nontrivial solution. In section 2, we will prove that $\mathcal{M}$ is a smooth Hilbert manifold with codimension 2 under the condition $0 < \beta < \sqrt{\mu_1\mu_2}$ and a nontrivial solution will be obtained as a minimizer of I on $\mathcal{M}$ (see section 6).

We remark that for problems with constant coefficients Sirakov [20] introduced manifolds in the space of radially symmetric functions:

$$\mathcal{N}_r := \{u \in H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N) | u \not\equiv (0,0),\ I'(u)u = 0\},$$
$$\mathcal{M}_r := \{u \in H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N) | u_1, u_2 \not\equiv 0,\ I'(u)(u_1, 0) = I'(u)(0, u_2) = 0\}.$$

He obtained a nontrivial solution as a minimizer of I on $\mathcal{N}_r$ ($\mathcal{M}_r$ respectively) when $\beta > 0$ is large ($\beta > 0$ is small respectively). We also remark that when $\beta > 0$ is small Ambrosetti–Colorado [2] develops a mountain pass argument in $\mathcal{N}_r$ to find a nontrivial solution. We also remark that in these works, the compactness of the embedding $H^1(\mathbf{R}^N) \hookrightarrow L^4(\mathbf{R}^N)$ is very important to get the Palais–Smale condition ((PS) condition).

In our setting, we cannot work in the space of radially symmetric functions and due to non–compactness of the embedding $H^1(\mathbf{R}^N) \hookrightarrow L^4(\mathbf{R}^N)$, the corresponding functional I does not satisfy the (PS) condition. To solve this difficulty we will develop a concentration–compactness type result and give the estimates of critical value of I.

Finally, we also give a mention to a work of Wei [22]. Wei considered (E) with variable coefficients, but under different conditions of $V_j(x)$ from ours. He considered the case where $V_j(x)$ is smooth, positive and $V_j(x) \to \infty$ as $|x| \to \infty$. The functional I is considered on

$$\tilde{V} = \left\{ u \in H \,\middle|\, \int_{\mathbf{R}^N} V_j(x) u_j^2 \, dx < \infty \quad \text{for} \quad j = 1, 2 \right\}.$$

In this case, the embedding $\tilde{V} \hookrightarrow L^4(\mathbf{R}^N) \times L^4(\mathbf{R}^N)$ is compact (See Rabinowitz [19], and Bartsch–Wang [5]), which implies that I satisfies the (PS) condition on $\tilde{V}$.

This paper is organized as follows: In sections 2–3, we give some preliminaries: especially we give functional frameworks and introduce our variational settings. In section 4, we prove the achievement of $\inf_{\mathcal{N}} I$ for all $\beta > 0$. It is important to determine whether the minimizer is nontrivial or not. In sections 5–6, we give a proof to Theorems 1.1 and 1.2. In section 5, we deal with the case where β is large and it turns out that the minimizer of $\inf_{\mathcal{N}} I$ is a nontrivial solution. In section 6, we study the case where β is small. In this case the Nehari type manifold $\mathcal{M}$ plays a role. Moreover we will show that for sufficiently small β, the least energy solution of (E) is a semitrivial solution. In section 7, we prove Theorem 1.3.

2. Preliminaries

In this section, we prove some preliminary results to prove Theorem 1.1.

2.1. Function spaces and functionals. We set $H = H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ and denote elements of H by $u = (u_1, u_2)$. For $u = (u_1, u_2)$, $v = (v_1, v_2) \in H$, we define inner products and norms in $H^1(\mathbf{R}^N)$ and H as follows:

$$\begin{aligned}
\langle u_j, v_j \rangle_j &= \int_{\mathbf{R}^N} (\nabla u_j \cdot \nabla v_j + V_j(x) u_j v_j)\, dx \quad (j = 1, 2)\,, \\
\langle u_j, v_j \rangle_{\infty,j} &= \int_{\mathbf{R}^N} (\nabla u_j \cdot \nabla v_j + V_{\infty,j} u_j v_j)\, dx \quad (j = 1, 2)\,, \\
\langle u, v \rangle &= \langle u_1, v_1 \rangle_1 + \langle u_2, v_2 \rangle_2\,, \\
\langle u, v \rangle_\infty &= \langle u_1, v_1 \rangle_{\infty,1} + \langle u_2, v_2 \rangle_{\infty,2}\,, \\
\|u_j\|_j^2 &= \langle u_j, u_j \rangle_j, \quad \|u_j\|_{\infty,j}^2 = \langle u_j, u_j \rangle_{\infty,j} \quad (j = 1, 2)\,, \\
\|u\|^2 &= \|u_1\|_1^2 + \|u_2\|_2^2, \quad \|u\|_\infty^2 = \|u_1\|_{\infty,1}^2 + \|u_2\|_{\infty,2}^2\,.
\end{aligned}$$

We remark that $\|\cdot\|_j$, $\|\cdot\|_{\infty,j}$ are equivalent to the standard $H^1(\mathbf{R}^N)$ norm under the conditions (V1)–(V2). We define the functional $I : H \to \mathbf{R}$ as follows:

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbf{R}^N} (\mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4) dx\,.$$

Differentiating I, we have

$$I'(u)v = \langle u, v \rangle - \int_{\mathbf{R}^N} (\mu_1 u_1^3 v_1 + \beta u_1 u_2^2 v_1 + \beta u_1^2 u_2 v_2 + \mu_2 u_2^3 v_2) dx\,.$$

It is easily seen that any critical point of I is a solution of (E). We also use a notation $\nabla I(u) \in H$, where $\nabla I(u)$ is a unique element such that

$$I'(u)v = \langle \nabla I(u), v \rangle \quad \text{for} \quad v \in H\,.$$

We also define the functional $I_\infty : H \to \mathbf{R}^N$ as follows:

$$I_\infty(u) = \frac{1}{2}\|u\|_\infty^2 - \frac{1}{4}\int_{\mathbf{R}^N} (\mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4) dx\,.$$

I_∞ is corresponding to the problem 'at infinity':

$$\begin{cases}
-\Delta u_1 + V_{\infty,1} u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in} \quad \mathbf{R}^N\,, \\
-\Delta u_2 + V_{\infty,2} u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 & \text{in} \quad \mathbf{R}^N\,, \\
u_1, u_2 \in H^1(\mathbf{R}^N)\,.
\end{cases} \tag{E_∞}$$

Any critical point of I_∞ is also a solution of (E_∞).

It is easily seen that the following equalities hold:

$$I'(u)u = \|u\|^2 - \mu_1\|u_1\|_{L^4}^4 - 2\beta\|u_1u_2\|_{L^2}^2 - \mu_2\|u_2\|_{L^4}^4 \,,$$

$$I'(u)(u_1,0) = \|u_1\|_1^2 - \mu_1\|u_1\|_{L^4}^4 - \beta\|u_1u_2\|_{L^2}^2 \,,$$

$$I'(u)(0,u_2) = \|u_2\|_2^2 - \beta\|u_1u_2\|_{L^2}^2 - \mu_2\|u_2\|_{L^4}^4 \,.$$

2.2. Nehari manifold and Nehari type manifold. In this subsection we introduce the Nehari manifold $\mathcal{N}$ and the Nehari type manifold $\mathcal{M}$ and state some properties of $\mathcal{N}$ and $\mathcal{M}$.

We define $J,\ J_1,\ J_2 : H \to \mathbf{R}$ as follows:

$$J(u) = I'(u)u\,, \quad J_1(u) = I'(u)(u_1,0)\,, \quad J_2(u) = I'(u)(0,u_2)\,.$$

DEFINITION 2.1. We define the Nehari manifold $\mathcal{N}$ and the Nehari type manifold $\mathcal{M}$ as follows:

$$\mathcal{N} = \{u \in H \mid u \not\equiv (0,0),\ J(u) = 0\}\,,$$

$$\mathcal{M} = \{u \in H \mid u_1 \not\equiv 0,\ u_2 \not\equiv 0\,,\ J_1(u) = J_2(u) = 0\}\,.$$

We also define $\mathcal{N}_\infty$ and $\mathcal{M}_\infty$ which are corresponding to (E_∞):

$$\mathcal{N}_\infty = \{u \in H \mid u \not\equiv (0,0),\ J_\infty(u) = 0\}\,,$$

$$\mathcal{M}_\infty = \{u \in H \mid u_1 \not\equiv 0,\ u_2 \not\equiv 0,\ J_{\infty,1}(u) = J_{\infty,2}(u) = 0\}\,.$$

REMARK 2.1. (i) $\mathcal{M} \subset \mathcal{N}$ and $\mathcal{M}_\infty \subset \mathcal{N}_\infty$.
(ii) Except for $(0,0)$, any solution of (E) belongs to $\mathcal{N}$.
(iii) If u is a nontrivial solution of (E), then $u \in \mathcal{M}$.

REMARK 2.2. We set $|u| := (|u_1|, |u_2|)$, then the following hold.
(i) If $u \in \mathcal{N}$, then $|u| \in \mathcal{N}$.
(ii) If $u \in \mathcal{M}$, then $|u| \in \mathcal{M}$.

Next, we state the fundamental properties of $\mathcal{N}$ and $\mathcal{N}_\infty$.

PROPOSITION 2.1. (i) *For each $u \in H, u \not\equiv (0,0)$, there exist unique $\theta_0 > 0$ and $\theta_{\infty,0} > 0$ such that $\theta_0 u \in \mathcal{N},\ \theta_{\infty,0}u \in \mathcal{N}_\infty$.*
(ii) $I(u) = \dfrac{1}{4}\|u\|^2$ on $\mathcal{N}$, $I_\infty(u) = \dfrac{1}{4}\|u\|_\infty^2$ on $\mathcal{N}_\infty$.
(iii) *There exist $\delta_0 > 0$ and $\delta_\infty > 0$ such that*

$$\|u\| \ge \delta_0 \quad \text{for} \quad u \in \mathcal{N}, \quad \|v\|_\infty \ge \delta_\infty \quad \text{for} \quad v \in \mathcal{N}_\infty\,.$$

PROOF. We only prove for $\mathcal{N}$.
(i) Suppose that $u \in H, u \not\equiv (0,0)$ and set

$$f(\theta) = I(\theta u) = \frac{\theta^2}{2}\|u\|^2 - \frac{\theta^4}{4}\int_{\mathbf{R}^N} \mu_1 u_1^4 + 2\beta u_1^2u_2^2 + \mu_2 u_2^4\,dx\,.$$

Then we see

$$f'(\theta) = I'(\theta u)u = \theta\{\|u\|^2 - \theta^2(\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4)\}\,.$$

Thus $f'(\theta) = 0$ holds if and only if $\theta = \theta_0$, where

$$\theta_0 = \frac{\|u\|}{\sqrt{\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4}} > 0\,.$$

(ii) Let $u \in \mathcal{N}$. Then it follows that

$$\|u\|^2 = \mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4\,.$$

From the above equality, we obtain

$$I(u) = \frac{\|u\|^2}{2} - \frac{\|u\|^2}{4} = \frac{\|u\|^2}{4}\,.$$

(iii) Let $u \in \mathcal{N}$. By using Hölder's inequality and Sobolev's embedding theorem, we have

$$\begin{aligned}\|u\|^2 &= \mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4\\ &\le \mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1\|_{L^4}^2\|u_2\|_{L^4}^2 + \mu_2\|u_2\|_{L^4}^4\\ &\le C(\mu_1\|u_1\|_1^4 + 2\beta\|u_1\|_1^2\|u_2\|_2^2 + \mu_2\|u_2\|_2^4)\\ &\le C(\|u_1\|_1^2 + \|u_2\|_2^2)^2 = C\|u\|^4\,.\end{aligned}$$

Therefore it follows that

$$\frac{1}{C} \le \|u\|^2.$$

□

Next, we prove that $\mathcal{N}$ and $\mathcal{M}$ are smooth Hilbert manifolds.

