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Endomorphism Rings of Split (B, N)-pairs

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Introduction.

This note concerns a certain property of endomorphism rings of subgroups of a split (B, N)-pair (G, B, N, R, U) of characteristic p and rank n containing U. Let $\mathscr{A}$ be a subgroup of G containing U and K be an arbitrary field. Let $K\mathscr{A}$ be the group algebra of $\mathscr{A}$ over K and $\bar{U}=\sum_{u\in U} u$. Then we have the following theorem (Theorem 2.4): Let $E=\mathrm{End}_{K\mathscr{A}}(K\mathscr{A}\bar{U})$, then

(i) E is a Frobenius algebra, and

(ii) if $\mathrm{ch}(K)\neq p$, E is also a symmetric algebra.

(i) of the above theorem was announced by the author at The Kyoto Conference on Permutation Groups with a complicated proof when $E=\mathrm{End}_{KG}(KG\bar{U})$ and K is an algebraically closed field of characteristic P (see [9]). However the author improved the proof as is done in this note soon after the conference, and he received letters from Professor James A. Green, who had already got the same simple proof of (i) of Theorem 2.4 and proved more successful theorems on these kinds of Frobenius endomorphism rings with his student Ms. Nalsey Tinberg in their papers [12] [13]. Hence (i) of Theorem 2.4 may be considered as a generalization of his result.

In § 1 we show the classification of the subgroups $\mathscr{A}$ which was announced by the author at The 23rd Symposium on Algebra (see [8]), but the classification of those subgroups had been first shown by T. Yokonuma in case of Chevalley groups (see [11]).

In § 2 we construct certain linear maps from E into K by which E is a Frobenius or symmetric algebra, respectively.

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§ 1. Subgroups of a split (B, N)-pair (G, B, N, R, U) containing U.

Let (G, B, N, R, U) be a split (B, N)-pair of characteristic p and rank n. Let $H=B\cap N$ and $W=N/H$, then from the definitions of (B, N)-pairs and split (B, N)-pairs (see [3, Definitions 2.1 and 3.1]) (W, R) is a Coxeter system, and H is a p'-abelian group and $B=UH\triangleright U$ respectively. In this section we classify all the subgroups of G containing U (Theorem 1.6).

Let Φ be a root system of (W, R) with the base $\Delta=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ which corresponds to R. Let w_R be a unique element of maximal length in W and $R=\{w_1, w_2, \cdots, w_n\}$. We define the subgroups of G as follows.

$U_i=U_{\alpha_i}=U\cap {}^{w_i w_R}U$ for $\alpha_i\in\Delta$, where ${}^{w_i w_R}U=w_i w_R U(w_i w_R)^{-1}$.

$U_w^+=U\cap {}^{w^{-1}}U$, $U_w^-=U\cap {}^{w^{-1}w_R}U$ for $w\in W$.

$V={}^{w_R}U$

$V_i=U_{-\alpha_i}={}^{w_i}U_{\alpha_i}$ for $\alpha_i\in\Delta$.

$H_i=H\cap\langle U_i, V_i\rangle$ for $\alpha_i\in\Delta$, where $\langle U_i, V_i\rangle$ is the subgroup of G generated by U_i and V_i.

Let (w_i) be an element of $U_i V_i U_i$ such that $(w_i)H=w_i$ for all $1\leqq i\leqq n$ (see, for example, [2, Lemma 2.2]).

Let $(J, \mathscr{H})$ be a pair such that J is a subset of R and $\mathscr{H}$ is a subgroup of H containing H_i for each $w_i\in J$. We shall show that there exists a certain bijective correspondence between the set of all such pairs $\mathscr{P}=\{(J, \mathscr{H})\}$ and the set of subgroups of G containing U.

LEMMA 1.1. *Let h be an element of H, then there exists $h_0\in H_i$ for each $w_i\in R$ such that $(w_i)h(w_i)^{-1}=hh_0$.*

PROOF. This is a straight consequence from the fact that $h^{-1}U_ih\subset U_i$ and $h^{-1}V_ih\subset V_i$ for all $h\in H$. Q.E.D.

LEMMA 1.2. *Let g_0 be a mapping of Φ into $\{{}^w\langle U_i, V_i\rangle | w\in W, \alpha_i\in\Delta\}$ such that*

$$\begin{array}{ccc} g_0\colon \Phi & \longrightarrow & \{{}^w\langle U_i, V_i\rangle \mid w\in W,\ \alpha_i\in\Delta\}\,. \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ w(\alpha_i) & \longmapsto & {}^w\langle U_i, V_i\rangle \end{array}$$

Then g_0 is well-defined and surjective, and $g_0(w(\alpha))={}^w g_0(\alpha)$ for all $\alpha\in\Phi$ and $w\in W$.

