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On Bounded Solutions of $x''=t^{\beta}x^{1+\alpha}$

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§ **1.** In this paper, we consider a second order nonlinear differential equation

$$(1) \qquad x''=t^{\beta}x^{1+\alpha} \qquad ('=d/dt)$$

where α and β are real numbers and $\alpha>0$. This equation includes, as its special case, the equation

$$x''=t^{1-m}x^{m}\ , \qquad 1<m<3\ ,$$

which is known as Emden's equation [1].

The solutions of (1) considered here are those which assume real values for real t. Therefore, for any given α and β, t^{β} and $x^{1+\alpha}$ must be regarded as representing real-valued branches. So it is quite natural to assume that

(1) the domain in which the equation (1) is considered is

$$G\colon 0<t<\infty\ , \qquad 0\leqq x<\infty\ ,$$

(2) $x^{1+\alpha}$ and t^{β} represent their nonnegative-valued branches in G.

The purpose of the present paper is to show that the equation (1) has a one-parameter family of (positive) bounded solutions if β satisfies a certain condition. Here, by a bounded solution, we mean a solution $x(t)$ such that $x(t)$ and $x'(t)$ are both bounded for $0<t<\infty$.

§ **2.** Let $x(t)$ be a bounded solution of (1). Since

$$x''(t)=t^{\beta}(x(t))^{1+\alpha}\geqq 0$$

in G by our assumptions given at the outset, $x'(t)$ is a nondecreasing function of t. So if $x'(a)>0$ for some $a>0$, we have

$$x'(t)\geqq x'(a) \qquad \text{for} \quad t\geqq a\ .$$

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Integrating both sides from a to t ($>a$), we get the following inequality:

$$x(t)-x(a) \geqq x'(a)(t-a)$$

which obviously contradicts the boundedness of $x(t)$. Hence $x'(t)$ must be nonpositive for every bounded solution $x(t)$.

Therefore every bounded solution $x(t)$ turns out to be nonincreasing. Consequently if we exclude the trivial solution $x(t)=0$, $x(t)$ has a positive limit as $t \to 0$. Thus we get

PROPOSITION 1. *If $x(t)$ is a nontrivial bounded solution of* (1), *then*

$$x'(t) \leqq 0 \quad \textit{and} \quad \lim_{t \to 0} x(t) > 0 .$$

For the existence of a nontrivial bounded solution, β cannot be an arbitrary real number. This is shown by the following

PROPOSITION 2. *If $\beta \leqq -1$, the equation* (1) *has no bounded nontrivial solution.*

PROOF. Suppose that $\beta \leqq -1$. If there exists a nontrivial bounded solution $x(t)$ of (1), then $\lim_{t \to 0} x(t) > 0$ by Proposition 1. Hence for suitably small $\delta > 0$, there exists $m > 0$ such that

$$(x(t))^{1+\alpha} > m \qquad \text{for} \quad 0 < t < \delta .$$

Then

$$x''(t) = t^{\beta}(x(t))^{1+\alpha} > m t^{\beta} , \qquad 0 < t < \delta .$$

Integrating both sides from ε to t where $0 < \varepsilon < t < \delta$, we obtain

$$x'(t) - x'(\varepsilon) > (m/(\beta+1)) \cdot (t^{\beta+1} - \varepsilon^{\beta+1}) , \qquad \text{if} \quad \beta < -1 ,$$

and

$$x'(t) - x'(\varepsilon) > m(\log t - \log \varepsilon) , \qquad \text{if} \quad \beta = -1 .$$

In both cases, the right-hand members of these inequalities tend to ∞ as $\varepsilon \to 0$. This implies

$$\lim_{\varepsilon \to 0} x'(\varepsilon) = -\infty$$

in contradiction with the boundedness of the solution $x(t)$.

So hereafter we assume that $\beta > -1$ and write the equation (1) in a form:

(2) $$x''=t^{\alpha\lambda-2}x^{1+\alpha}\,, \quad \alpha>0\,, \quad \lambda>0\,, \quad \alpha\lambda>1\,.$$

§ **3.** First we notice that

$$x=\psi(t)=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda}$$

is a solution of (2). This can be verified by direct calculation.

PROPOSITION 3. *Let $x=\phi(t)$ be an arbitrary nontrivial bounded solution of* (2) *and $y(t)$ be a function defined by*

(3) $$\phi(t)=\psi(t)(y(t))^{1/\alpha}\,,$$

or, written explicitly, by

(4) $$y(t)=[\lambda(\lambda+1)]^{-1}t^{\alpha\lambda}(\phi(t))^{\alpha}\,.$$

Then the function $z(y)$ defined by

(5) $$y=y(t)\,, \quad z=ty'(t)\,, \qquad 0<t<\infty\,,$$

is a solution of the differential equation

(6) $$\frac{dz}{dy}=\frac{-\lambda(\lambda+1)\alpha^2y^2+(2\lambda+1)\alpha yz-(1-\alpha)z^2+\lambda(\lambda+1)\alpha^2y^3}{\alpha yz}$$

such that

(1) *$z(y)$ is defined for $0<y<1$,*
(2) *$z(y)\geqq 0$ for $0<y<1$,*
(3) $\lim_{y\to 0} z(y)=\lim_{y\to 1} z(y)=0$.