LEMMA 2.2. (i) *For each* $\beta > 0$, $\mathcal{N}$ *and* $\mathcal{N}_\infty$ *are smooth Hilbert manifolds with codimension* 1.

(ii) *If* $0 < \beta < \sqrt{\mu_1\mu_2}$, *then* $\mathcal{M}$ *and* $\mathcal{M}_\infty$ *are smooth Hilbert manifolds with codimension* 2.

(iii) $T_u\mathcal{N} = \{v \in H \mid J'(u)v = 0\}$.

(iv) $T_u\mathcal{M} = \{v \in H \mid J_1'(u)v = J_2'(u)v = 0\}$.

The above lemma will be derived from the following well known lemma. For example, see Ambrosetti–Malchiodi [3].

LEMMA 2.3. *Let* $O \subset H$ *be open set. Suppose* $G, G_1, G_2 \in C^m(O, \mathbf{R})$ *and set* $M = G^{-1}(0)$, $\tilde{M} = G_1^{-1}(0) \cap G_2^{-1}(0)$. *Then the following hold*:

(i) *If* $G'(p) \neq 0$ *for each* $p \in M$, *then* M *is a* C^m *Hilbert manifold with codimension* 1.

(ii) *If* $G_1'(p)$ *and* $G_2'(p)$ *are linearly independent for each* $p \in \tilde{M}$, *then* $\tilde{M}$ *is a* C^m *Hilbert manifold with codimension* 2.

(iii) $T_p M = \{q \in H \mid G'(p)q = 0\}$.

(iv) $T_p \tilde{M} = \{q \in H \mid G_1'(p)q = G_2'(p)q = 0\}$.

We prove Lemma 2.2 with the aid of Lemma 2.3.

PROOF OF LEMMA 2.2. We only prove (i) and (ii) since (iii) and (iv) are directly derived from Lemma 2.3.

(i) For $u \in \mathcal{N}$, we have

$$\begin{aligned} J'(u)u &= 2\|u\|^2 - 4(\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4) \\ &= -2\|u\|^2 < 0. \end{aligned}$$

In particular, we have $J'(u) \neq 0$ for $u \in \mathcal{N}$. Thus applying Lemma 2.3 to $J : H\backslash\{0\} \to \mathbf{R}$, we have (i) of Lemma 2.2.

(ii) Next we apply (ii) of Lemma 2.3 to $J_1, J_2 : H\backslash\{u_1 = 0 \text{ or } u_2 = 0\} \to \mathbf{R}$. For $u \in \mathcal{M}$, we have

$$\begin{aligned} &J_1'(u)(u_1, 0) = -2\mu_1\|u_1\|_{L^4}^4\,, \quad J_2'(u)(0, u_2) = -2\mu_2\|u_2\|_{L^4}^4\,, \\ &J_1'(u)(0, u_2) = J_2'(u)(u_1, 0) = -2\beta\|u_1u_2\|_{L^2}^2\,. \end{aligned}$$

Define $A(u)$ by

$$A(u) = \begin{pmatrix} J_1'(u)(u_1, 0) & J_1'(u)(0, u_2) \\ J_2'(u)(u_1, 0) & J_2'(u)(0, u_2) \end{pmatrix} = \begin{pmatrix} -2\mu_1\|u_1\|_{L^4}^4 & -2\beta\|u_1u_2\|_{L^2}^2 \\ -2\beta\|u_1u_2\|_{L^2}^2 & -2\mu_2\|u_2\|_{L^4}^4 \end{pmatrix},$$

and we see

$$\begin{aligned} \det A(u) &= 4(\mu_1\mu_2\|u_1\|_{L^4}^4\|u_2\|_{L^4}^4 - \beta^2\|u_1u_2\|_{L^2}^4) \\ &\geq 4(\mu_1\mu_2 - \beta^2)\|u_1\|_{L^4}^4\|u_2\|_{L^4}^4 > 0\,. \end{aligned}$$

The above inequality implies $J_1'(u), J_2'(u)$ are linearly independent. Thus Lemma 2.3 infers that $\mathcal{M}$ is a smooth Hilbert manifold with codimension 2. □

Lastly we state some properties of the level sets of $\mathcal{N}$ and $\mathcal{M}$. For each $\alpha > 0$, we define $\mathcal{N}^\alpha$ and $\mathcal{M}^\alpha$ as follows:

$$\mathcal{N}^\alpha = \{u \in \mathcal{N} \mid I(u) \leq \alpha\}\,, \quad \mathcal{M}^\alpha = \{u \in \mathcal{M} \mid I(u) \leq \alpha\}\,.$$

PROPOSITION 2.4 (Properties of $\mathcal{N}$). (i) *$\mathcal{N}$ is a closed subset of H and $\mathcal{N}^\alpha$ is a bounded closed subset of H. In particular,*

$$0 < \delta_0 \leq \|u\| \leq 2\sqrt{\alpha} \quad \text{for} \quad u \in \mathcal{N}^\alpha,$$

where δ_0 is given in Proposition 2.1.

(ii) *For each* $\alpha > 0$, *there holds*

$$0 < 2\delta_0 \le \|\nabla J(u)\| \le c_1(\alpha) \quad \text{for} \quad u \in \mathcal{N}^\alpha ,$$

where $c_1(\alpha)$ *depends on* α *but not on* $u \in \mathcal{N}^\alpha$.

PROOF. (i) It is clear from Proposition 2.1 (ii) and (iii).

(ii) Since $J'(u)u = -2\|u\|^2$ and $\|u\| \ge \delta_0$, we have $2\delta_0 \le \|J'(u)\|$. On the other hand, since $J' : H \to H^*$ maps bounded sets to bounded sets and $\mathcal{N}^\alpha$ is bounded, we infer the conclusion of Proposition 2.4. □

We define $T_u\mathcal{N}^\perp$ and $T_u\mathcal{M}^\perp$ as the orthonormal complement of $T_u\mathcal{N}$ and $T_u\mathcal{M}$, respectively:

$$T_u\mathcal{N}^\perp := \{v \in H \mid \langle v, h\rangle = 0 \text{ for } h \in T_u\mathcal{N}\} ,$$
$$T_u\mathcal{M}^\perp := \{v \in H \mid \langle v, h\rangle = 0 \text{ for } h \in T_u\mathcal{M}\} .$$

We also define $P_{T_u\mathcal{N}^\perp}$ and $P_{T_u\mathcal{M}^\perp}$ as the projections from H to $T_u\mathcal{N}^\perp$ and $T_u\mathcal{M}^\perp$, respectively:

$$P_{T_u\mathcal{N}^\perp} : H \to T_u\mathcal{N}^\perp , \quad P_{T_u\mathcal{M}^\perp} : H \to T_u\mathcal{M}^\perp .$$

By Lemma 2.2, we have $T_u\mathcal{N}^\perp = \operatorname{span}\{\nabla J(u)\}$. Thus

$$P_{T_u\mathcal{N}^\perp} u = \left\langle \frac{\nabla J(u)}{\|\nabla J(u)\|}, u \right\rangle \frac{\nabla J(u)}{\|\nabla J(u)\|} .$$

By Lemma 2.2 and Proposition 2.4, we have the following corollary.

COROLLARY 2.5. *For each* $\alpha > 0$, *there holds*

$$0 < c_1(\alpha) \le \|P_{T_u\mathcal{N}^\perp} u\| \le c_2(\alpha) \quad \textit{for} \quad u \in \mathcal{N}^\alpha ,$$

where $c_1(\alpha)$, $c_2(\alpha)$ *are positive constants and depend on* α.

Next we state the properties of $\mathcal{M}$.

PROPOSITION 2.6 (Properties of $\mathcal{M}$). *Let* $\alpha > 0$.

(i) *There exist* $\beta_1(\alpha) \in (0, \sqrt{\mu_1\mu_2})$, $c_1(\alpha)$, $c_2(\alpha) > 0$ *such that for each* $\beta \in (0, \beta_1(\alpha))$ *and* $u \in \mathcal{M}^\alpha$,

$$c_1(\alpha) \le \|u_j\|_{L^4} \le c_2(\alpha) , \quad c_1(\alpha) \le \|u_j\|_j \le c_2(\alpha) ,$$
$$c_1(\alpha) \le \|\nabla J_j(u)\| \le c_2(\alpha) \quad (j = 1, 2) .$$

(ii) *If* $\beta \in (0, \beta_1(\alpha))$, *then* $\mathcal{M}^\alpha$ *is a closed subset of* H.

(iii) *There exists an* $\varepsilon_1(\alpha) > 0$ *such that for each* $u \in \mathcal{M}^\alpha$ *and* $\beta \in (0, \beta_1(\alpha))$,

$$|\langle \nabla J_1(u), \nabla J_2(u)\rangle| \le (1 - \varepsilon_1(\alpha))\|\nabla J_1(u)\|\|\nabla J_2(u)\| .$$

(iv) *There exist* $c_3(\alpha) > 0$ *and* $c_4(\alpha) > 0$ *such that for each* $\beta \in (0, \beta_1(\alpha))$ *and* $u = (u_1, u_2) \in \mathcal{M}^\alpha$,

$$0 < c_3(\alpha) \le \|P_{T_u\mathcal{M}^\perp} U_j\| \le c_4(\alpha) \quad (j = 1, 2)\,,$$

where $U_1 = (u_1, 0)$ *and* $U_2 = (0, u_2)$. *Moreover, there exists an* $\varepsilon_2(\alpha) > 0$ *such that*

$$|\langle P_{T_u\mathcal{M}^\perp} U_1, P_{T_u\mathcal{M}^\perp} U_2\rangle| \le (1 - \varepsilon_2(\alpha))\|P_{T_u\mathcal{M}^\perp} U_1\|\|P_{T_u\mathcal{M}^\perp} U_2\|$$

for all $u \in \mathcal{M}^\alpha$.