PROOF. From the proof of Lemma 1.1, ${}^w\langle U_i, V_i\rangle$ is well-defined. Assume $w(\alpha_i)=w'(\alpha_j)$ where $\alpha_i, \alpha_j\in\Delta$ and $w, w'\in W$, then ${}^wU_i={}^{w'}U_j$ and

${}^{ww_i}U_i = {}^{w'w_j}U_j$ from [6, Theorem 3.6]. Hence g_0 is well-defined. Q.E.D.

Let N_J be the inverse image of W_J of the natural mapping $N \to W = N/H$. From Lemmas 1.1 and 1.2 and [1, Proposition 5 on p. 16] we have the following lemma.

LEMMA 1.3. *Let* $(J, \mathscr{H}) \in \mathscr{P}$, *and* $\tau(w) = (w_{j_1})(w_{j_2}) \cdots (w_{j_t})$ *where* $w \in W_J$ *and* $w = w_{j_1} w_{j_2} \cdots w_{j_t}$ *is a reduced expression of* w. *Then*

(i) $N_J \triangleright \mathscr{H}$;

(ii) $T: W_J \to N_J/\mathscr{H}$ *is a well-defined injective homomorphism*;
$$\begin{array}{ccc} W_J & \to & N_J/\mathscr{H} \\ \cup & & \cup \\ w & \to & \tau(w)\mathscr{H} \end{array}$$

(iii) $\mathscr{N}(J, \mathscr{H}) = \bigcup_{w \in W_J} \tau(w)\mathscr{H}$ *forms a subgroup of* N;

(iv) $\mathscr{N}(J, \mathscr{H}) \triangleright \mathscr{H}$ *and* $W_J \cong \mathscr{N}(J, \mathscr{H})/\mathscr{H}$.

PROOF.

(i) is clear from Lemma 1.1.

(ii) Let $w_i, w_j \in J$ and n_{ij} be the order of $w_i w_j$.

Assume n_{ij} be even. Then since

$$\underbrace{w_i w_j w_i w_j \cdots w_i w_j}_{n_{ij}} = \underbrace{w_j w_i w_j w_i \cdots w_j w_i}_{n_{ij}}, \text{ we have}$$

$$\underbrace{w_i w_j \cdots w_i w_j w_i}_{n_{ij}-1} w_j \underbrace{w_i^{-1} w_j^{-1} \cdots w_i^{-1} w_j^{-1} w_i^{-1}}_{n_{ij}-1} = w_j .$$

Let $w' = \underbrace{w_i w_j \cdots w_i w_j w_i}_{n_{ij}-1}$, then $r_{w'(\alpha_j)} = r_{\alpha_j}$ where r_α means the reflection of a root $\alpha \in \Phi$ (see [3, 1.1]). Hence $w'(\alpha_j) = \pm \alpha_j$. From Lemma 1.2 we have

$$\underbrace{(w_i)(w_j) \cdots (w_i)(w_j)}_{n_{ij}} \underbrace{(w_i)^{-1}(w_j)^{-1} \cdots (w_i)^{-1}(w_j)^{-1}}_{n_{ij}} \in H_j .$$

Therefore

$$\underbrace{(w_i)(w_j) \cdots (w_i)(w_j)}_{n_{ij}} \mathscr{H} = \underbrace{(w_j)(w_i) \cdots (w_j)(w_i)}_{n_{ij}} \mathscr{H} .$$

Similarly we have $\underbrace{(w_i)(w_j) \cdots (w_i)(w_j)(w_i)}_{n_{ij}} \mathscr{H} = \underbrace{(w_j)(w_i) \cdots (w_j)}_{n_{ij}} \mathscr{H}$ when n_{ij} is odd. Since (W_J, J) is a Coxeter system, T is a well-defined mapping from [1, Proposition 5, p. 16]. It is clear that T is an injective homomorphism.

(iii) is clear from (ii).

(iv) Since $T(W_J)=\{\tau(w)\mathscr{H}\mid w\in W_J\}$, (iv) is also clear from (ii). Q.E.D.