Proof of this proposition is divided into several steps. First it is easy to show that $z(y)$ defined by (5) satisfies (6). Indeed, substituting the expression (3) into the equation (2), we obtain

$$t^2yy''+(\alpha^{-1}-1)t^2y'^2-2\lambda tyy'+\alpha\lambda(\lambda+1)(y^2-y^3)=0\,.$$

Then, if we notice that

$$ty'=z\,, \qquad t^2y''=z\,\frac{dz}{dy}-z\,,$$

we are immediately led to the equation (6).

To prove that $z(y)$ has the properties (1), (2) and (3), we begin with the following two lemmas.

LEMMA 1. *If $\phi(t)$ is a nontrivial bounded solution of* (2), *then*

$$\phi(t)<\psi(t) \qquad \textit{for} \quad 0<t<\infty ,$$

and hence $\lim_{t\to\infty}\phi(t)=0$.

Proof. Since

$$\lim_{t\to 0}\psi(t)=\infty$$

and $\phi(t)$ is bounded, we have $\phi(t)<\psi(t)$ if t is sufficiently small. If the conclusion of the lemma does not hold, then there exists $\tau>0$ such that

$$\begin{aligned}&\phi(t)<\psi(t) , \qquad 0<t<\tau ,\\ &\phi(\tau)=\psi(\tau) .\end{aligned}$$

Then obviously $\phi'(\tau)\geqq\psi'(\tau)$. However, since $\phi(\tau)=\psi(\tau)$ and $\phi'(\tau)=\psi'(\tau)$ imply $\phi(t)=\psi(t)$ by the uniqueness of the solution, the case $\phi'(\tau)=\psi'(\tau)$ must be excluded and we have $\phi'(\tau)>\psi'(\tau)$. This is turn implies that $\phi(t)>\psi(t)$ if $t>\tau$ and $t-\tau$ is suitably small.

Suppose that there exists $\tau_1>\tau$ such that

$$\begin{aligned}&\phi(t)>\psi(t) , \qquad \tau<t<\tau_1 ,\\ &\phi(\tau_1)=\psi(\tau_1) .\end{aligned}$$

Then, by the same reasoning as above, we get

$$\phi'(\tau_1)<\psi'(\tau_1) . \tag{7}$$

Since $\phi(t)>\psi(t)$ for $\tau<t<\tau_1$, the following inequality must hold for $\tau<t<\tau_1$:

$$\phi''(t)=t^{\alpha\lambda-2}(\phi(t))^{1+\alpha}>t^{\alpha\lambda-2}(\psi(t))^{1+\alpha}=\psi''(t) .$$

Integrating both sides of this inequality from τ to t $(\tau<t<\tau_1)$, we get

$$\phi'(t)-\phi'(\tau)>\psi'(t)-\psi'(\tau) ,$$

or

$$\phi'(t)>\psi'(t)+(\phi'(\tau)-\psi'(\tau)) . \tag{8}$$

As (8) is valid for $\tau<t<\tau_1$ and $\phi'(\tau)>\psi'(\tau)$ we obtain

$$\phi'(\tau_1)>\psi'(\tau_1)$$

in contradiction with (7).

Hence no such τ_1 exists and (8) holds for $\tau<t<\infty$. Then the integration of (8) from τ to t $(>\tau)$ will yield

$$\phi(t) > \psi(t) + (\phi'(\tau) - \psi'(\tau))(t-\tau) .$$

Since $\psi(t) \to 0$ as $t \to \infty$ and $\phi'(\tau) - \psi'(\tau) > 0$, this implies

$$\lim_{t\to\infty} \phi(t) = \infty .$$

This contradicts with the boundedness of $\phi(t)$ and the lemma is proved.

For any nontrivial bounded solution $\phi(t)$ of (2), we define a function $f(t, \tau)$ by

$$f(t, \tau) = \tau^\lambda \phi(\tau) t^{-\lambda} ,$$

where τ is an arbitrary positive number. Evidently $f(\tau, \tau) = \phi(\tau)$.

LEMMA 2. *For any* $\tau > 0$, *we have*

$$\phi(t) \geqq f(t, \tau) , \qquad \tau \leqq t .$$

PROOF. Direct calculation will give

$$\phi''(t) - f''(t, \tau) = t^{\alpha\lambda-2}(\phi(t))^{1+\alpha}\tau^{-\alpha\lambda}(\phi(\tau))^{-\alpha}\left[(\tau^\lambda\phi(\tau))^\alpha - \lambda(\lambda+1)\left(\frac{\tau^\lambda\phi(\tau)}{t^\lambda\phi(t)}\right)^{1+\alpha}\right] .$$

Since $\phi(t) < \psi(t) = (\lambda(\lambda+1))^{1/\alpha} t^{-\lambda}$ by Lemma 1,

$$\tau^\lambda \phi(\tau) < \tau^\lambda \psi(\tau) = (\lambda(\lambda+1))^{1/\alpha} .$$