PROOF. (i) Since $\mathcal{M}^\alpha \subset \mathcal{N}^\alpha$, $\mathcal{N}^\alpha$ is a bounded set in H and J_j' maps bounded sets to bounded sets, it is sufficient to show that

$$0 < c_1(\alpha) \le \|u_j\|_{L^4}\,, \quad 0 < c_1(\alpha) \le \|u_j\|_j\,, \quad 0 < c_1(\alpha) \le \|\nabla J_j(u)\|$$

for each $u \in \mathcal{M}^\alpha$. We only show the statements for u_1 and ∇J_1 since the same argument is valid for u_2 and ∇J_2.

Since

$$\|u_1\|_1^2 = \mu_1\|u_1\|_{L^4}^4 + \beta\|u_1u_2\|_{L^2}^2\,,$$

using Hölder's inequality and Sobolev's embedding theorem, it follows that

$$\|u_1\|_{L^4}^2 \le C\|u_1\|_1^2 \le C(\mu_1\|u_1\|_{L^4}^4 + \beta\|u_1\|_{L^4}^2\|u_2\|_{L^4}^2)\,.$$

This implies that

$$\frac{1}{C} - \beta\|u_2\|_{L^4}^2 \le \mu_1\|u_1\|_{L^4}^2\,.$$

Since $\|u_j\|_j$ are bounded, there exists a $\beta(\alpha) > 0$ such that if $\beta \in (0, \beta(\alpha))$, then

$$0 < c_1(\alpha) \le \|u_1\|_{L^4}\,.$$

By Sobolev's embedding theorem, we have

$$c_1(\alpha) \le \|u_1\|_{L^4} \le C\|u_1\|_1\,.$$

Since $J_1'(u)(u_1, 0) = -2\mu_1\|u_1\|_{L^4}^4$, we have $c_1(\alpha) \le \|\nabla J_1(u)\|$.

(ii) By (i) and the continuity of $J_j(u)$, it is easy to check that (ii) holds.

(iii) Let $u \in \mathcal{M}^\alpha$ and set

$$\xi_1 = \frac{\nabla J_1(u)}{\|\nabla J_1(u)\|}\,, \qquad \xi_2 = \frac{\nabla J_2(u)}{\|\nabla J_2(u)\|}\,,$$

$$\tilde{\xi}_2 = \xi_2 - \langle \xi_1, \xi_2\rangle\xi_1\,, \quad \xi_3 = \frac{\tilde{\xi}_2}{\|\tilde{\xi}_2\|}\,.$$

Since $\mathcal{M}^\alpha$ is bounded and ∇J_1, ∇J_2 map bounded sets into bounded sets, we only prove that there exists a $c(\alpha) = c > 0$ such that

$$0 < c \leq \|\tilde{\xi}_2\|^2 \quad \text{for all } u \in \mathcal{M}^\alpha . \tag{1}$$

Indeed, since

$$\|\tilde{\xi}_2\|^2 = 1 - \langle \xi_1, \xi_2 \rangle^2 = \frac{\|\nabla J_1(u)\|^2 \|\nabla J_2(u)\|^2 - \langle \nabla J_1(u), \nabla J_2(u) \rangle^2}{\|\nabla J_1(u)\|^2 \|\nabla J_2(u)\|^2} ,$$

(iii) follows from (1).

Set $U_1 = (u_1, 0)$, $U_2 = (0, u_2)$ and define $A(u)$ as follows:

$$A(u) = \begin{pmatrix} \langle U_1, \xi_1 \rangle & \langle U_1, \xi_3 \rangle \\ \langle U_2, \xi_1 \rangle & \langle U_2, \xi_3 \rangle \end{pmatrix} .$$

Since

$$\begin{aligned} \det A(u) &= \frac{1}{\|\tilde{\xi}_2\|} \det \begin{pmatrix} \langle U_1, \xi_1 \rangle & \langle U_1, \tilde{\xi}_2 \rangle \\ \langle U_2, \xi_1 \rangle & \langle U_2, \tilde{\xi}_2 \rangle \end{pmatrix} \\ &= \frac{1}{\|\tilde{\xi}_2\|} \det \begin{pmatrix} \langle U_1, \xi_1 \rangle & \langle U_1, \xi_2 - \langle \xi_1, \xi_2 \rangle \xi_1 \rangle \\ \langle U_2, \xi_1 \rangle & \langle U_2, \xi_2 - \langle \xi_1, \xi_2 \rangle \xi_1 \rangle \end{pmatrix} \\ &= \frac{1}{\|\tilde{\xi}_2\|} \det \begin{pmatrix} \langle U_1, \xi_1 \rangle & \langle U_1, \xi_2 \rangle \\ \langle U_2, \xi_1 \rangle & \langle U_2, \xi_2 \rangle \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \langle U_1, \xi_1 \rangle &= -\frac{2\mu_1 \|u_1\|_{L^4}^4}{\|\nabla J_1(u)\|} , \qquad \langle U_1, \xi_2 \rangle = -\frac{2\beta \|u_1 u_2\|_{L^2}^2}{\|\nabla J_2(u)\|} , \\ \langle U_2, \xi_1 \rangle &= -\frac{2\beta \|u_1 u_2\|_{L^2}^2}{\|\nabla J_1(u)\|} , \qquad \langle U_2, \xi_2 \rangle = -\frac{2\mu_2 \|u_2\|_{L^4}^4}{\|\nabla J_2(u)\|} , \end{aligned}$$

we have

$$\begin{aligned} \det A(u) &= \frac{4(\mu_1 \mu_2 \|u_1\|_{L^4}^4 \|u_2\|_{L^4}^4 - \beta^2 \|u_1 u_2\|_{L^2}^4)}{\|\tilde{\xi}_2\| \|\nabla J_1(u)\| \|\nabla J_2(u)\|} \\ &\geq \frac{4(\mu_1 \mu_2 - \beta^2) \|u_1\|_{L^4}^4 \|u_2\|_{L^4}^4}{\|\tilde{\xi}_2\| \|\nabla J_1(u)\| \|\nabla J_2(u)\|} . \end{aligned}$$

By (i) and the assumption of β,

$$\det A(u) \geq \frac{C(\alpha)}{\|\tilde{\xi}_2\|} \quad \text{for all } u \in \mathcal{M}^\alpha . \tag{2}$$

On the other hand, the components of $A(u)$ are bounded, which implies that there exists a $C_1 = C_1(\alpha) > 0$ such that

$$\det A(u) \leq C_1(\alpha) \quad \text{for all } u \in \mathcal{M}^\alpha . \tag{3}$$

From (2)–(3), there exits a $c = c(\alpha) > 0$ such that

$$0 < c \le \|\tilde{\xi}_2\| \quad \text{for all } u \in \mathcal{M}^\alpha .$$

(iv) Since

$$P_{T_u\mathcal{M}^\perp} U_1 = \langle U_1, \xi_1\rangle \xi_1 + \langle U_1, \xi_3\rangle \xi_3 , \tag{4}$$

where ξ_j are given in (iii), it follows that

$$\|U_1\|^2 \ge \|P_{T_u\mathcal{M}^\perp} U_1\|^2 = \langle U_1, \xi_1\rangle^2 + \langle U_1, \xi_3\rangle^2 \ge \langle U_1, \xi_1\rangle^2 = \frac{4\mu_1^2 \|u_1\|_{L^4}^8}{\|\nabla J_1(u)\|^2} .$$

By (i), it follows that there exist $c_3(\alpha) > 0$ and $c_4(\alpha) > 0$ such that

$$c_3(\alpha) \le \|P_{T_u\mathcal{M}^\perp} U_1\| \le c_4(\alpha) \quad \text{for all } u \in \mathcal{M}^\alpha . \tag{5}$$

Similarly we have (5) for U_2. Since (4) and

$$P_{T_u\mathcal{M}^\perp} U_2 = \langle U_2, \xi_1\rangle \xi_1 + \langle U_2, \xi_3\rangle \xi_3 ,$$

we have

$$\|P_{T_u\mathcal{M}^\perp} U_1\|^2 \|P_{T_u\mathcal{M}^\perp} U_2\|^2 - |\langle P_{T_u\mathcal{M}^\perp} U_1, P_{T_u\mathcal{M}^\perp} U_2\rangle|^2 = (\det A(u))^2 .$$

By (2) and the boundness of $(P_{T_u\mathcal{M}^\perp} U_j)$, for sufficiently small $\varepsilon_2(\alpha) > 0$, the conclusion of (iv) holds. □

2.3. $(\mathrm{PS})_c$ **sequence.** At first, we introduce the important values to obtain a nontrivial solution of (E).

We define $b_\mathcal{N}, \hat{b}_\mathcal{M}, b_{\mathcal{N}_\infty}, \hat{b}_{\mathcal{M}_\infty}$ as follows.

$$b_\mathcal{N} = \inf_{u\in\mathcal{N}} I(u) , \quad \hat{b}_\mathcal{M} = \inf_{u\in\mathcal{M}} I(u) ,$$

$$b_{\mathcal{N}_\infty} = \inf_{u\in\mathcal{N}_\infty} I_\infty(u) , \quad \hat{b}_{\mathcal{M}_\infty} = \inf_{u\in\mathcal{M}_\infty} I_\infty(u) .$$

REMARK 2.3. By Remark 2.1, it follows that

$$0 < b_\mathcal{N} \le \hat{b}_\mathcal{M} , \quad 0 < b_{\mathcal{N}_\infty} \le \hat{b}_{\mathcal{M}_\infty} .$$

To obtain a solution of (E), we see that $b_\mathcal{N}$ or $\hat{b}_\mathcal{M}$ is attained. So it is important to see the behavior of the minimizing sequence on $\mathcal{N}$ or $\mathcal{M}$.

DEFINITION 2.2. Let $c \in \mathbf{R}$.

(i) $(u_n) \subset H$ is said to be a Palais–Smale sequence of I on H at level c (in short $(\mathrm{PS})_{c,H}$ sequence), if it satisfies

$$I(u_n) \to c , \quad \|I'(u_n)\|_{H^*} \to 0 ,$$

where

$$\|I'(u)\|_{H^*} := \sup_{h\in H,\ \|h\|=1} I'(u)h\,.$$

(ii) $(u_n) \subset \mathcal{N}$ is said to be a $(\mathrm{PS})_{c,\mathcal{N}}$ sequence of I, if it satisfies

$$I(u_n) \to c\,, \quad \|I'(u_n)\|_{T_{u_n}\mathcal{N}^*} \to 0\,,$$

where

$$\|I'(v)\|_{T_v\mathcal{N}^*} := \sup_{h\in T_v\mathcal{N},\ \|h\|=1} I'(v)h\,.$$

(iii) Let $\beta < \sqrt{\mu_1\mu_2}$. $(u_n) \subset \mathcal{M}$ is said to be a $(\mathrm{PS})_{c,\mathcal{M}}$ sequence of I on $\mathcal{M}$, if it satisfies

$$I(u_n) \to c\,, \quad \|I'(u_n)\|_{T_{u_n}\mathcal{M}^*} \to 0\,,$$

where

$$\|I'(w)\|_{T_w\mathcal{M}^*} := \sup_{h\in T_w\mathcal{M},\ \|h\|=1} I'(w)h\,.$$

Next we see the relationships between a $(\mathrm{PS})_{c,H}$, $(\mathrm{PS})_{c,\mathcal{N}}$ and $(\mathrm{PS})_{c,\mathcal{M}}$ sequence.