LEMMA 1.4. *Let W be the Weyl group of a root system Φ and $\Delta=\{\alpha_1,\cdots,\alpha_n\}$ be a base of Φ. Let $w_i=r_{\alpha_i}$ for all $1\leqq i\leqq n$, $R=\{w_1, w_2,\cdots, w_n\}$ and w_R be a unique element of maximal length in W where r_{α_i} is the reflection with respect to α_i.*

(i) *Let w be an element of W, then there exists $w'\in W$ such that $ww'=w_R$ and $l(ww')=l(w)+l(w')$ where $l(w)=\mathrm{Min}\{k\mid w=w_{j_1}\cdots w_{j_k}, w_{j_i}\in R\}$.*

(ii) *Let w be an element of W, then there exists $w'\in W$ such that $w'w=w_R$ and $l(w'w)=l(w')+l(w)$.*

PROOF. For any $\alpha_i\in\Delta$ and $w\in W$,
$w(\alpha_i)>0$ if and only if $l(ww_i)=l(w)+1$ and
$w(\alpha_i)<0$ if and only if $l(ww_i)=l(w)-1$ (see [10, Appendix]).
Since $w=w_R$ if and only if $w(\alpha_i)<0$ for any $\alpha_i\in\Delta$, (i) and (ii) are clear from the above fact. Q.E.D.

Let $J\subset R$ and $\Phi_J=\Phi\cap\sum_{w_i\in J}\boldsymbol{R}\alpha_i$ where $\boldsymbol{R}$ is the real number field, then it is clear that Φ_J is a root system with a base $\Delta_J=\{\alpha_i\mid w_i\in J\}$ and the Weyl group W_J. Let $\Phi_w^-=\{\alpha\in\Phi^+\mid w(\alpha)\in\Phi^-\}$ where $w\in W$, Φ^+ and Φ^- are the sets of positive roots and negative roots of Φ respectively.

Let $\alpha\in\Phi$, then there exists $w\in W$ and $\alpha_i\in\Delta$ such that $\alpha=w(\alpha_i)$. We write $U_\alpha={}^wU_i$, which does not depend on w and α_i (see [6, Theorem 3.6]).

LEMMA 1.5. *Let J be a subset of R and w_J be a unique element of maximal length in W_J. Let $n\in N_J$ and $w_i\in J$, then*

$$(w_i)U_{w_J}^-n\subset U_{w_J}^-(w_i)nU_{w_J}^-\cup(\bigcup_{u\in U_i^*}U_{w_J}^-h_i(u)nU_{w_J}^-)$$

where $U_i^*=U_i-\{1\}$ and h_i is the same mapping defined in [3, *Lemma* 4.4].

PROOF. Let $w=nH$. Suppose $l(w_iw)>l(w)$ i.e. $l(w^{-1}w_i)>l(w^{-1})$. Since $U_{w_J}^-=U_{w_i}^{-\,w_i}(U_{w_Jw_i}^-)$ from [2, Lemma 2.3],

$$(w_i)U_{w_J}^-n=(w_i)U_{w_i}^{-\,w_i}(U_{w_Jw_i}^-)n=(w_i)^{w_i}(U_{w_Jw_i}^-)U_{w_i}^-n=U_{w_Jw_i}^-(w_i)n^{n^{-1}}U_{w_i}^- .$$

Since $(\Phi_J)_w^-=\Phi_w^-$ for all $w\in W_J$, $\Phi_{w_J}^-$ is the set of all positive roots in Φ_J. Hence $U_{w_Jw_i}^-\subset U_{w_J}^-$ and ${}^{n^{-1}}U_{w_i}^-\subset U_{w_J}^-$ from [4, Proposition 1.4], because $U_{w_J}^-$ is generated by the subgroups U_β where $\beta\in\Phi_{w_J}^-$. Therefore $(w_i)U_{w_J}^-n\subset U_{w_J}^-(w_i)nU_{w_J}^-$.

Assume $l(w_iw)<l(w)$. Let $n'\in N_J$ such that $n=(w_i)n'$. Since

$U^-_{w_J} = {}^{w_i}(U^-_{w_J w_i})U^-_{w_i}$, we have

$$(w_i)U^-_{w_J}n = (w_i)U^-_{w_J}(w_i)n' = (w_i)^{w_i}(U^-_{w_J w_i})U^-_{w_i}(w_i)n'$$
$$= U^-_{w_J w_i}(w_i)U^-_{w_i}(w_i)n' \subset U^-_{w_J}(w_i)U_i(w_i)n' .$$

