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Analytic Functionals on the Lie Sphere

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Introduction

Suppose $S^1=\{z\in C; |z|=1\}$ is the unit circle. Let us denote by $L^2(S^1)$ the Hilbert space of square integrable functions on S^1 equipped with the inner product $(f, g)_{L^2(S^1)}=(f, \bar{g})_{S^1}$, where $(\ ,\ )_{S^1}$ is the bilinear form defined as follows:

$$(f, g)_{S^1}=\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})g(e^{i\theta})d\theta\ . \tag{0.1}$$

Let us denote by $\mathscr{H}^{(m)}(S^1)$ the one dimensional subspace of $L^2(S^1)$ spanned by the exponential function $e^{im\theta}$. Then we have the direct sum decomposition:

$$L^2(S^1)=\bigoplus_{m\in Z}\mathscr{H}^{(m)}(S^1) \tag{0.2}$$

and the orthogonal projection of $L^2(S^1)$ onto $\mathscr{H}^{(m)}(S^1)$ is given by

$$f(e^{i\theta})\longmapsto c_m e^{im\theta}\ , \tag{0.3}$$

where

$$c_m=\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-im\theta}d\theta \tag{0.4}$$

is the m-th Fourier coefficient of f.

More generally, suppose S^{n-1} is the $n-1$ dimensional unit sphere. $d\Omega_n$ denotes the invariant measure on S^{n-1} and Ω_n is the volume of S^{n-1}. Denote by $L^2(S^{n-1})$ the Hilbert space of square integrable functions on S^{n-1} equipped with the inner product $(f, g)_{L^2(S^{n-1})}=(f, \bar{g})_{S^{n-1}}$, where $(\ ,\ )_{S^{n-1}}$ is the bilinear form defined as follows:

$$(f, g)_{S^{n-1}}=\frac{1}{\Omega_n}\int_{S^{n-1}} f(\omega)g(\omega)d\Omega_n(\omega)\ . \tag{0.5}$$

If we denote by $\mathscr{H}^k(S^{n-1})$ the space of spherical harmonics of degree k,

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we have the direct sum decomposition:

$$L^2(S^{n-1}) = \bigoplus_{k \in Z_+} \mathscr{H}^k(S^{n-1}) . \tag{0.6}$$

Let us remark that

$$\mathscr{H}^k(S^1) = \mathscr{H}^{(k)}(S^1) \oplus \mathscr{H}^{(-k)}(S^1) \quad \text{for} \quad k \neq 0 .$$

The orthogonal projection of $L^2(S^{n-1})$ onto $\mathscr{H}^k(S^{n-1})$ is given by

$$f(\omega) \longmapsto S_k(f;\omega) , \tag{0.7}$$

where

$$S_k(f;\omega) = \frac{N(n, k)}{\Omega_n} \int_{S^{n-1}} f(\tau) P_k(n; \langle \omega, \tau \rangle) d\Omega_n(\tau) , \tag{0.8}$$

where $P_k(n; t)$ is the Legendre polynomial of degree k and of dimension n and $N(n, k) = \dim \mathscr{H}^k(S^{n-1})$.

In this paper, we will study the Lie sphere Σ^n:

$$\Sigma^n = \{e^{i\theta}\omega;\ \theta \in \boldsymbol{R},\ \omega \in S^{n-1}\} . \tag{0.9}$$

Remark that $\Sigma^1 = S^1$ and that Σ^2 is a polycircle: $\Sigma^2 \approx S^1 \times S^1$. Let us denote by $L^2(\Sigma^n)$ the Hilbert space of square integral functions on Σ^n equipped with the inner product $(f, g)_{L^2(\Sigma^n)} = (f, \bar{g})_{\Sigma^n}$, where $(\ ,\ )_{\Sigma^n}$ is the bilinear form defined as follows:

$$(f, g)_{\Sigma^n} = \frac{1}{\pi \Omega_n} \int_0^\pi \int_{S^{n-1}} f(e^{i\theta}\omega) g(e^{i\theta}\omega) d\theta d\Omega_n(\omega) . \tag{0.10}$$

Let us define the set Λ by

$$\Lambda = \{(m, k) \in \boldsymbol{Z} \times \boldsymbol{Z}_+;\ m \equiv k \pmod 2\} . \tag{0.11}$$