Hence

$$\phi''(t) - f''(t, \tau) < t^{\alpha\lambda-2}(\phi(t))^{1+\alpha}\tau^{-\alpha\lambda}(\phi(\tau))^{-\alpha}\lambda(\lambda+1)\left[1 - \left(\frac{\tau^\lambda\phi(\tau)}{t^\lambda\phi(t)}\right)^{1+\alpha}\right] .$$

Therefore, if

$$t^\lambda \phi(t) \leqq \tau^\lambda \phi(\tau) ,$$

or equivalently if

(9) $$\phi(t) \leqq \tau^\lambda \phi(\tau) t^{-\lambda} = f(t, \tau) ,$$

we have

(10) $$\phi''(t) < f''(t, \tau) .$$

If we assume that the conclusion of the lemma does not hold, we are led to the following alternative:

(i) there exist τ_1 and τ_2 $(\tau \leqq \tau_1 < \tau_2)$ such that

$$\text{(11)} \qquad \begin{cases} \phi(\tau_1)=f(\tau_1, \tau)\ , \\ \phi(t)<f(t, \tau)\ , & \tau_1<t<\tau_2\ , \\ \phi(\tau_2)=f(\tau_2, \tau)\ , \end{cases}$$

(ii) there exists $\tau_1 \geqq \tau$ such that

$$\text{(12)} \qquad \begin{cases} \phi(\tau_1)=f(\tau_1, \tau)\ , \\ \phi(t)<f(t, \tau)\ , & \tau_1<t<\infty\ . \end{cases}$$

In the first case, the inequality (9) and hence the inequality (10) holds for $\tau_1 \leqq t \leqq \tau_2$. Integrating both sides of (10) from τ_1 to t ($\tau_1<t<\tau_2$), we get

$$\phi'(t)-\phi'(\tau_1)<f'(t, \tau)-f'(\tau_1, \tau)\ ,$$

or

$$\text{(13)} \qquad \phi'(t)<f'(t, \tau)+(\phi'(\tau_1)-f'(\tau_1, \tau))\ , \qquad \tau_1<t<\tau_2\ .$$

If we integrate (13) from τ_1 to τ_2 and notice the relation (11), then we get

$$0<(\phi'(\tau_1)-f'(\tau_1, \tau))(\tau_2-\tau_1)\ .$$

Since $\tau_2-\tau_1>0$, this yields

$$\phi'(\tau_1)>f'(\tau_1, \tau)$$

which is absurd since the first two relations of (11) obviously imply

$$\text{(14)} \qquad \phi'(\tau_1)\leqq f'(\tau_1, \tau)\ .$$

Next consider the case (ii) in which (12) holds. Then the inequality (13) holds for $\tau_1<t<\infty$. Integrating it from τ_1 to t ($t>\tau_1$) and taking the relation (12) into account, we get

$$\phi(t)<f(t, \tau)+(\phi'(\tau_1)-f'(\tau_1, \tau))(t-\tau_1)\ .$$

If $\phi'(\tau_1)<f'(\tau_1, \tau)$, this leads us to

$$\lim_{t\to\infty} \phi(t)=-\infty$$

which is impossible. Hence, by (14), we must have

$$\phi'(\tau_1)=f'(\tau_1, \tau)$$

and the inequality (13) will become

$$\phi'(t)<f'(t, \tau) , \qquad \tau_1<t<\infty .$$

So, if we choose $\tau_1'>\tau_1$, we have

$$\phi'(\tau_1')<f'(\tau_1', \tau) . \tag{15}$$

On the other hand,

$$\begin{aligned} f'(\tau_1', \tau_1')-f'(\tau_1', \tau) &= -\lambda\tau_1'^{\lambda}\phi(\tau_1')\tau_1'^{-\lambda-1}+\lambda\tau^{\lambda}\phi(\tau)\tau_1'^{-\lambda-1} \\ &=\lambda\tau_1'^{-1}(\tau^{\lambda}\phi(\tau)\tau_1'^{-\lambda}-\phi(\tau_1')) . \end{aligned}$$

As we have assumed that

$$\phi(t)<f(t, \tau) , \qquad \tau_1<t<\infty ,$$

we have

$$\phi(\tau_1')<f(\tau_1', \tau)=\tau^{\lambda}\phi(\tau)\tau_1'^{-\lambda} .$$

Thus it follows that

$$f'(\tau_1', \tau_1')>f'(\tau_1', \tau) .$$

From this and (15), we get an inequality

$$\phi'(\tau_1')<f'(\tau_1', \tau_1') . \tag{16}$$

Since $\phi(\tau_1')=f(\tau_1', \tau_1')$, it follows from (16) that

$$\phi(t)<f(t, \tau_1') , \qquad \tau_1'<t<\tau_2'\leqq\infty ,$$

and this implies

$$\phi''(t)<f''(t, \tau_1') , \qquad \tau_1'<t<\tau_2'\leqq\infty .$$

So we can repeat the argument given above replacing τ_1 and τ_2 by τ_1' and τ_2' respectively. Since the inequality (16) excludes the case

$$\phi'(\tau_1')=f'(\tau_1', \tau_1') ,$$

the final part of our discussion is now unnecessary and we are led to the contradiction. Thus we have proved the lemma.