Hence

$$(w_i)U^-_{w_J}n \subset U^-_{w_J}(w_i)^2 n' \cup (\bigcup_{u \in U_i^*} U^-_{w_J} f_i(u)h_i(u)(w_i)g_i(u)n')$$
$$\subset U^-_{w_J}(w_i)nU^-_{w_J} \cup (\bigcup_{u \in U_i^*} U^-_{w_J}h_i(u)(w_i)U^-_{w_J}n') ,$$

because $f_i(u), g_i(u) \in U_i \subset U^-_{w_J}$ (see [3, Lemma 4.4]). Since $l(w_i w') > l(w')$ where $w' = n'H$, we have $(w_i)U^-_{w_J}n' \subset U^-_{w_J}(w_i)n'U^-_{w_J}$. Therefore

$$(w_i)U^-_{w_J}n \subset U^-_{w_J}(w_i)nU^-_{w_J} \cup (\bigcup_{u \in U_i^*} U^-_{w_J}h_i(u)U^-_{w_J}nU^-_{w_J})$$
$$= U^-_{w_J}(w_i)nU^-_{w_J} \cup (\bigcup_{u \in U_i^*} U^-_{w_J}h_i(u)nU^-_{w_J}) . \qquad \text{Q.E.D.}$$

THEOREM 1.6. (See [8, Theorem 2.6].) *Let* $\mathscr{A}(J, \mathscr{H}) = U\mathscr{N}(J, \mathscr{H})U$ *where* $(J, \mathscr{H}) \in \mathscr{P}$. *Then,*

(i) $\mathscr{A}(J, \mathscr{H})$ *is a subgroup of* G *containing* U *and* $\mathscr{A}(J, \mathscr{H}) \supset \mathscr{A}(J', \mathscr{H}')$ *if and only if* $(J, \mathscr{H}) \supset (J', \mathscr{H}')$ i.e. $J \supset J'$ *and* $\mathscr{H} \supset \mathscr{H}'$, *where* $(J', \mathscr{H}') \in \mathscr{P}$;

(ii) *if* $\mathscr{A}$ *is a subgroup of* G *containing* U, *then there exists a pair* $(J, \mathscr{H}) \in \mathscr{P}$ *such that* $\mathscr{A} = \mathscr{A}(J, \mathscr{H})$.

PROOF. (i) Since $\mathscr{A}(J, \mathscr{H}) = \bigcup_{w \in W_J} U\tau(w)\mathscr{H}U$, $\tau(w)\mathscr{A}(J, \mathscr{H}) = \mathscr{A}(J, \mathscr{H})$ for any $w \in W_J$ from Lemma 1.5. Further from (iii) of Lemma 1.3 $x^{-1} \in \mathscr{A}(J, \mathscr{H})$ for each $x \in \mathscr{A}(J, \mathscr{H})$. Hence $\mathscr{A}(J, \mathscr{H})$ is a subgroup of G containing U.

Assume $\mathscr{A}(J, \mathscr{H}) \supset \mathscr{A}(J' \mathscr{H}')$, then we have $\mathscr{N}(J, \mathscr{H}) \supset \mathscr{N}(J', \mathscr{H}')$, because $UnU = Un'U$ if and only if $n = n'$ for all $n, n' \in N$. Hence we have $\bigcup_{w \in W_J} \tau(w)\mathscr{H} \supset \bigcup_{w' \in W_{J'}} \tau(w')\mathscr{H}'$. Since $\tau(w)\mathscr{H} \cap H = \varnothing$ for each $w \in W_J - \{1\}$, we have $\mathscr{H} \supset \mathscr{H}'$ and $J \supset J'$.

(ii) Since $\mathscr{A} \supset U$, there exists a subset $\mathscr{N}_0$ of N such that $\mathscr{A} = \bigcup_{n \in \mathscr{N}_0} UnU$. It is clear that $\mathscr{N}_0 = \mathscr{A} \cap N$. Since $\mathscr{A}H = H\mathscr{A}$, $\mathscr{A}H$ is a parabolic subgroup of G and there exists $J \subset R$ such shat $\mathscr{A}H = BW_JB$. Since $\mathscr{A}H$ has a (B, N)-pair $(\mathscr{A}H, B, \mathscr{N}_0H, J)$, $\mathscr{N}_0H/H \cong \mathscr{N}_0/\mathscr{N}_0 \cap H \cong W_J$. Let $\mathscr{H} = \mathscr{N}_0 \cap H$. Since $U \supset U_i$ and $\mathscr{A} \supset V_i$ for all $w_i \in J$, $\mathscr{A} \supset \langle U_i, V_i \rangle$ and $\mathscr{N}_0 \cap H \supset H_i$. Further $\mathscr{N}_0 = \bigcup_{w \in W_J} \tau(w)\mathscr{H}$, because

$$(w_i)hUh^{-1}(w_i)^{-1} \subset U \cup (\bigcup_{u \in U_i^*} Uh_i(u)(w_i)^{-1}U) .$$

Hence $(J, \mathscr{H}) \in \mathscr{P}$ and $\mathscr{A} = \mathscr{A}(J, \mathscr{H})$. Q.E.D.