LEMMA 2.7. (i) *Any* $(\mathrm{PS})_{c,H}$ *sequence* (u_n) *is a bounded sequence on* H.
(ii) *Any* $(\mathrm{PS})_{c,\mathcal{N}}$ *sequence* (u_n) *is a* $(\mathrm{PS})_{c,H}$ *sequence.*
(iii) *If* $c < \alpha$ *and* $\beta \in (0, \beta_1(\alpha))$, *then any* $(\mathrm{PS})_{c,\mathcal{M}}$ *sequence is a* $(\mathrm{PS})_{c,H}$ *sequence, where* $\beta_1(\alpha)$ *appeared in Proposition* 2.6.

PROOF. (i) Let (u_n) be a $(\mathrm{PS})_{c,H}$ sequence. Since $\|I(u_n)\|_{H^*} \to 0$, there exists an $n_1 \in \mathbf{N}$ such that

$$|I'(u_n)u_n| \le \|u_n\| \quad \text{for } n \ge n_1\,.$$

On the other hand, there hold

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{4}(\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_{n,1}u_{n,2}\|_{L^2}^2 + \mu_2\|u_{n,2}\|_{L^4}^4)\,,$$
$$I'(u_n)u_n = \|u_n\|^2 - (\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_{n,1}u_{n,2}\|_{L^2}^2 + \mu_2\|u_{n,2}\|_{L^4}^4)\,,$$

which implies

$$I(u_n) - \frac{1}{4}I'(u_n)u_n = \frac{1}{4}\|u_n\|^2\,.$$

Thus we conclude that for sufficiently large n,

$$\frac{1}{4}\|u_n\|^2 \le \|u_n\| + c + o(1)\,,$$

which implies that (u_n) is a bounded sequence.
(ii) By $\|I'(u_n)\|_{T_{u_n}\mathcal{N}^*} \to 0$ and $I(u_n) \to c$, it is sufficient to prove $\|I'(u_n)\|_{H^*} \to 0$. Let $\alpha > c$. Since we may assume that $(u_n) \subset \mathcal{N}^\alpha$, (u_n) is bounded sequence. By Lemma 2.2, $H = \operatorname{span}\{\nabla J(u_n)\} \oplus T_{u_n}\mathcal{N}$. So we prove that $I'(u_n)\zeta_n \to 0$ where $\zeta_n = \nabla J(u_n)/\|\nabla J(u_n)\|$ and it is equivalent to

$$I'(u_n)\left[\frac{P_{T_{u_n}\mathcal{N}^\perp} u_n}{\|P_{T_{u_n}\mathcal{N}^\perp} u_n\|}\right] \to 0\,. \tag{6}$$

Firstly we prove that $I'(u_n)[P_{T_{u_n}\mathcal{N}^\perp} u_n] \to 0$. Since $I'(u_n)u_n = J(u_n) = 0$ and $u_n - P_{T_{u_n}\mathcal{N}^\perp} u_n \in T_{u_n}\mathcal{N}$, it follows that

$$\begin{aligned}
|I'(u_n)[P_{T_{u_n}\mathcal{N}^\perp} u_n]| &= |I'(u_n)u_n - I'(u_n)[u_n - P_{T_{u_n}\mathcal{N}^\perp} u_n]| \\
&= |I'(u_n)[u_n - P_{T_{u_n}\mathcal{N}^\perp} u_n]| \\
&\le \|I'(u_n)\|_{T_{u_n}\mathcal{N}^*} \|u_n - P_{T_{u_n}\mathcal{N}^\perp} u_n\| \to 0\,.
\end{aligned}$$

By Corollary 2.5, $(\|P_{T_{u_n}\mathcal{N}^\perp} u_n\|)$ is bounded below away from 0. Thus (6) holds.
(iii) Let (u_n) be a $(\mathrm{PS})_{c,\,\mathcal{M}}$ sequence and $c < \alpha$. We remark that (u_n) is bounded in H and $(U_{n,j})$ also. As in (ii), by Lemma 2.2 and Proposition 2.6, we prove that

$$I'(u_n)\xi_{n,1} \to 0\,, \quad I'(u_n)\xi_{n,3} \to 0\,, \tag{7}$$

where $(\xi_{n,1})$ and $(\xi_{n,3})$ are given in the proof of Proposition 2.6. Since $I'(u_n)U_{n,1} = I'(u_n)U_{n,2} = 0$, $U_{n,j} - P_{T_{u_n}\mathcal{M}^\perp} U_{n,j} \in T_{u_n}\mathcal{M}$ and $\|I'(u_n)\|_{T_{u_n}\mathcal{M}} \to 0$, we have

$$\begin{aligned}
|I'(u_n)[P_{T_{u_n}\mathcal{M}^\perp} U_{n,j}]| &= |I'(u_n)U_{n,j} - I'(u_n)[U_{n,j} - P_{T_{u_n}\mathcal{M}^\perp} U_{n,j}]| \\
&= |I'(u_n)[U_{n,j} - P_{T_{u_n}\mathcal{M}^\perp} U_{n,j}]| \\
&\le \|I'(u_n)\|_{T_{u_n}\mathcal{M}^*} \|U_{n,j} - P_{T_{u_n}\mathcal{M}^\perp} U_{n,j}\| \to 0\,.
\end{aligned}$$

By Proposition 2.6, $\|P_{T_{u_n}\mathcal{M}^\perp} U_{n,j}\|$ are bounded below away from 0, it follows that

$$I'(u_n)\xi_{n,1} \to 0\,, \quad I'(u_n)\xi_{n,2} \to 0\,.$$

Using Proposition 2.6 again, it follows that $\|\xi_{n,2} - \langle \xi_{n,2}, \xi_{n,1}\rangle \xi_{n,1}\|$ is bounded below away from 0, which implies (7). □

The following lemma tells us that we obtain a $(\mathrm{PS})_{b_{\mathcal{N}},\,H}$ sequence and a $(\mathrm{PS})_{\hat{b}_{\mathcal{M}},\,H}$ sequence from the minimizing sequence, respectively.

LEMMA 2.8. (i) *For each* $\beta > 0$, *there exists a* $(\mathrm{PS})_{b_{\mathcal{N}},\,H}$ *sequence.*

(ii) *Suppose that* $\alpha > \hat{b}_{\mathcal{M}}$ *for all* $\beta \in (0, \sqrt{\mu_1\mu_2})$. *Then there exists a* $0 < \tilde{\beta}(\alpha) \le \sqrt{\mu_1\mu_2}$ *such that if* $\beta \in (0, \tilde{\beta}(\alpha))$, *then there exists a* $(\mathrm{PS})_{\hat{b}_{\mathcal{M}},\,H}$ *sequence.*

REMARK 2.4. We remark that $\hat{b}_{\mathcal{M}}$ depends on β. In Proposition 6.1, we will prove $\sup_{\beta\in[0,\infty)} \hat{b}_{\mathcal{M}} < \infty$. In particular, there exists an α which satisfies the assumption of Lemma 2.8 (ii).

We can prove Lemma 2.8 by applying Ekeland's variational principle. (See Ekeland [8] and Mahwin–Willem [17].) So we omit the proof.

The following lemma is so-called Concentration-Compactness Lemma. This lemma plays an important role in analysing a $(\mathrm{PS})_{c,\,H}$ sequence.

LEMMA 2.9 (Concentration-Compactness Lemma). *Let (u_n) be a $(\mathrm{PS})_{c,\,H}$ sequence. Then there exist a subsequence (u_{n_k}), an $\ell \in \mathbf{N}$, a critical point u_0 of I, critical points $\omega_i\,(1 \le i \le \ell)$ of I_∞, $(y_k^i) \subset \mathbf{R}^N\,(1 \le i \le \ell)$ which satisfy the following*:

(i) $|y_k^i| \to \infty\ (1 \le i \le \ell), \quad |y_k^i - y_k^j| \to \infty\ (i \ne j)$.

(ii) $\left\|u_{n_k} - u_0 - \sum_{i=1}^{\ell} \omega_i(x - y_k^i)\right\| \to 0$.

(iii) $I(u_{n_k}) \to c = I(u_0) + \sum_{i=1}^{\ell} I_\infty(\omega_i)$.

See Bahri–Lions [4] and Jeanjean–Tanaka [11] for a proof of Lemma 2.9.

REMARK 2.5. If $\ell = 0$ in the above lemma, then u_{n_k} converges to u_0 strongly.

3. Semitrivial solutions

Here, we consider some properties of semitrivial solutions, i.e., the solution of a form $(u_1, 0)$ or $(0, u_2)$.

The functionals $u_1 \mapsto I(u_1, 0)$ and $u_2 \mapsto I(0, u_2)$ are corresponding to

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 \quad \text{in} \quad \mathbf{R}^N\,, \\ \qquad u_1 \in H^1(\mathbf{R}^N)\,, \end{cases} \tag{E_1}$$

$$\begin{cases} -\Delta u_2 + V_2(x)u_2 = \mu_2 u_2^3 \quad \text{in} \quad \mathbf{R}^N\,, \\ \qquad u_2 \in H^1(\mathbf{R}^N)\,. \end{cases} \tag{E_2}$$

We define d_j as the least energy of (E_j):

$$d_1 = \inf_{(u_1,0)\in\mathcal{N}} I(u_1, 0)\,, \quad d_2 = \inf_{(0,u_2)\in\mathcal{N}} I(0, u_2)\,.$$

Similarly, we set

$$d_{\infty,1} = \inf_{(u_1,0)\in\mathcal{N}_\infty} I_\infty(u_1, 0)\,, \quad d_{\infty,2} = \inf_{(0,u_2)\in\mathcal{N}_\infty} I_\infty(0, u_2)\,.$$

REMARK 3.1. By the definition of d_j, we have

$$b_{\mathcal{N}} \le \min\{d_1, d_2\}\,. \tag{8}$$

If the inequality (8) is strict, we can see the critical point corresponding to $b_{\mathcal{N}}$ is nontrivial. We will see in section 5 that this is the case when β is large.

The following lemma shows that d_j is attained and (E) has a semitrivial solution.