PROOF OF PROPOSITION 3. By the two lemmas just proved, we have

$$f(t, \tau)=\tau^{\lambda}\phi(\tau)t^{-\lambda}\leqq\phi(t)<\psi(t)=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda} , \qquad \tau\leqq t<\infty .$$

This can easily be rewritten as

$$\frac{1}{\lambda(\lambda+1)}\tau^{\alpha\lambda}(\phi(\tau))^{\alpha} \leqq \frac{1}{\lambda(\lambda+1)} t^{\alpha\lambda}(\phi(t))^{\alpha} < 1 .$$

Recalling the definition of $y(t)$ given by (4), this means that

$$(17) \qquad y(\tau) \leqq y(t) < 1 , \qquad \tau \leqq t < \infty ,$$

which shows that $y(t)$ is a nondecreasing function of t. Hence

$$z(y) = ty'(t) \geqq 0 , \qquad 0 < t < \infty .$$

Consequently, in order to show that $y(t)$ has the properties (1) and (2) stated in the Proposition 3, it remains to prove that

$$\lim_{t\to 0} y(t) = 0 , \quad \lim_{t\to\infty} y(t) = 1 .$$

Since $y(t)$ is defined by (4):

$$y(t) = [\lambda(\lambda+1)]^{-1} t^{\alpha\lambda}(\phi(t))^{\alpha} , \qquad \alpha > 0 , \quad \lambda > 0 ,$$

and $\phi(t)$ is bounded, the first relation $\lim_{t\to 0} y(t) = 0$ is immediate.

Since (17) shows that $y(t)$ is nondecreasing and bounded from above, the existence of $\lim_{t\to\infty} y(t)$ is obvious. Let us denote this limit by c. Then since

$$\lim_{t\to\infty} t^{\lambda}\phi(t) = \lim_{t\to\infty} [\lambda(\lambda+1)y(t)]^{1/\alpha} = (\lambda(\lambda+1))^{1/\alpha} c^{1/\alpha} ,$$
$$\lim_{t\to\infty} t^{\lambda}\psi(t) = [\lambda(\lambda+1)]^{1/\alpha} ,$$

the limit

$$\lim_{t\to\infty} \frac{t^{\lambda}\phi(t)}{t^{\lambda}\psi(t)} = \lim_{t\to\infty} \frac{\phi(t)}{\psi(t)}$$

exists and is equal to $c^{1/\alpha}$. As $\lim_{t\to\infty} \phi(t) = 0$ by Lemma 1 and

$$\lim_{t\to\infty} \psi(t) = \lim_{t\to\infty} (\lambda(\lambda+1))^{1/\alpha} t^{-\lambda} = 0$$

we can apply the well-known l'Hospital's theorem and get

$$\lim_{t\to\infty} \frac{\phi(t)}{\psi(t)} = \lim_{t\to\infty} \frac{\phi'(t)}{\psi'(t)} = c^{1/\alpha} .$$

However, as

$$\lim_{t\to\infty} \psi'(t) = \lim_{t\to\infty} [-\lambda(\lambda(\lambda+1))^{1/\alpha} t^{-\lambda-1}] = 0 ,$$

l'Hospital's theorem can be applied again and

$$\lim_{t\to\infty}\frac{\phi(t)}{\psi(t)}=\lim_{t\to\infty}\frac{\phi''(t)}{\psi''(t)}=c^{1/\alpha}\ . \tag{18}$$

On the other hand, from

$$\phi''(t)=t^{\alpha\lambda-2}(\phi(t))^{1+\alpha}\ ,\qquad \psi''(t)=t^{\alpha\lambda-2}(\psi(t))^{1+\alpha}\ ,$$

we also get

$$\lim_{t\to\infty}\frac{\phi''(t)}{\psi''(t)}=\lim_{t\to\infty}\frac{(\phi(t))^{1+\alpha}}{(\psi(t))^{1+\alpha}}=(c^{1/\alpha})^{1+\alpha}=c^{1/\alpha+1}\ .$$

Comparing this with (18), we obtain

$$c=\lim_{t\to\infty} y(t)=1$$

which is the required result.

Finally we have to prove (3). Direct calculation shows that

$$z(y(t))=ty'(t)=\alpha\lambda y(t)+[\lambda(\lambda+1)]^{-1}t^{\alpha\lambda+1}[(\phi(t))^{\alpha}]'\ . \tag{19}$$

As c was proved to be equal to 1, (18) implies that

$$\lim_{t\to\infty}\frac{(\phi(t))^{\alpha}}{(\psi(t))^{\alpha}}=1\ .$$

So, by applying l'Hospital's theorem again,

$$\lim_{t\to\infty}\frac{[(\phi(t))^{\alpha}]'}{[(\psi(t))^{\alpha}]'}=1\ .$$

From this we get

$$\lim_{t\to\infty}\frac{t^{\alpha\lambda+1}[(\phi(t))^{\alpha}]'}{t^{\alpha\lambda+1}[(\psi(t))^{\alpha}]'}=1\ .$$

As direct calculation shows

$$\lim_{t\to\infty} t^{\alpha\lambda+1}[(\psi(t))^{\alpha}]'=\lim_{t\to\infty} t^{\alpha\lambda+1}[\lambda(\lambda+1)t^{-\alpha\lambda}]'=-\alpha\lambda^{2}(\lambda+1)\ .$$