THEOREM 1.7. *Let* $(J, \mathscr{H}) \in \mathscr{P}$ *and* $\mathscr{G}(J, \mathscr{H}) = U^-_{w_J} \mathscr{N}(J, \mathscr{H}) U^-_{w_J}$. *Let* $\mathscr{G} = \mathscr{G}(J, \mathscr{H})$, $\mathscr{B} = U^-_{w_J} \mathscr{H}$, $\mathscr{N} = \mathscr{N}(J, \mathscr{H})$ *and* $\mathscr{U} = U^-_{w_J}$. *Then*

(i) $\mathscr{G}$ *is a subgroup of* G *with a split* (B, N)*-pair* $(\mathscr{G}, \mathscr{B}, \mathscr{N}, J, \mathscr{U})$;

(ii) $\mathscr{G}(J, \mathscr{H}) = \bigcup_{w \in W_J} U^-_{w_J} \tau(w) \mathscr{H} U^-_w$;

(iii) $\mathscr{A}(J, \mathscr{H}) = \mathscr{G}(J, \mathscr{H}) U^+_{w_J}$ *and* $\mathscr{G}(J, \mathscr{H}) \cap U^+_{w_J} = \{1\}$.

PROOF. (i) Since $\mathscr{N}(J, \mathscr{H})$ is a subgroup of N and $(w_i)\mathscr{G}(J, \mathscr{H}) \subset \mathscr{G}(J, \mathscr{H})$ for any $w_i \in J$ from Lemma 1.5, $\mathscr{G}$ is a subgroup of G. It can be easily checked according to the definition that $\mathscr{G}$ has a split (B, N)-pair $(\mathscr{G}, \mathscr{B}, \mathscr{N}, J, \mathscr{U})$.

(ii) Let $w \in W_J$, then from Lemma 1.4 and [2, Lemma 2.3] there exists $w' \in W_J$ such that $w_J = w'w$, $l(w_J) = l(w') + l(w)$ and $U^-_{w_J} = U^-_w (U^-_{w'})^{\tau(w)} = (U^-_{w'})^{\tau(w)} U^-_w$ where $(U^-_{w'})^{\tau(w)} = \tau(w)^{-1} U^-_{w'} \tau(w)$. Hence

$$U^-_{w_J} \tau(w) \mathscr{H} U^-_{w_J} = U^-_{w_J} \tau(w) \mathscr{H} \tau(w)^{-1} U^-_{w'} \tau(w) U^-_w$$
$$= U^-_{w_J} U^-_{w'} \tau(w) \mathscr{H} U^-_w = U^-_{w_J} \tau(w) \mathscr{H} U^-_w .$$

(iii) Since $\mathscr{A}(J, \mathscr{H}) = \bigcup_{w \in W_J} U \tau(w) \mathscr{H} U = \bigcup_{w \in W_J} U^+_{w_J} U^-_{w_J} \tau(w) \mathscr{H} U^-_w$, we have

$\mathscr{A}(J, \mathscr{H}) = U^+_{w_J} \mathscr{G}(J, \mathscr{H}) = \mathscr{G}(J, \mathscr{H}) U^+_{w_J}$ from (ii). It is clear that $\mathscr{G}(J, \mathscr{H}) \cap U^+_{w_J} = \{1\}$. Q.E.D.

§ 2. Endomorphism rings of split (B, N)-pairs.

Let (G, B, N, R, U) be a split (B, N)-pair of characteristic p and rank n. Let K be an arbitrary field and KG be the group algebra of G over K. If T is a subset of G, we write $\bar{T} = \sum_{t \in T} t$ in KG. Let $\mathscr{A}$ be a subgroup of G containing U, then there exists a unique pair $(J, \mathscr{H}) \in \mathscr{P}$ such that $\mathscr{A} = \mathscr{A}(J, \mathscr{H})$ from Theorem 1.6. In this section we show that the endomorphism ring $E(J, \mathscr{H}) = \mathrm{End}_{K\mathscr{A}}(K\mathscr{A}\bar{U})$ is a Frobenius algebra and further $E(J, \mathscr{H})$ is a symmetric algebra when the characteristic of K is not equal to p.

Let $n \in \mathscr{N}(J, \mathscr{H})$ and $w = n\mathscr{H}$, then $w \in W_J$ from (iv) of Lemma 1.3 and the mapping A_n also belongs to $E(J, \mathscr{H})$ from [7, §§ 1, 2] where $A_n(x) = xn\bar{U}^-_w$ for any $x \in K\mathscr{A}\bar{U}$. It is also clear from [7, §§ 1, 2] that $\{A_n | n \in \mathscr{N}(J, \mathscr{H})\}$ forms a basis for $E(J, \mathscr{H})$. We write $\mathscr{N} = \mathscr{N}(J, \mathscr{H})$.