LEMMA 3.1. *Let $V_j(x)$ satisfy* (V1)–(V3). *Then,*

(i) (E_j) *has the least energy solution which is positive in $\mathbf{R}^N$.*

(ii) *$d_j \le d_{\infty,j}$ holds. Moreover if $V_j(x) \not\equiv V_{\infty,j}$, then $d_j < d_{\infty,j}$.*

A proof of Lemma 3.1 is standard, so we omit it.

4. Achievements of $b_{\mathcal{N}}$, $b_{\mathcal{N}_\infty}$

In this section, we prove that $b_{\mathcal{N}}$ and $b_{\mathcal{N}_\infty}$ are attained for each $\beta > 0$. These facts are useful to prove the existence of nontrivial solutions of (E) in section 5.

At first we recall the following result.

PROPOSITION 4.1 (Ambrosetti–Colorado [2] and Sirakov [20]). (i) *For each $\beta > 0$, $b_{\mathcal{N}_\infty}$ is attained.*

(ii) *There exists a $\beta_0 > 0$ such that if $\beta > \beta_0$, then $b_{\mathcal{N}_\infty}$ is attained by a nontrivial function.*

This Proposition is proved in Ambrosetti–Colorado [2] and Sirakov [20]. For reader's convenience, we will give a proof of (i). To prove Proposition 4.1, we need the Schwarz symmetrization. We denote u^* the Schwarz symmetrization of u:

$$u^* = (u_1^*, u_2^*)\,.$$

It is well-known that the Schwarz symmetrization satisfies the following: (See Lieb–Loss [13])

$$\|u_j^*\|_{L^p} = \|u_j\|_{L^p}\,, \quad \|\nabla u_j^*\|_{L^2} \le \|\nabla u_j\|_{L^2}\,, \quad \|u_1^* u_2^*\|_{L^2} \ge \|u_1 u_2\|_{L^2}\,.$$

PROOF OF PROPOSITION 4.1. Suppose that $(u_n) \subset \mathcal{N}_\infty$ satisfies $I_\infty(u_n) \to b_{\mathcal{N}_\infty}$. Then (u_n) is a bounded sequence. By the above properties of u^*, (u_n^*) is also a bounded sequence. Let $H_r^1(\mathbf{R}^N)$ be the space of radially symmetric functions in $H^1(\mathbf{R}^N)$. Since the embedding $H_r^1(\mathbf{R}^N) \hookrightarrow L^4(\mathbf{R}^N)$ is compact, there exists a subsequence (write still (u_n)) such that

$$\begin{aligned} u_n^* &\rightharpoonup u_0 \quad \text{weakly} \quad \text{in} \quad H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N)\,, \\ u_n^* &\to u_0 \quad \text{strongly} \quad \text{in} \quad L^4(\mathbf{R}^N) \times L^4(\mathbf{R}^N)\,. \end{aligned}$$

Then it follows that

$$\begin{aligned}\|u_0\|_\infty^2 &\le \liminf_{n\to\infty} \|u_n^*\|_\infty^2 \le \liminf_{n\to\infty} \|u_n\|_\infty^2 \\ &= \liminf_{n\to\infty} (\mu_1 \|u_{n,1}\|_{L^4}^4 + 2\beta \|u_{n,1}u_{n,2}\|_{L^2}^2 + \mu_2 \|u_{n,2}\|_{L^4}^4) \\ &= \mu_1 \|u_{0,1}\|_{L^4}^4 + 2\beta \|u_{0,1}u_{0,2}\|_{L^2}^2 + \mu_2 \|u_{0,2}\|_{L^4}^4 .\end{aligned}$$

By the above inequality, there exists a unique $\theta_0 \in (0, 1]$ such that $\theta_0 u_0 \in \mathcal{N}_\infty$. Thus we see

$$b_{\mathcal{N}_\infty} \le \frac{\theta_0^2}{4} \|u_0\|_\infty^2 \le \liminf_{n\to\infty} \frac{\theta_0^2}{4} \|u_n\|_\infty^2 = \theta_0^2 b_{\mathcal{N}_\infty} ,$$

which implies $\theta_0 = 1$, $u_0 \in \mathcal{N}_\infty$, $I_\infty(u_0) = b_{\mathcal{N}_\infty}$. □

Next we prove that $b_{\mathcal{N}}$ is attained.

PROPOSITION 4.2. *For each $\beta > 0$, $b_{\mathcal{N}}$ is attained.*

PROOF. Firstly, we prove the inequality $b_{\mathcal{N}} \le b_{\mathcal{N}_\infty}$. By Proposition 4.1, there exists a $u_\infty \in \mathcal{N}_\infty$ such that $I_\infty(u_\infty) = b_{\mathcal{N}_\infty}$. With the assumption of $V_j(x)$ we obtain

$$\|u_\infty\|^2 \le \|u_\infty\|_\infty^2 = \mu_1 \|u_{\infty,1}\|_{L^4}^4 + 2\beta \|u_{\infty,1}u_{\infty,2}\|_{L^2}^2 + \mu_2 \|u_{\infty,2}\|_{L^4}^4 ,$$

which implies that there exists a $\theta_\infty \in (0, 1]$ such that $\theta_\infty u_\infty \in \mathcal{N}$. Then it follows that

$$b_{\mathcal{N}} \le I(\theta_\infty u_\infty) = \frac{\theta_\infty^2}{4} \|u_\infty\|^2 \le \frac{1}{4} \|u_\infty\|_\infty^2 = I_\infty(u_\infty) = b_{\mathcal{N}_\infty} . \tag{9}$$

Thus we obtain $b_{\mathcal{N}} \le b_{\mathcal{N}_\infty}$.

Next we consider two cases: $b_{\mathcal{N}} = b_{\mathcal{N}_\infty}$ and $b_{\mathcal{N}} < b_{\mathcal{N}_\infty}$.

If $b_{\mathcal{N}} = b_{\mathcal{N}_\infty}$ takes place, then by (9), we have $\theta_\infty = 1$. This implies that $u_\infty \in \mathcal{N}$ and $I(u_\infty) = b_{\mathcal{N}}$. This is our conclusion.

If $b_{\mathcal{N}} < b_{\mathcal{N}_\infty}$ takes place, then by Lemma 2.8, there exists a $(\mathrm{PS})_{b_{\mathcal{N}}, H}$ sequence (u_n). By Lemma 2.9, there exist subsequence (u_{n_k}), $\ell \in \mathbf{N}$, u_0 ($I'(u_0) = 0$), $\omega_i \ne 0$ ($I'_\infty(\omega_i) = 0$) and $(y_k^i) \subset \mathbf{R}^N$ such that

$$\left\| u_{n_k} - u_0 - \sum_{i=1}^{\ell} \omega_i(x - y_k^i) \right\| \to 0, \quad I(u_{n_k}) \to b_{\mathcal{N}} = I(u_0) + \sum_{i=1}^{\ell} I_\infty(\omega_i) .$$

Since $\omega_i \ne (0, 0)$, we have $b_{\mathcal{N}_\infty} \le I_\infty(\omega_i)$. By $b_{\mathcal{N}} < b_{\mathcal{N}_\infty}$, it follows that $\ell = 0$, which implies

$$u_{n_k} \to u_0 \quad \text{strongly} \quad \text{in} \quad H .$$

This shows that $u_0 \in \mathcal{N}$, $I(u_0) = b_{\mathcal{N}}$. □

REMARK 4.1. We consider the situation $b_{\mathcal{N}} = b_{\mathcal{N}_\infty}$ more precisely. We deal with the two cases. (a) $b_{\mathcal{N}_\infty}$ is attained by nontrivial functions u_0. In this case, we can show that both of $V_j(x)$ are constant functions. (b) $b_{\mathcal{N}_\infty}$ is attained by semitrivial functions u_0. We may assume that $u_0 = (u_1, 0)$. Then we can show that $V_1(x)$ is a constant function. Moreover, we can prove the equality $b_{\mathcal{N}} = b_{\mathcal{N}_\infty} = d_{\infty,1} = d_1$.

5. Proof of Theorem 1.1 (ii) (when β is large)

In this section, we prove the existence of a nontrivial positive solution of (E) when β is large. By Proposition 4.2, there exists a $u_0 = (u_{0,1}, u_{0,2}) \in \mathcal{N}$ such that $I(u_0) = b_{\mathcal{N}}$. We need to prove $u_{0,1},\ u_{0,2} \not\equiv 0$.

Following Ambrosetti–Colorado [2], let us define the constants which are related to the stability of semitrivial solution on $\mathcal{N}$.

DEFINITION 5.1. We define $\hat{\beta}_1$ and $\hat{\beta}_2$ as follows:

$$\hat{\beta}_1 := \inf_{(u_1,0)\in S_1} \inf_{\varphi_2 \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\|\varphi_2\|_2^2}{\int_{\mathbf{R}^N} u_1^2 \varphi_2^2 dx},$$

$$\hat{\beta}_2 := \inf_{(0,u_2)\in S_2} \inf_{\varphi_1 \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\|\varphi_1\|_1^2}{\int_{\mathbf{R}^N} u_2^2 \varphi_1^2\, dx}.$$

Here, S_1 and S_2 are defined by

$$S_1 = \{(u_1, 0) \in \mathcal{N} \mid I(u_1, 0) = d_1\},$$

$$S_2 = \{(0, u_2) \in \mathcal{N} \mid I(0, u_2) = d_2\}.$$

Main result in this section is following:

THEOREM 5.1. *If $\beta > \max\{\hat{\beta}_1,\ \hat{\beta}_2\}$, then both components of any minimizer of I on $\mathcal{N}$ are not zero, i.e.,*

$$I(u_0) = b_{\mathcal{N}}, \quad u_0 \in \mathcal{N} \ \Rightarrow\ u_{0,1},\ u_{0,2} \not\equiv 0.$$

PROOF. It suffices to prove $b_{\mathcal{N}} < \min\{d_1, d_2\}$. Since $\beta > \max\{\hat{\beta}_1, \hat{\beta}_2\}$, there exist $(u_1, 0) \in S_1$, $(0, u_2) \in S_2$, $\varphi_1, \varphi_2 \in H^1(\mathbf{R}^N)$ such that

$$\frac{\|\varphi_1\|_1^2}{\int_{\mathbf{R}^N} u_2^2 \varphi_1^2\, dx} < \beta, \quad \frac{\|\varphi_2\|_2^2}{\int_{\mathbf{R}^N} u_1^2 \varphi_2^2\, dx} < \beta.$$

We remark that $\{0\} \times H^1(\mathbf{R}^N) \subset T_{(u_1,0)}\mathcal{N}$ and $H^1(\mathbf{R}^N) \times \{0\} \subset T_{(0,u_2)}\mathcal{N}$. In fact, for each $\psi_1, \psi_2 \in H^1(\mathbf{R}^N)$, we have

$$J'(u_1, 0)[(0, \psi_2)] = 0, \quad J'(0, u_2)[(\psi_1, 0)] = 0.$$

Thus, by Lemma 2.2, $\{0\} \times H^1(\mathbf{R}^N) \subset T_{(u_1,0)}\mathcal{N}$ and $H^1(\mathbf{R}^N) \times \{0\} \subset T_{(0,u_2)}\mathcal{N}$ hold.