Hence

$$\lim_{t\to\infty} t^{\alpha\lambda+1}[(\phi(t))^{\alpha}]'=\lim_{t\to\infty} t^{\alpha\lambda+1}[(\psi(t))^{\alpha}]'=-\alpha\lambda^{2}(\lambda+1)\ .$$

Thus we have

$$\lim_{t\to\infty} z(y(t)) = \alpha\lambda \lim_{t\to\infty} y(t) + [\lambda(\lambda+1)]^{-1} \lim_{t\to\infty} t^{\alpha\lambda+1}[(\phi(t))^{\alpha}]'$$
$$= \alpha\lambda + [\lambda(\lambda+1)]^{-1}(-\alpha\lambda^2(\lambda+1)) = 0 .$$

Since $y(t) \to 1$ as $t \to \infty$, we get

$$\lim_{y\to 1} z(y) = 0 .$$

To prove that

$$\lim_{y\to 0} z(y) = 0 ,$$

we have to show that

$$\lim_{t\to 0} z(y(t)) = 0 .$$

However, since

$$\lim_{t\to 0} y(t) = 0$$

and $\phi(t)$ and $\phi'(t)$ are bounded as $t \to 0$, this follows immediately from (19).

This completes the proof of the Proposition 3.

§ 4. This section is devoted to the proof of

PROPOSITION 4. *The differential equation* (6):

$$\frac{dz}{dy} = \frac{-\lambda(\lambda+1)\alpha^2 y^2 + (2\lambda+1)\alpha yz - (1-\alpha)z^2 + \lambda(\lambda+1)\alpha^2 y^3}{\alpha yz}$$

has one and only one solution $z(y)$ *such that*

$$z(y) > 0 \quad \textit{for} \quad 0 < y < 1 ,$$

and

$$\lim_{y\to 0} z(y) = \lim_{y\to 1} z(y) = 0 .$$

PROOF. Introducing a new parameter s, we write the equation (6) in the following form:

$$(20) \qquad \begin{aligned} \frac{dy}{ds} &= \alpha yz , \\ \frac{dz}{ds} &= -\lambda(\lambda+1)\alpha^2 y^2 + (2\lambda+1)\alpha yz - (1-\alpha)z^2 + \lambda(\lambda+1)\alpha^2 y^3 . \end{aligned}$$

As can easily be seen, $y=z=0$ and $y=1$, $z=0$ are the only critical points of (20).

First let us investigate the behaviour of orbits near $y=1$, $z=0$. For that purpose, we put

$$y=1+\eta\,,\quad z=\zeta\,.$$

Then (20) is transformed into

$$\text{(20}'\text{)}\qquad \begin{aligned}\frac{d\eta}{ds}&=\alpha\zeta+\cdots\,,\\ \frac{d\zeta}{ds}&=\lambda(\lambda+1)\alpha^2\eta+(2\lambda+1)\alpha\zeta+\cdots\,,\end{aligned}$$

where the unwritten part represents the terms whose degrees are greater than 1. Since the eigenvalues of a matrix

$$\begin{pmatrix} 1 & \alpha \\ \lambda(\lambda+1)\alpha^2 & (2\lambda+1)\alpha\end{pmatrix}$$

are

$$\mu=\frac{1}{2}[(2\lambda+1)\alpha-\sqrt{(2\lambda+1)^2\alpha^2+4\lambda(\lambda+1)\alpha^3}]<0\,,$$

and

$$\mu'=\frac{1}{2}[(2\lambda+1)\alpha+\sqrt{(2\lambda+1)^2\alpha^2+4\lambda(\lambda+1)\alpha^3}]>0\,,$$

$\eta=\zeta=0$ is a saddle point. Two orbits tending to this saddle point as $s\to\infty$ can be expressed as

$$\text{(21)}\qquad \begin{aligned}\eta&=a_1(ce^{\mu s})+a_2(ce^{\mu s})^2+\cdots\,,\\ \zeta&=b_1(ce^{\mu s})+b_2(ce^{\mu s})^2+\cdots\,,\end{aligned}$$

where the power series in $ce^{\mu s}$ in the right-hand members are convergent in the neighbourhood of $s=\infty$. Substituting the above expression into (20′) and comparing the coefficients, we get

$$\frac{b_1}{a_1}=\frac{\mu}{\alpha}\,.$$

Therefore the curve consisting of these two orbits together with the saddle point $\eta=\zeta=0$ is expressed in a form

$$\zeta=\frac{\mu}{\alpha}\eta+\cdots$$

where the right-hand member is a power series in η convergent in the neighbourhood of $\eta=0$.