The next proposition is a straight consequence of [7, Proposition 2.6].

PROPOSITION 2.1. *Let* $E(J, \mathscr{H})$ *and* $\{A_n | n \in \mathscr{N}\}$ *be as above. Assume*

$n\mathscr{H}=w\in W_J$, *then*

(i) $A_hA_n=A_{nh}$ *and* $A_nA_h=A_{hn}$, *for all* $h\in\mathscr{H}$ *and* $n\in\mathscr{N}$;

(ii) $A_{(w_i)}A_n=A_{n(w_i)}$ *if* $l(ww_i)>l(w)$ *and*

$$A_nA_{(w_i)}=A_{(w_i)n} \text{ if } l(w_iw)>l(w)\,;$$

(iii) $A_{(w_i)}A_n=|U_i|A_{n(w_i)}+(\sum_{u\in U_i^*}A_{(w_i)^{-1}h_i(u)(w_i)})A_n$ *if* $l(ww_i)<l(w)$ *and*

$$A_nA_{(w_i)}=|U_i|A_{(w_i)n}+A_n(\sum_{u\in U_i^*}A_{h_i(u)}) \quad \text{if} \quad l(w_iw)<l(w)\,.$$

Let w_J be a unique element in W_J of maximal length and ω_J be a fixed element in $\mathscr{N}$ such that $\omega_J\mathscr{H}=w_J$.

LEMMA 2.2. *Let* ν *be a linear mapping of* $E(J, \mathscr{H})$ *into* K *such that*

$$\begin{array}{ccc} \nu\colon E(J, \mathscr{H}) & \longrightarrow & K, \text{ then} \\ \cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid \\ \sum_{n\in\mathscr{N}} c_nA_n & \longmapsto & c_{\omega_J} \end{array}$$

ν *is well-defined and the kernel of* ν *contains no left or right ideals different from zero.*

PROOF. It is clear that ν is a well-defined non-trivial linear mapping.

When $\sum_{n\in\mathscr{N}}c_nA_n$ is a non-zero element of $E(J, \mathscr{H})$, let $\mathfrak{h}(\sum_{n\in\mathscr{N}}c_nA_n)$ denote the maximal length of $n\mathscr{H}\in W_J$ with non-zero c_n, i.e.,

$$\mathfrak{h}(\sum_{n\in\mathscr{N}}c_nA_n)=\mathrm{Max}\{l(nH)\,|\,n\in\mathscr{N},\ c_n\neq 0\}\,.$$

Assume that there exists a non-zero left ideal L of $E(J, \mathscr{H})$ in $\mathrm{Ker}\,\nu$. Let $x_0=\sum_{n\in\mathscr{N}}c_nA_n$ be a non-zero element in L.

If $\mathfrak{h}(x_0)=l(w_J)$, then there exists $h_0\in\mathscr{H}$ such that $c_{\omega_Jh_0}\neq 0$. However

$$A_{h_0^{-1}}x_0=A_{h_0^{-1}}(c_{\omega_Jh_0}A_{\omega_Jh_0}+\sum_{n\neq\omega_Jh_0}c_nA_n)=c_{\omega_Jh_0}A_{\omega_J}+\sum_{n\neq\omega_Jh_0}c_nA_{nh_0^{-1}}\in L\,.$$

Hence $\nu(A_{h_0^{-1}}x_0)=c_{\omega_Jh_0}\neq 0$, contrary to our assumption.

If $1\leqq\mathfrak{h}(x_0)<l(w_J)$, then there exists $\omega\in\mathscr{N}$ such that $c_\omega\neq 0$ and $l(\omega\mathscr{H})=\mathfrak{h}(x_0)$. However there exists an element $w'\in W_J$ such that $ww'=w_J$ and $l(ww')=l(w)+l(w')$ from Lemma 1.4, where $w=\omega\mathscr{H}$. Hence there exists $h'\in\mathscr{H}$ such that $\omega(w_{j_1})\cdots(w_{j_t})h'=\omega_J$ where $w'=w_{j_1}\cdots w_{j_t}$ is a reduced expression of w', and we have

$$\begin{aligned} A_{(w_{j_1})\cdots(w_{j_t})h'}x_0&=A_{h'}A_{(w_{j_t})}A_{(w_{j_{t-1}})}\cdots A_{(w_{j_1})}(c_\omega A_\omega+\sum_{n\neq\omega}c_nA_n) \\ &=c_\omega A_{\omega_J}+\sum_{n\neq\omega}c_nA_{h'}A_{(w_{j_t})}A_{(w_{j_{t-1}})}\cdots A_{(w_{j_1})}A_n\,. \end{aligned}$$

Hence $\nu(A_{(w_{j_1})\cdots(w_{j_l})h'}x_0)=c_\omega\neq 0$ from Proposition 2.1, and this contradicts to our assumption.