Let $\gamma_1, \gamma_2 \in C^2((-\varepsilon, \varepsilon), \mathcal{N})$ satisfy

$$\gamma_1(0) = (u_1, 0), \quad \gamma_1'(0) = (0, \varphi_2), \quad \gamma_2(0) = (0, u_2), \quad \gamma_2'(0) = (\varphi_1, 0).$$

By the Taylor expansion of $I(\gamma_j(t))$ and $I'(u_1, 0) = I'(0, u_2) = 0$, we obtain

$$I(\gamma_j(t)) = I(\gamma_j(0)) + \frac{1}{2} I''(\gamma_j(0))[\gamma_j'(0), \gamma_j'(0)]t^2 + o(t^2) .$$

Since

$$I''(u_1, 0)[(0, \varphi_2), (0, \varphi_2)] = \|\varphi_2\|_2^2 - \beta \int_{\mathbf{R}^N} u_1^2 \varphi_2^2 \, dx < 0 ,$$
$$I''(0, u_2)[(\varphi_1, 0), (\varphi_1, 0)] < 0 ,$$

it follows that for sufficiently small $t > 0$

$$I(\gamma_j(t)) - I(\gamma_j(0)) < 0 .$$

Thus we have $b_{\mathcal{N}} < \min\{d_1, d_2\}$. □

Next, we give a proof of Theorem 1.1 (ii).

PROOF OF THEOREM 1.1 (II). By Theorem 5.1, there exists a u_0 such that $b_{\mathcal{N}} = I(u_0)$, $u_{0,1} \neq 0$, $u_{0,2} \neq 0$. By Remark 2.2, we have

$$|u_0| = (|u_{0,1}|, |u_{0,2}|) \in \mathcal{N} , \quad b_{\mathcal{N}} = I(u_0) = I(|u_0|) ,$$

which implies that $|u_0|$ is also a minimizer of I on $\mathcal{N}$. Thus we may assume that $u_{0,1} \geq 0$, $u_{0,1} \not\equiv 0$, $u_{0,2} \geq 0$, $u_{0,2} \not\equiv 0$. By the maximum principle we have $u_{0,1}, u_{0,2} > 0$. □

6. Proofs of Theorem 1.1 (i) and Theorem 1.2. (when β is small)

6.1. Proof of Theorem 1.1 (i). The aim of this subsection is to prove the existence of a nontrivial positive solution of (E) when β is small.

The following two propositions give some estimates of $\hat{b}_{\mathcal{M}}$.

PROPOSITION 6.1. *For each $\beta > 0$,*

(i) $\hat{b}_{\mathcal{M}} < d_1 + d_{\infty,2}$, $\hat{b}_{\mathcal{M}} < d_{\infty,1} + d_2$.

(ii) $\hat{b}_{\mathcal{M}_\infty} < d_{\infty,1} + d_{\infty,2}$.

REMARK 6.1. $\hat{b}_{\mathcal{M}}$ depends on β but $d_1, d_2, d_{\infty,1}, d_{\infty,2}$ are independent of β.

PROPOSITION 6.2. *There exists a $\tilde{\beta}_1 > 0$ such that for each $\beta \in (0, \tilde{\beta}_1)$*

$$\hat{b}_{\mathcal{M}} < \hat{b}_{\mathcal{M}_\infty} .$$

Proofs of Propositions 6.1 and 6.2 will be given in subsection 6.2.

THEOREM 6.3. *There exists a $\tilde{\beta}_2 > 0$ such that for each $\beta \in (0, \tilde{\beta}_2)$, $\hat{b}_{\mathcal{M}}$ is attained.*

PROOF OF THEOREM 6.3. Set $\alpha_0 = \min\{d_1 + d_{\infty,2}, d_{\infty,1} + d_2\}$. By Proposition 6.1, $\mathcal{M}^{\alpha_0} \neq \emptyset$ for all $\beta \in (0, \sqrt{\mu_1\mu_2})$. By Proposition 2.6, there exist $\hat{\beta}_0 > 0$ and $\delta_1 > 0$ such that for each $u \in \mathcal{M}^{\alpha_0}$ and $\beta \in (0, \hat{\beta}_0)$,

$$\|u_1\|_1 \geq \delta_1 , \quad \|u_2\|_2 \geq \delta_1 . \tag{10}$$

Suppose $0 < \beta < \min\{\tilde{\beta}_1, \hat{\beta}_0\}$. Then we remark that there exists a $(\mathrm{PS})_{\hat{b}_{\mathcal{M}}, H}$ sequence (u_n) by Lemmas 2.7 and 2.8. Then by Lemma 2.9, we have

$$\left\| u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(x - y_n^i) \right\| \to 0 , \tag{11}$$

$$I(u_n) \to \hat{b}_{\mathcal{M}} = I(u_0) + \sum_{i=1}^{\ell} I_\infty(\omega_i) . \tag{12}$$

We shall show that $u_0 = (u_{0,1}, u_{0,2})$, $u_{0,1} \neq 0$, $u_{0,2} \neq 0$ and $\ell = 0$. We divide our argument into three steps.

Step 1. $u_0 \not\equiv (0, 0)$.

We prove indirectly and we assume that $u_0 \equiv (0, 0)$. By (12), it follows that

$$\hat{b}_{\mathcal{M}} = \sum_{i=1}^{\ell} I_\infty(\omega_i) .$$

By $\hat{b}_{\mathcal{M}} > 0$, we obtain $\ell \neq 0$. Since $\hat{b}_{\mathcal{M}} < \hat{b}_{\mathcal{M}_\infty}$, we conclude that one of the components of ω_i equals 0. Moreover if $\ell \geq 2$, we have

$$\omega_{i,1} \equiv 0 \quad (1 \leq i \leq \ell) \quad \text{or} \quad \omega_{i,2} \equiv 0 \quad (1 \leq i \leq \ell) . \tag{13}$$

Otherwise, we have $\hat{b}_{\mathcal{M}} \geq d_{\infty,1} + d_{\infty,2}$, which contradicts Proposition 6.1.

Suppose that $\omega_{i,1} \equiv 0 \quad (1 \leq i \leq \ell)$. By (11), we obtain $\|u_{n,1}\|_1 \to 0$, which contradicts (10). In a similar way, $\omega_{i,2} \equiv 0$ $(1 \leq i \leq \ell)$ does not take place. This implies that $u_0 \not\equiv (0, 0)$.

Step 2. $u_0 \notin (H^1(\mathbf{R}^N) \times \{0\}) \cup (\{0\} \times H^1(\mathbf{R}^N))$.

We prove indirectly and we assume that $u_0 \in H^1(\mathbf{R}^N) \times \{0\}$. By (12) we have

$$\hat{b}_{\mathcal{M}} = I(u_0) + \sum_{i=1}^{\ell} I_\infty(\omega_i) .$$

Since $\hat{b}_{\mathcal{M}} < \hat{b}_{\mathcal{M}_\infty}$, one of the components of ω_i is equal to 0 for $1 \leq i \leq \ell$. Since $\hat{b}_{\mathcal{M}} < d_1 + d_{\infty,2}$ and $d_1 \leq I(u_0)$, we have

$$\omega_{i,2} \equiv 0 \quad \text{for} \quad 1 \leq i \leq \ell . \tag{14}$$

From (11) and (14), it follows that $\|u_{n,2}\|_2 \to 0$, which contradicts (10). So, we conclude $u_0 \notin H^1(\mathbf{R}^N) \times \{0\}$. In a similar way, we can prove that $u_0 \notin \{0\} \times H^1(\mathbf{R}^N)$.

Step 3. Conclusion.

Now we complete a proof of Theorem 6.3. By Steps 1 and 2, it follows that $u_{0,1},\ u_{0,2} \neq 0$. Since $\hat{b}_{\mathcal{M}} \leq I(u_0)$ and $I_\infty(\omega_i) > 0$, we have $\ell = 0$. By Remark 2.5, (u_n) converges to u_0 strongly in H, so $I(u_0) = \inf_{\mathcal{M}} I$. □

We give the proof of Theorem 1.1 (i).

PROOF OF THEOREM 1.1 (i). As in the proof of Theorem 1.1 (ii), we obtain a nontrivial positive solution of (E) by Theorem 6.3 and the maximum principle. □

6.2. Proofs of Propositions 6.1 and 6.2. Before proving Propositions 6.1 and 6.2, we state a useful lemma. For $u \in H$, $u_1 \neq 0$, $u_2 \neq 0$, we set

$$\begin{aligned} f_u(s_1, s_2) &= I(\sqrt{s_1}u_1, \sqrt{s_2}u_2) \\ &= \frac{s_1}{2}\|u_1\|_1^2 + \frac{s_2}{2}\|u_2\|_2^2 - \frac{s_1^2}{4}\mu_1\|u_1\|_{L^4}^4 - \frac{s_1 s_2}{2}\beta\|u_1u_2\|_{L^2}^2 - \frac{s_2^2}{4}\mu_2\|u_2\|_{L^4}^4 . \end{aligned}$$

LEMMA 6.4. *Let $u \in H$, $u_1 \neq 0$, $u_2 \neq 0$. Then the following hold.*

(i) *Let $0 \leq \beta < \sqrt{\mu_1\mu_2}$. Then $f_u(s_1, s_2)$ is strictly concave in $[0, \infty) \times [0, \infty)$.*

(ii) *Let $u \in \mathcal{M}$ and $0 \leq \beta < \sqrt{\mu_1\mu_2}$. Then $(1, 1)$ is an unique maximum point of $f_u(s_1, s_2)$. Namely, it follows*

$$I(u) = f_u(1, 1) = \max_{[0,\infty)\times[0,\infty)} I(\sqrt{s_1}u_1, \sqrt{s_2}u_2) .$$

(iii) *Let $\beta \geq 0$ and $(s_{0,1}, s_{0,2}) \in (0, \infty) \times (0, \infty)$ be a maximum point of $f_u(s_1, s_2)$. Then $(\sqrt{s_{0,1}}u_1, \sqrt{s_{0,2}}u_2) \in \mathcal{M}$.*

REMARK 6.2. Similar results hold for I_∞ and $\mathcal{M}_\infty$.

PROOF. This lemma is proved in Lin–Wei [14], however, for reader's convenience, we give a proof.