In the same way, the curve representing another two orbits tending to $\eta=\zeta=0$ as $s\to-\infty$ together with the saddle point $\eta=\zeta=0$ is expressed as

$$\zeta=\frac{\mu'}{\alpha}\eta+\cdots .$$

Returning to the original variables y and z, these two curves passing through the saddle point $y=1$, $z=0$ are expressed by the following power series in $y-1$:

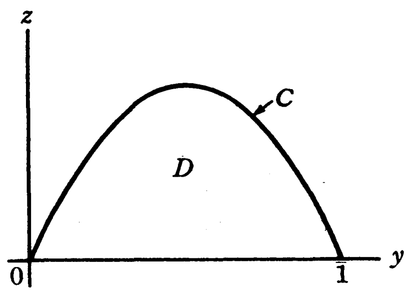
(22) $$z=\frac{\mu}{\alpha}(y-1)+\cdots ,$$

(22′) $$z=\frac{\mu'}{\alpha}(y-1)+\cdots .$$

Since $\mu/\alpha<0$ and $\mu'/\alpha>0$, only the curve (22) intersects the domain

$$0<y<1 , \qquad z\geqq 0 .$$

So we have only to consider the curve (22), and especially a part of the curve lying to the left of $y=1$. As is clear from the expression (21), this left-half part of the curve represents an orbit which approaches the critical point (1, 0) from the left as $s\to\infty$. We shall now show that this orbit tends to (0, 0) as $s\to-\infty$.



We denote by C the arc of a parabola

$$z=f(y)=\alpha(\lambda+1)(y-y^2)$$

lying between $y=0$ and $y=1$ and by D the domain bounded by C and the segment $0 \leqq y \leqq 1$ on the y-axis.

On the open segment $0<y<1$, $z=0$, we have

$$\frac{dy}{ds}=0\ ,\quad \frac{dz}{ds}=\lambda(\lambda+1)\alpha^2(y^3-y^2)<0$$

which shows that the orbits of (20) passing through this segment go out of D as s increases.

On the arc C, we have

$$\frac{d}{ds}(z-f(y))=\alpha^2(\alpha+1)(\lambda+1)^2y^2(y-y^2)>0\ .$$

Hence the orbits passing through C also go out of D as s increases. Therefore every orbit starting from inside of D can never leave D as s decreases to $-\infty$.

Now let us return to the orbit which approaches (1, 0) from the left as $s\to\infty$ and show that this orbit lies in D. Since the slope of the curve (22) and that of the curve C at (1, 0) are μ/α (<0) and $f'(1)=-\alpha(\lambda+1)$ (<0) respectively, what we have to show is $\mu/\alpha>-\alpha(\lambda+1)$.

As was already given

$$\mu=\frac{1}{2}\,[(2\lambda+1)\alpha-\sqrt{(2\lambda+1)^2\alpha^2+4\lambda(\lambda+1)\alpha^3}]$$

and so

$$\frac{\mu}{\alpha}=\left(\lambda+\frac{1}{2}\right)[1-\sqrt{1+4\lambda(\lambda+1)\alpha(2\lambda+1)^{-2}}]\ .$$

Since $(2\lambda+1)^2>4\lambda(\lambda+1)$, we have

$$\frac{\mu}{\alpha}>\left(\lambda+\frac{1}{2}\right)(1-\sqrt{1+\alpha})\ .$$

Hence

$$\begin{aligned}\frac{\mu}{\alpha}-(-\alpha(\lambda+1))&>\left(\lambda+\frac{1}{2}\right)(1-\sqrt{1+\alpha})+\alpha(\lambda+1)\\&=(\sqrt{1+\alpha}-1)\left(\sqrt{1+\alpha}\lambda+\sqrt{1+\alpha}+\frac{1}{2}\right)>0\end{aligned}$$

and we get the required result.

Therefore the orbit with which we are now concerned belongs to D

if s is sufficiently large. So, from what we have shown above, it can never leave D as $s \to -\infty$ and hence its α-limit set belongs to D. Since

$$\frac{dy}{ds} = \alpha yz > 0$$

inside D, a point $(y(s), z(s))$ on the orbit keeps on moving to the left as $s \to -\infty$. Therefore its α-limit set cannot be a closed curve. Hence, from the well-known Poincaré-Bendixson theory, it tends to a critical point as $s \to -\infty$. Since the only critical point other than $(1, 0)$ is $y = z = 0$, we get the desired result

$$\lim_{s \to -\infty} y = \lim_{s \to -\infty} z = 0 .$$

Also, since the curve $z = z(y)$ lies in the inside of D for $0 < y < 1$, it is obvious that $z(y) > 0$ for $0 < y < 1$.

So, if we denote by $z(y)$ the restriction within $0 < y < 1$ of the function given by the power series (22), it fulfills all the requirements stated in the proposition.

§ 5. By Proposition 4, we now know the unique existence of the solution $z(y)$ of the differential equation (6):

$$\frac{dz}{dy} = \frac{-\lambda(\lambda+1)\alpha^2 y^2 + (2\lambda+1)\alpha yz - (1-\alpha)z^2 + \lambda(\lambda+1)\alpha^2 y^3}{\alpha yz}$$

such that

$$z(y) > 0 \qquad \text{for} \quad 0 < y < 1 ,$$
$$\lim_{y \to 0} z(y) = \lim_{y \to 1} z(y) = 0 .$$

Let $y(t)$ be an arbitrary solution of the equation

$$t\frac{dy}{dt} = z(y) . \tag{23}$$

Then, if we put

$$\phi(t) = \psi(t)(y(t))^{1/\alpha} = (\lambda(\lambda+1))^{1/\alpha} t^{-\lambda} (y(t))^{1/\alpha} , \tag{24}$$

$x = \phi(t)$ is evidently a solution of the equation (2). So it remains for us to prove that

(1) $y(t)$ is defined for $0 < t < \infty$, and

(2) $\phi(t)$ and $\phi'(t)$ are bounded for $0 < t < \infty$.