If $\mathfrak{h}(x_0)=0$ i.e. $x_0=\sum_{h\in\mathscr{H}} c_h A_h$, similarly we have $\nu(A_{\omega_J}A_{h_0^{-1}}x_0)\neq 0$ for some $h_0\in\mathscr{H}$ contrary to the assumption.

Hence there is no non-zero left ideal of $E(J, \mathscr{H})$ in Ker ν. Similarly we can prove that there is no non-zero right ideal of $E(J, \mathscr{H})$ in Ker ν. Q.E.D.

LEMMA 2.3. *Assume that the characteristic of K is not equal to p. Let σ be a linear mapping of $E(J, \mathscr{H})$ into K such that*

$$\begin{array}{ccc} \sigma\colon E(J, \mathscr{H}) & \longrightarrow & K\,, \\ \cup\!\!\!| & & \cup\!\!\!| \\ \sum_{n\in\mathscr{N}} c_n A_n & \longmapsto & c_1 \end{array} \qquad \textit{then}$$

σ is a well-defined and the kernel of σ contains no left or right ideals different from zero. Further $\sigma(xy-yx)=0$ for all $x, y\in E(J, \mathscr{H})$.

PROOF. It is clear that σ is a well-defined non-trivial linear mapping.

When $\sum_{n\in\mathscr{N}} c_n A_n$ is a non-zero element in $E(J, \mathscr{H})$, let $\mathfrak{h}'(\sum_{n\in\mathscr{N}} c_n A_n)$ denote the minimal length of $n\mathscr{H}\in W_J$ with non-zero c_n, i.e.,

$$\mathfrak{h}'(\sum_{n\in\mathscr{N}} c_n A_n)=\mathrm{Min}\{l(n\mathscr{H})\,|\,n\in\mathscr{N},\ c_n\neq 0\}\,.$$

Let $x_0=\sum_{n\in\mathscr{N}} c_n A_n\in E(J, \mathscr{H})-\{0\}$, then we can prove that there exists $a\in E(J, \mathscr{H})$ such that $\sigma(ax_0)\neq 0$ from the induction of $\mathfrak{h}'(x_0)$ as follows.

If $\mathfrak{h}'(x_0)\neq 0$, then there exists $h_0\in\mathscr{H}$ such that $c_{h_0}\neq 0$. Hence

$$\sigma(A_{h_0^{-1}}x_0)=\sigma(A_{h_0^{-1}}(c_{h_0}A_{h_0}+\sum_{n\neq h_0} c_n A_n))=\sigma(c_{h_0}A_1+\sum_{n\neq h_0} c_n A_{nh_0^{-1}})=c_{h_0}\neq 0\,.$$

Assume that there exists $a\in E(J, \mathscr{H})$ such that $\sigma(ax_0')\neq 0$ when $\mathfrak{h}'(x_0')<k$, where $x_0'\in E(J, \mathscr{H})-\{0\}$ and k is a natural number. Let us take $x_0\in E(J, \mathscr{H})-\{0\}$ such that $\mathfrak{h}'(x_0)=k$. Then there exists $\omega\in\mathscr{N}$ such that $c_\omega\neq 0$ and $l(\omega\mathscr{H})=\mathfrak{h}'(x_0)$. Let $\omega\mathscr{H}=w_{j_1}w_{j_2}\cdots w_{j_k}$ be a reduced expression of $\omega\mathscr{H}$, then $\omega=h(w_{j_1})(w_{j_2})\cdots(w_{j_k})$ for some $h\in\mathscr{H}$. Now, since

$$A_{(w_{j_k})^{-1}}A_\omega=|U_{j_k}|A_{h(w_{i_1})(w_{j_2})\cdots(w_{j_{k-1}})}+(\sum_{u\in U_{j_k}^*} A_{(w_{j_k})h_{j_k}(u)(w_{j_k})^{-1}})A_\omega$$

and $|U_{j_k}|\neq 0$, we have $\mathfrak{h}'(A_{(w_{j_k})^{-1}}x_0)\leq k-1$ and there exists $a\in E(J, \mathscr{H})$ such that $\sigma(aA_{(w_{j_k})^{-1}}x_0)\neq 0$ from the induction. Hence Ker σ contains no left ideals different from zero.