(i) Differentiating $f_u(s_1, s_2)$, we have

$$\begin{aligned} \frac{\partial f_u}{\partial s_1} &= \frac{1}{2}\|u_1\|_1^2 - \frac{s_1}{2}\mu_1\|u_1\|_{L^4}^4 - \frac{s_2}{2}\beta\|u_1u_2\|_{L^2}^2 , \\ \frac{\partial f_u}{\partial s_2} &= \frac{1}{2}\|u_2\|_2^2 - \frac{s_1}{2}\beta\|u_1u_2\|_{L^2}^2 - \frac{s_2}{2}\mu_2\|u_2\|_{L^4}^4 , \\ \frac{\partial^2 f_u}{\partial s_j^2} &= -\frac{1}{2}\mu_j\|u_j\|_{L^4}^4 \ (j = 1, 2) , \quad \frac{\partial^2 f_u}{\partial s_1 \partial s_2} = -\frac{1}{2}\beta\|u_1u_2\|_{L^2}^2 . \end{aligned} \tag{15}$$

Since $0 \leq \beta < \sqrt{\mu_1\mu_2}$, the matrix

$$\begin{pmatrix} \frac{\partial^2 f_u}{\partial s_1^2}(s_1, s_2) & \frac{\partial^2 f_u}{\partial s_1 \partial s_2}(s_1, s_2) \\ \frac{\partial^2 f_u}{\partial s_1 \partial s_2}(s_1, s_2) & \frac{\partial^2 f_u}{\partial s_2^2}(s_1, s_2) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\mu_1\|u_1\|_{L^4}^4 & -\beta\|u_1u_2\|_{L^2}^2 \\ -\beta\|u_1u_2\|_{L^2}^2 & -\mu_2\|u_2\|_{L^4}^4 \end{pmatrix}$$

is negative definite. Thus $f_u(s_1, s_2)$ is strictly concave in $[0, \infty) \times [0, \infty)$.

(ii) Suppose $u \in \mathcal{M}$. By (15) and $\beta \in [0, \sqrt{\mu_1\mu_2})$, we have

$$\nabla f_u(s_1, s_2) = (0, 0) \;\Leftrightarrow\; (s_1, s_2) = (1, 1)\,.$$

Since $f_u(s_1, s_2)$ is strictly concave, $(1, 1)$ is an unique maximum point and

$$I(u) = f_u(1, 1) = \max_{[0,\infty)\times[0,\infty)} I(\sqrt{s_1}u_1, \sqrt{s_2}u_2)\,.$$

(iii) Suppose $(s_{0,1}, s_{0,2}) \in (0, \infty) \times (0, \infty)$ is a maximum point of $f_u(s_1, s_2)$. Since $\nabla f_u(s_{0,1}, s_{0,2}) = (0, 0)$, we have

$$s_{0,1}\|u_1\|_1^2 = s_{0,1}^2\mu_1\|u_1\|_{L^4}^4 + s_{0,1}s_{0,2}\beta\|u_1u_2\|_{L^2}^2\,,$$
$$s_{0,2}\|u_2\|_2^2 = s_{0,1}s_{0,2}\beta\|u_1u_2\|_{L^2}^2 + s_{0,2}^2\mu_2\|u_2\|_{L^4}^4\,.$$

Thus this implies $(\sqrt{s_{0,1}}u_{0,1}, \sqrt{s_{0,2}}u_{0,2}) \in \mathcal{M}$. □

Firstly we prove Proposition 6.1.

PROOF OF PROPOSITION 6.1. We only prove $\hat{b}_{\mathcal{M}} < d_1 + d_{\infty,2}$ since we can prove other inequalities in a similar way. By Lemma 3.1, we suppose that $(\varphi_{0,1}, 0) \in \mathcal{N}$, $(0, \varphi_{\infty,2}) \in \mathcal{N}_\infty$ satisfy

$$I(\varphi_{0,1}, 0) = d_1\,, \quad I_\infty(0, \varphi_{\infty,2}) = d_{\infty,2}\,, \quad \varphi_{0,1} > 0, \quad \varphi_{\infty,2} > 0\,.$$

We remark that for a $k \in \mathbf{N}$, it follows $\|\varphi_{0,1}(x)\varphi_{\infty,2}(x - ke_1)\|_{L^2}^2 \to 0$ as $k \to \infty$ where $e_1 = (1, 0, \dots, 0)$. Thus we have

$$g_k(s_1, s_2) \equiv I(\sqrt{s_1}\varphi_{0,1}(x), \sqrt{s_2}\varphi_{\infty,2}(x - ke_1)) \to g(s_1, s_2) \quad \text{in} \quad C_{\mathrm{loc}}^2((\mathbf{R}_+)^2)$$

where

$$g(s_1, s_2) = \frac{s_1}{2}\|\varphi_{0,1}\|_1^2 - \frac{s_1^2}{4}\mu_1\|\varphi_{0,1}\|_{L^4}^4 + \frac{s_2}{2}\|\varphi_{\infty,2}\|_{\infty,2}^2 - \frac{s_2^2}{4}\|\varphi_{\infty,2}\|_{L^4}^4\,.$$

Since $g(s_1, s_2)$ has an unique maximum point $(1, 1)$ and $g_k(s_1, s_2) \le g(s_1, s_2)$, $g_k(s_1, s_2)$ has a maximum point $(s_{k,1}, s_{k,2}) \in (0, \infty) \times (0, \infty)$ for a sufficiently large k. By Lemma 6.4 we have $(\sqrt{s_{k,1}}\varphi_{0,1}(x), \sqrt{s_{k,2}}\varphi_{\infty,2}(x - ke_1)) \in \mathcal{M}$.

Thus we have

$$\begin{aligned}
\hat{b}_{\mathcal{M}} &\le I(\sqrt{s_{k,1}}\varphi_{0,1}, \sqrt{s_{k,2}}\varphi_{\infty,2}(x - ke_1)) \\
&= \frac{1}{2}s_{k,1}\|\varphi_{0,1}\|_1^2 + \frac{1}{2}s_{k,2}\|\varphi_{\infty,2}(x - ke_1)\|_2^2 - \frac{1}{4}s_{k,1}^2\mu_1\|\varphi_{0,1}\|_{L^4}^4 \\
&\quad - \frac{1}{2}\beta s_{k,1}s_{k,2}\|\varphi_{0,1}\varphi_{\infty,2}(x - ke_1)\|_{L^2}^2 - \frac{1}{4}s_{k,2}^2\mu_2\|\varphi_{\infty,2}\|_{L^4}^4
\end{aligned}$$

$$\leq \frac{1}{2}s_{k,1}\|\varphi_{0,1}\|_1^2 + \frac{1}{2}s_{k,2}\|\varphi_{\infty,2}\|_{\infty,2}^2 - \frac{1}{4}s_{k,1}^2\mu_1\|\varphi_{0,1}\|_{L^4}^4$$
$$-\frac{1}{2}\beta s_{k,1}s_{k,2}\|\varphi_{0,1}\varphi_{\infty,2}(x-ke_1)\|_{L^2}^2 - \frac{1}{4}s_{k,2}^2\mu_2\|\varphi_{\infty,2}\|_{L^4}^4$$
$$= d_1 + d_{\infty,2} + \frac{1}{2}(s_{k,1}-1)\|\varphi_{0,1}\|_1^2 + \frac{1}{2}(s_{k,2}-1)\|\varphi_{\infty,2}\|_{\infty,2}^2$$
$$+\frac{\mu_1}{4}(1-s_{k,1}^2)\|\varphi_{0,1}\|_{L^4}^4 + \frac{\mu_2}{4}(1-s_{k,2}^2)\|\varphi_{\infty,2}\|_{L^4}^4$$
$$-\frac{1}{2}\beta s_{k,1}s_{k,2}\|\varphi_{0,1}\varphi_{\infty,2}(x-ke_1)\|_{L^2}^2 .$$

Since

$$\|\varphi_{0,1}\|_1^2 = \mu_1\|\varphi_{0,1}\|_{L^4}^4 , \quad \|\varphi_{\infty,2}\|_{\infty,2} = \mu_2\|\varphi_{\infty,2}\|_{L^4}^4 ,$$

we obtain

$$\frac{1}{2}(s_{k,1}-1)\|\varphi_{0,1}\|_1^2 + \frac{\mu_1}{4}(1-s_{k,1}^2)\|\varphi_{0,1}\|_{L^4}^4 = \frac{\|\varphi_{0,1}\|_1^2}{4}(-s_{k,1}^2 + 2s_{k,1} - 1)$$
$$= -\frac{\|\varphi_{0,1}\|_1^2}{4}(s_{k,1}-1)^2 \leq 0 ,$$
$$\frac{1}{2}(s_{k,2}-1)\|\varphi_{\infty,2}\|_{\infty,2}^2 + \frac{\mu_2}{4}(1-s_{k,2}^2)\|\varphi_{\infty,2}\|_{L^4}^4 \leq 0 .$$

Moreover, since $\varphi_{0,1},\ \varphi_{\infty,2} > 0$, it follows that $\|\varphi_{0,1}\varphi_{\infty,2}(x-ke_1)\|_{L^2}^2 > 0$. Hence we have

$$\hat{b}_{\mathcal{M}} < d_1 + d_{\infty,2} .$$

□

The following lemma is related to the existence of minimizer for $\hat{b}_{\mathcal{M}_\infty}$, which is due to Lin–Wei [14] and Sirakov [20].

LEMMA 6.5 (Lin–Wei [14] and Sirakov [20]). *There exists a $\bar{\beta} \in (0, \sqrt{\mu_1\mu_2}]$ such that if $\beta \in (0, \bar{\beta})$, then $\hat{b}_{\mathcal{M}_\infty}$ is attained by a nontrivial positive function $\omega = (\omega_1, \omega_2)$.*

Now we prove Proposition 6.2.