The solution of (23) such that $y(t_0)=y_0$ $(0<t_0<\infty,\ 0<y_0<1)$ is given implicitly by

$$\int_{y_0}^{y(t)}\frac{dy}{z(y)}=\int_{t_0}^{t}\frac{dt}{t}=\log t-\log t_0 .$$

From this we can easily see that the inverse function $t(y)$ of $y(t)$ is an increasing function defined for $0<y<1$, because $z(y)>0$ for $0<y<1$. Therefore, to prove (1), it is sufficient to prove that

$$\lim_{y\to 0} t(y)=0\ ,\ \lim_{y\to 1} t(y)=\infty .$$

For that purpose, we need the explicit expression of $z(y)$ in the neighbourhood of $y=0$ and $y=1$. For $y=1$, this has already been obtained and is given by (22):

$$z=\frac{\mu}{\alpha}(y-1)+\cdots$$

where the unwritten terms are the power series in $y-1$ beginning with the term whose degree is greater than 1.

To obtain the expression of $z(y)$ at $y=0$, we put

$$z=yu .$$

Then (6) is transformed into

$$\frac{du}{dy}=\frac{-\lambda(\lambda+1)\alpha^2+(2\lambda+1)\alpha u-u^2+\lambda(\lambda+1)\alpha^2 y}{\alpha y u} .$$

Next we put

$$u=\alpha\lambda+v ,$$

and the equation is again transformed into

$$y\frac{dv}{dy}=\frac{\lambda(\lambda+1)\alpha^2 y+\alpha v-v^2}{\alpha^2\lambda+\alpha v}=(\lambda+1)y+\frac{1}{\alpha\lambda}v+\cdots$$

where the power series in y and v on the right-hand side is convergent in the neighbourhood of $y=v=0$.

$y=0$ is a well-known Briot-Bouquet type singularity. Since $\alpha\lambda>1$, $1/\alpha\lambda$ is positive and is not an integer. So the general solution of this equation can be expressed in a form

$$v=\sum_{m+n>0} v_{mn}y^m(Cy^{1/\alpha\lambda})^n$$

where C is an arbitrary constant. Therefore, by giving C an adequate value C_0, we get the following expression of $z(y)$ valid in the neighbourhood of $y=0$:

$$z(y)=y\cdot(\alpha\lambda+v)=\alpha\lambda y+y\sum_{m+n>0} v_{mn}y^m(C_0y^{1/\alpha\lambda})^n\,. \tag{25}$$

From (22) we get

$$\int_{y_0}^{y}\frac{dy}{z(y)}=\frac{\alpha}{\mu}\log(y-1)+\cdots \tag{26}$$

in the neighbourhood of $y=1$ where the unwritten terms are bounded as $y\to 1$, and from (25) we also get

$$\int_{y_0}^{y}\frac{dy}{z(y)}=\frac{1}{\alpha\lambda}\log y+\cdots \tag{27}$$

where the unwritten terms are bounded as $y\to 0$. As $\alpha/\mu<0$ and $1/\alpha\lambda>0$, we have

$$\lim_{y\to 1}\int_{y_0}^{y}\frac{dy}{z(y)}=\infty\,,\qquad \lim_{y\to 0}\int_{y_0}^{y}\frac{dy}{z(y)}=-\infty\,.$$

From this and the relation

$$\int_{y_0}^{y}\frac{dy}{z(y)}=\log t-\log t_0\,, \tag{28}$$

we get

$$\lim_{y\to 1} t(y)=\infty\,,\quad \lim_{y\to 0} t(y)=0\,. \tag{29}$$

Thus we have proved (1).

From (26), (27), (28) and (29), we can immediately derive the boundedness of $\phi(t)$ at $t=0$ and $t=\infty$. For example, from (27) and (28), we get

$$t/t_0=y^{1/\alpha\lambda}e^{F(y)}\quad\text{or}\quad y=(t/t_0)^{\alpha\lambda}e^{-\alpha\lambda F(y)}$$

where $F(y)$ is bounded as $y\to 0$. Since

$$\phi(t)=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda}(y(t))^{1/\alpha}$$

we have

$$\phi(t)=(\lambda(\lambda+1))^{1/\alpha}t_0^{-\lambda}e^{-\lambda F(y)}\,.$$

Therefore

$$0<\lim_{t\to 0}\phi(t)=\lim_{y\to 0}(\lambda(\lambda+1))^{1/\alpha}t_0^{-\lambda}e^{-\lambda F(y)}<\infty .$$

Analogously we can show that

$$\lim_{t\to\infty}\phi(t)<\infty .$$

However, to prove (2), we need more detailed knowledge about $y(t)$ because we have still to show the boundedness of $\phi'(t)$.