Next we will show that $\sigma(A_{(w_i)}A_n)=\sigma(A_n A_{(w_i)})$ for any $w_i\in J$ and

$n \in \mathscr{N}$. In case $l(n\mathscr{H})=1$ we have $n=(w_j)h$ for some $w_j \in J$ and $h \in \mathscr{H}$. If $i \neq j$, then $\sigma(A_{(w_i)}A_n)=0=\sigma(A_n A_{(w_i)})$. If $i=j$, then

$$\begin{aligned}&\sigma(A_{(w_i)}A_{(w_i)}h)\\&\quad=\sigma(|U_i|A_{(w_i)h(w_i)}+(\sum_{u\in U_i^*}A_{(w_i)^{-1}h_i(u)(w_i)})A_{(w_i)h})=|U_i|\sigma(A_{(w_i)h(w_i)})\end{aligned}$$

and

$$\begin{aligned}&\sigma(A_{(w_i)h}A_{(w_i)})=\sigma(A_h A_{(w_i)}A_{(w_i)})\\&\quad=\sigma(A_h(|U_i|A_{(w_i)^2}+(\sum_{u\in U_i^*}A_{(w_i)^{-1}h_i(u)(w_i)}A_{(w_i)})=|U_i|\sigma(A_{(w_i)^2h})\ .\end{aligned}$$

Since the relation $(w_i)h(w_i)=1$ is equivalent to $(w_i)^2h=1$, we have $\sigma(A_{(w_i)}A_n)=\sigma(A_n A_{(w_i)})$. If $l(n\mathscr{H})>1$, then $\sigma(A_{(w_i)}A_n)=0=\sigma(A_n A_{(w_i)})$.

It is clear that $\sigma(A_{(w_i)}A_h)=0=\sigma(A_h A_{(w_i)})$ for any $h \in \mathscr{H}$. Hence we have $\sigma(A_{(w_i)}x)=\sigma(xA_{(w_i)})$ for any $w_i \in J$ and $x \in E(J, \mathscr{H})$. We can easily show that $\sigma(A_h x)=\sigma(xA_h)$ for any $h \in \mathscr{H}$ and $x \in E(J, \mathscr{H})$. Therefore we have $\sigma(A_n x)=\sigma(xA_n)$ for all $n \in \mathscr{N}$ and $x \in E(J, \mathscr{H})$. Hence $\sigma(xy-yx)=0$ for any $x, y \in E(J, \mathscr{H})$, and $\operatorname{Ker}\sigma$ contains no right ideals different from zero either. Q.E.D.

THEOREM 2.4. *Let K be an arbitrary field and (G, B, N, R, U) be a split (B, N)-pair of characteristic p and rank n. Let $\mathscr{A}$ be a subgroup of G containing U and $E=\operatorname{End}_{K\mathscr{A}}(K\mathscr{A}\bar{U})$. Then*

(i) *E is a Frobenius algebra, and*

(ii) *if $ch(K)\neq p$, E is also a symmetric algebra.*

PROOF. (i) and (ii) are clear from Lemmas 2.2 and 2.3 and from the definitions of those algebras (see [5, Chapter IX]). Q.E.D.

ADDENDUM TO THEOREM 2.4. *If K is an algebraically closed field of characteristic p, then $E=\operatorname{End}_{KG}(KG\bar{U})$ is not a symmetric algebra in general.*

PROOF. Assume that E is a symmetric algebra, then from [5, Exercise 83.1] $E\pi_i/(\operatorname{rad} E)\pi_i \cong l(\operatorname{rad} E)\pi_i$ for any primitive idempotent π_i in [7, Theorem 2.11], where $l(\operatorname{rad} E)=\{x \in E \mid x \operatorname{rad} E=0\}$. Let (J, χ) be a pair in P such that $A(J, \chi)\pi_i=A(J, \chi)$ i.e. $l(\operatorname{rad} E)\pi_i=EA(J, \chi)$, then we have $E\pi_i/(\operatorname{rad} E)\pi_i \cong EA({}^{w_0}J, {}^{w_0}\chi)$ from [7, Theorem 3.10]. However it is very easy to give an example of pairs (J, χ) such that $(J, \chi)\neq({}^{w_0}J, {}^{w_0}\chi)$ i.e. $EA(J, \chi)\not\cong EA({}^{w_0}J, {}^{w_0}\chi)$. Hence E is not a symmetric algebra in general. Q.E.D.

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