PROOF OF PROPOSITION 6.2. Set $\tilde{\beta}_1 = \bar{\beta}$ where $\bar{\beta}$ is given in Lemma 6.5. By Lemma 6.5, there exists $\omega \in \mathcal{M}_\infty$ such that $I_\infty(\omega) = \hat{b}_{\mathcal{M}_\infty}$ and $\omega_j > 0$ in $\mathbf{R}^N$. By Lemma 6.4 a function

$$h(s_1, s_2) \equiv I_\infty(\sqrt{s_1}\omega_1, \sqrt{s_2}\omega_2)$$

has an unique maximum point $(1, 1)$. Let $h_k(s_1, s_2) \equiv I(\omega_1(x-ke_1), \omega_2(x-ke_1))$. Since $h_k(s_1, s_2) \leq h(s_1, s_2)$ and

$$h_k(s_1, s_2) \equiv I(\omega_1(x-ke_1), \omega_2(x-ke_1)) \to h(s_1, s_2) \quad \text{in} \quad C^2_{\mathrm{loc}}((\mathbf{R}_+)^2) ,$$

$h_k(s_1, s_2)$ has a maximum point $(s_{k,1}, s_{k,2}) \in (0, \infty) \times (0, \infty)$ for a sufficiently large k. By Lemma 6.4, we have

$$(\sqrt{s_{k,1}}\omega_1(x - ke_1), \sqrt{s_{k,2}}\omega_2(x - ke_1)) \in \mathcal{M} .$$

By Lemma 6.4 again, we have

$$\begin{aligned}\hat{b}_{\mathcal{M}} &\le I(\sqrt{s_{k,1}}\omega_1(x - ke_1), \sqrt{s_{k,2}}\omega_2(x - ke_1)) \\ &< I_\infty(\sqrt{s_{k,1}}\omega_1(x - ke_1), \sqrt{s_{k,2}}\omega_2(x - ke_1)) = I_\infty(\sqrt{s_{k,1}}\omega_1(x), \sqrt{s_{k,2}}\omega_2(x)) \\ &\le \max_{(s_1,s_2)\in[0,\infty)\times[0,\infty)} I_\infty(\sqrt{s_1}\omega_1(x), \sqrt{s_2}\omega_2(x)) = \hat{b}_{\mathcal{M}_\infty} .\end{aligned}$$

We complete the proof. □

6.3. Proof of Theorem 1.2. In this subsection, we prove Theorem 1.2. When $\beta > 0$ is large, in other words, Theorem 1.2(ii) follows from the construction of a positive solution of (E). So we only prove (i). The main result in this section is the following.

PROPOSITION 6.6. *For each sufficiently small $\beta > 0$, it holds*

$$b_{\mathcal{N}} < \hat{b}_{\mathcal{M}} . \tag{16}$$

We remark that (16) shows that the minimizer of $\inf_{\mathcal{N}} I$ is a semitrivial solution and a proof of Theorem 1.2 easily follows.

PROOF OF PROPOSITION 6.6. We prove (16) indirectly. So we assume that there exists a sequence (β_n) such that $\beta_n \to 0$ and $\hat{b}_{\mathcal{M}_n} = b_{\mathcal{N}_n}$, where

$$\begin{aligned}I_n(u) &= \frac{\|u\|^2}{2} - \frac{1}{4}\int_{\mathbf{R}^N}(\mu_1 u_1^4 + 2\beta_n u_1^2 u_2^2 + \mu_2 u_2^4)\,dx , \\ \mathcal{N}_n &= \left\{u \in H \mid u \neq 0,\ I_n'(u)u = 0\right\} , \\ \mathcal{M}_n &= \{u \in H \mid u_1,\ u_2 \neq 0,\ I_n'(u)(u_1, 0) = I_n'(u)(0, u_2) = 0\} , \\ b_{\mathcal{N}_n} &= \inf_{u\in\mathcal{N}_n} I_n(u), \quad b_{\mathcal{M}_n} = \inf_{u\in\mathcal{M}_n} I_n(u) .\end{aligned}$$

By Theorem 6.3, there exists a $(u_n) \subset \mathcal{M}_n$ such that $I_n(u_n) = \hat{b}_{\mathcal{M}_n} = b_{\mathcal{N}_n}$. It is obvious that (u_n) is a bounded sequence. So we assume that $u_n \rightharpoonup u_0$ weakly in H. Since

$$\begin{cases} -\Delta u_{n,1} + V_1(x)u_{n,1} = \mu_1 u_{n,1}^3 + \beta_n u_{n,1}u_{n,2}^2 & \text{in} \quad \mathbf{R}^N , \\ -\Delta u_{n,2} + V_2(x)u_{n,2} = \beta_n u_{n,1}^2 u_{n,2} + \mu_2 u_{n,2}^3 & \text{in} \quad \mathbf{R}^N , \end{cases}$$

we have

$$\begin{cases} -\Delta u_{0,1} + V_1(x)u_{0,1} = \mu_1 u_{0,1}^3 & \text{in} \quad \mathbf{R}^N , \\ -\Delta u_{0,2} + V_2(x)u_{0,2} = \mu_2 u_{0,2}^3 & \text{in} \quad \mathbf{R}^N . \end{cases} \tag{17}$$

We prove the following claim.

CLAIM. $u_{0,1} \equiv 0$ or $u_{0,2} \equiv 0$.

PROOF OF CLAIM. We assume that $u_{0,1} \not\equiv 0$ and $u_{0,2} \not\equiv 0$. From (17), we have $d_1 + d_2 \le I_0(u_0)$. On the other hand, since $I_n(u_n) = \|u_n\|^2/4$ and $u_n \rightharpoonup u_0$, it follows that

$$I_0(u_0) \le \liminf_{n\to\infty} I_n(u_n) = \liminf_{n\to\infty} b_{\mathcal{N}_n} \le \min\{d_1, d_2\}\,.$$

This is contradiction, hence $u_{0,1} \equiv 0$ or $u_{0,2} \equiv 0$. □

Suppose that $u_{0,2} \equiv 0$. By Proposition 2.6, there exists a $\delta_1 > 0$ such that $\|u_{n,j}\|_{L^4} \ge \delta_1$ $(j = 1, 2)$. Developing a concentration–compactness type argument, we can find a sequence $(y_n) \subset \mathbf{R}^N$ such that

$$|y_n| \to \infty\,, \quad \|u_{n,2}\|_{L^4(Q+y_n)} \to c > 0\,,$$
$$u_{n,2}(x + y_n) \rightharpoonup \omega_2 \quad \text{weakly} \quad \text{in} \quad H^1(\mathbf{R}^N)\,,$$

where $Q = [0, 1]^N$. Moreover ω_2 satisfies that $\omega_2 \not\equiv 0$ and

$$-\Delta\omega_2 + V_{\infty,2}\omega_2 = \mu_2\omega_2^3\,.$$

Since

$$\begin{aligned} I_n(u_n) &= \frac{1}{4}\int_{\mathbf{R}^N} \mu_1 u_{n,1}^4 + 2\beta_n u_{n,1}^2 u_{n,2}^2 + \mu_2 u_{n,2}^4 dx \\ &\ge \frac{\mu_1}{4}\int_{\mathbf{R}^N} u_{n,1}^4 dx + \frac{\mu_2}{4}\int_{\mathbf{R}^N} u_{n,2}^4(x + y_n) dx\,, \end{aligned}$$

we have

$$\begin{aligned} I_{\infty,0}(0, \omega_2) = \frac{\mu_2}{4}\int_{\mathbf{R}^N} \omega_2^4 dx &< \frac{\mu_1}{4}\delta_1^4 + \liminf_{n\to\infty} \frac{\mu_2}{4}\int_{\mathbf{R}^N} u_{n,2}^4(x + y_n) dx \\ &\le \liminf_{n\to\infty} I_n(u_n) = \lim_{n\to\infty} b_{\mathcal{N}_n} \le \min\{d_1, d_2\}\,, \end{aligned}$$

which implies that $d_{\infty,2} < \min\{d_1, d_2\} \le d_2$. This is contradiction. The situation $u_{0,1} \equiv 0$ can be treated similarly. Thus we have (16). □

7. Proof of Theorem 1.3.

In this section, we prove Theorem 1.3.

PROOF OF THEOREM 1.3. We follow the idea in Tanaka [21]. We prove indirectly and we assume that (E) has a positive solution u. Since $V_j(x) \in C^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, we remark $u_j \in H^2(\mathbf{R}^N)$. Without loss of generality we may assume that $\nu = e_1 = (1, 0, \ldots, 0)$. Since $I'(u)\big[\big(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}\big)\big] = 0$, we have

$$\sum_{j=1}^{2}\left\langle u_j, \frac{\partial u_j}{\partial x_1}\right\rangle_j = \sum_{j=1}^{2}\int_{\mathbf{R}^N} \mu_j u_j^3 \frac{\partial u_j}{\partial x_1} dx + \beta\int_{\mathbf{R}^N}\left(u_1 u_2^2 \frac{\partial u_1}{\partial x_1} + u_1^2 u_2 \frac{\partial u_2}{\partial x_1}\right) dx\,. \tag{18}$$

Here, we remark that

$$\int_{\mathbf{R}^N} \nabla u_j \cdot \nabla \left(\frac{\partial u_j}{\partial x_1} \right) dx = 0\,, \qquad \int_{\mathbf{R}^N} \mu_j u_j^3 \frac{\partial u_j}{\partial x_1} dx = 0\,, \tag{19}$$

$$\int_{\mathbf{R}^N} \left(u_1 u_2^2 \frac{\partial u_1}{\partial x_1} + u_1^2 u_2 \frac{\partial u_2}{\partial x_1} \right) dx = 0\,, \tag{20}$$

$$\int_{\mathbf{R}^N} V_j(x) u_j \frac{\partial u_j}{\partial x_1} dx = -\frac{1}{2} \int_{\mathbf{R}^N} \frac{\partial V_j}{\partial x_1} u_j^2 dx\,. \tag{21}$$

Using (19)–(21), it follows from (18) that

$$-\frac{1}{2} \sum_{j=1}^{2} \int_{\mathbf{R}^N} \frac{\partial V_j}{\partial x_1} u_j^2 dx = 0\,.$$

By (V3′), (V4′) and $u_j > 0$, this is contradiction, so (E) has no positive solution.

Next we show (19)–(21). We only prove (20) since the proofs of other cases are similar. For $\varphi_1, \varphi_2 \in C_0^\infty(\mathbf{R}^N)$, we have

$$\left(\varphi_1 \varphi_2^2 \frac{\partial \varphi_1}{\partial x_1} + \varphi_1^2 \varphi_2 \frac{\partial \varphi_2}{\partial x_1} \right) = \frac{1}{2} \frac{\partial}{\partial x_1} (\varphi_1^2 \varphi_2^2)\,.$$

Thus

$$\begin{aligned} \int_{\mathbf{R}^N} \left(\varphi_1 \varphi_2^2 \frac{\partial \varphi_1}{\partial x_1} + \varphi_1^2 \varphi_2 \frac{\partial \varphi_2}{\partial x_1} \right) dx &= \int_{\mathbf{R}^N} \frac{\partial}{\partial x_1} (\varphi_1^2 \varphi_2^2) dx \\ &= \int_{\mathbf{R}^{N-1}} \int_{-\infty}^{\infty} \frac{\partial}{\partial x_1} (\varphi_1^2 \varphi_2^2) dx_1 dx' = 0\,. \end{aligned}$$

Since $C_0^\infty(\mathbf{R}^N)$ is dense in $H^2(\mathbf{R}^N)$ and the functional

$$(u_1, u_2) \mapsto \int_{\mathbf{R}^N} \left(u_1 u_2^2 \frac{\partial u_1}{\partial x_1} + u_1^2 u_2 \frac{\partial u_2}{\partial x_1} \right) dx : H^2(\mathbf{R}^N) \times H^2(\mathbf{R}^N) \to \mathbf{R}$$

is continuous, (20) holds. □

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