In the neighbourhood of $y=1$, the equation (23) can be written as

$$t\frac{dy}{dt}=\frac{\mu}{\alpha}(y-1)+\cdots$$

by (22). Hence the general solution of (23) is of the form

$$y(t)=1+Ct^{\mu/\alpha}+\cdots$$

in the neighbourhood of $t=\infty$ where the unwritten part is the power series in $Ct^{\mu/\alpha}$ starting with the term whose degree is at least 2 and C is an arbitrary constant. Therefore

$$\phi(t)=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda}(y(t))^{1/\alpha} \tag{30}$$
$$=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda}(1+(C/\alpha)t^{\mu/\alpha}+\cdots)$$

where the power series in $t^{\mu/\alpha}$ in the parenthesis is convergent in the vicinity of $t=\infty$. As λ is positive and μ/α is negative, we have

$$\lim_{t\to\infty}\phi(t)=\lim_{t\to\infty}\phi'(t)=0$$

which shows the boundedness of ϕ and ϕ' at $t=\infty$.

Next let us show the boundedness of $\phi'(t)$ at $t=0$. First we notice that

$$0<\lim_{t\to 0}t^{-\alpha\lambda}y(t)<\infty . \tag{31}$$

This follows immediately from the relation

$$0<\lim_{t\to 0}\phi(t)=\lim_{t\to 0}(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda}(y(t))^{1/\alpha}<\infty$$

which we have just proved above.

By direct calculation, we get

$$\phi'(t)=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda-1}y^{1/\alpha-1}(-\lambda y+\alpha^{-1}ty'(t))$$
$$=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda-1}y^{1/\alpha-1}(-\lambda y+\alpha^{-1}z(y)) .$$

Since $z=y\cdot(\alpha\lambda+v)$ by (25),

$$\begin{aligned}\phi'(t)&=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda-1}y^{1/\alpha}(v/\alpha)\\&=(\phi(t)/\alpha)\cdot t^{-1}v\ .\end{aligned}$$

As we already know that $\phi(t)$ is bounded as $t\to 0$, what we have to show is

$$\lim_{t\to 0} t^{-1}v<\infty\ .$$

Since v is expressed as

$$v=\sum_{m+n>0} v_{mn}y^m(C_0y^{1/\alpha\lambda})^n$$

in the neighbourhood of $y=0$ and $\lim_{y\to 0} t(y)=0$, it is sufficient to prove that

$$\lim_{t\to 0} t^{-1}y<\infty \quad \text{and} \quad \lim_{t\to 0} t^{-1}y^{1/\alpha\lambda}<\infty\ .$$

However, since $\alpha\lambda>1$, we have only to prove the latter inequality. But this is obvious from (31). Thus we have proved the boundedness of $\phi'(t)$ at $t=0$.

As the expression of $\phi(t)$ given by (30) shows, $\phi(t)$ contains an arbitrary constant C. Hence the totality of nontrivial bounded solutions of (2) constitutes a one-parameter family.

Summarizing the results obtained so far, we get the following theorem.

THEOREM. *The differential equation*

$$x''=t^\beta x^{1+\alpha}\ , \qquad \alpha>0\ ,$$

has a one-parameter family of nontrivial bounded solutions if and only if $\beta>-1$.

Finally we add a short remark about the analytical expression of $\phi(t)$.

At $t=\infty$, $\phi(t)$ is expressed in a form (30):

$$\phi(t)=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda}(1+Ct^{\mu/\alpha}+\cdots)\ .$$

(Here we replaced the arbitrary constant C/α by C.) This shows that every nontrivial bounded solution of (2) is asymptotic to the solution $\psi(t)=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda}$ as $t\to\infty$.

The analytical expression of $\phi(t)$ at $t=0$ can be obtained from (25):

$$z(y)=ty'=y\cdot(\alpha\lambda+\sum_{m+n>0} v_{mn}y^m(C_0y^{1/\alpha\lambda})^n)\ .$$

As can easily be verified, this equation is equivalent to a system

$$ty'=\alpha\lambda y(1+\sum_{m+n>0} c_{mn}y^m w^n)$$
$$tw'=w(1+\sum_{m+n>0} c_{mn}y^m w^n)\ , \qquad c_{mn}=v_{mn}/\alpha\lambda\ .$$

Since $t=0$ is a Briot-Bouquet type singularity of this system and $\alpha\lambda>1$, $y(t)$ can be expressed as a double power series in t and $t^{\alpha\lambda}$ convergent in the neighbourhood of $t=0$ if $\alpha\lambda$ is not an integer. Since $t^{-\alpha\lambda}y(t)$ has a finite positive limit as $t\to 0$ by (31), $y(t)$ has the form

$$y(t)=t^{\alpha\lambda}\Phi(t)$$

where $\Phi(t)$ is a convergent double power series in t and $t^{\alpha\lambda}$ starting with the positive constant term. Hence

$$\phi(t)=(\lambda(\lambda+1))^{1/\alpha}t^{-\lambda}(y(t))^{1/\alpha}$$

admits following double power series expansion in the neighbourhood of $t=0$:

$$\phi(t)=\sum_{m+n\geq 0} a_{mn}t^m t^{\alpha\lambda n}\ , \qquad a_{00}>0\ ,$$

unless $\alpha\lambda$ is an integer.

Reference

[1] E. Hille, Ordinary differential equations in the complex domain, Wiley, 1976.

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