If we define, for $(m, k) \in \Lambda$, the space

$$\mathscr{H}^{m,k}(\Sigma^n) = \{e^{im\theta} S_k(\omega);\ S_k \in \mathscr{H}^k(S^{n-1})\} , \tag{0.12}$$

we have the direct sum decomposition:

$$L^2(\Sigma^n) = \bigoplus_{(m,k) \in} \mathscr{H}^{m,k}(\Sigma^n) \tag{0.13}$$

and the orthogonal projection of $L^2(\Sigma^n)$ onto $\mathscr{H}^{m,k}(\Sigma^n)$ is given by

$$f \longmapsto e^{im\theta} S_{m,k}(f;\omega) , \tag{0.14}$$

where

$$(0.15)\quad S_{m,k}(f;\omega)=\frac{N(n,k)}{\pi\Omega_n}\int_0^{\pi}\int_{S^{n-1}} f(e^{i\theta}\tau)e^{-im\theta}P_k(n;\langle\omega,\tau\rangle)d\theta d\Omega_n(\tau)\ .$$

The function $S_{m,k}(f;\omega)$ is called the (m,k)-component of the function f.

Put, for $R>1$,

$$(0.16)\qquad K^0_{R,R}=\{z\in \boldsymbol{C};\ R^{-1}<|z|<R\}\ .$$

Then $\{K^0_{R,R};\ R>1\}$ is a fundamental system of complex neighborhoods of S^1. Let us denote by $\tilde{S}^{n-1}$ the complex sphere:

$$(0.17)\qquad \tilde{S}^{n-1}=\{z\in \boldsymbol{C}^n;\ z^2=1\}\ ,$$

where we put $z^2=\sum_{j=1}^n z_j^2$. The Lie norm $L(z)$ on $\boldsymbol{C}^n$ is defined as follows:

$$(0.18)\qquad L(z)^2=\|z\|^2+[\|z\|^4-|z^2|^2]^{1/2}\ ,$$

$\|z\|$ being the Euclidean norm on $\boldsymbol{C}^n$. Put, for $R>1$,

$$(0.19)\qquad \tilde{S}^{n-1}(R)=\{z\in\tilde{S}^{n-1};\ L(z)<R\}\ .$$

Then it has been proved in our previous paper [7] that $\{\tilde{S}^{n-1}(R);\ R>1\}$ is a fundamental system of complex neighborhoods of the sphere S^{n-1}. Put, for $R>1$,

$$(0.20)\qquad \tilde{V}(R,R;R)=\{z\in\boldsymbol{C}^n;\ R^{-2}<|z^2|<R^2,\ L(z)^2<|z^2|R^2\}\ .$$

We shall prove in this paper that $\{\tilde{V}(R,R;R);\ R>1\}$ is a fundamental system of complex neighborhoods of the Lie sphere Σ^n.

Let us denote by $C^\infty(S^1)$, $C^\infty(S^{n-1})$ and $C^\infty(\Sigma^n)$ the spaces of C^∞ functions on S^1, S^{n-1} and Σ^n. Let us denote by $\mathscr{A}(S^1)$, $\mathscr{A}(S^{n-1})$ and $\mathscr{A}(\Sigma^n)$ the spaces of real analytic functions on S^1, S^{n-1} and Σ^n. Let us denote by $\mathscr{O}(K^0_{R,R})$, $\mathscr{O}(\tilde{S}^{n-1}(R))$ and $\mathscr{O}(\tilde{V}(R,R;R))$ the spaces of holomorphic functions on $K^0_{R,R}$, $\tilde{S}^{n-1}(R)$ and $\tilde{V}(R,R;R)$. Then we have the following inclusion relations:

$$(0.21)\qquad \mathscr{O}(K^0_{R,R})\subset\mathscr{A}(S^1)\subset C^\infty(S^1)\subset L^2(S^1)\ ,$$

$$(0.22)\qquad \mathscr{O}(\tilde{S}^{n-1}(R))\subset\mathscr{A}(S^{n-1})\subset C^\infty(S^{n-1})\subset L^2(S^{n-1})$$

and

$$(0.23)\qquad \mathscr{O}(\tilde{V}(R,R;R))\subset\mathscr{A}(\Sigma^n)\subset C^\infty(\Sigma^n)\subset L^2(\Sigma^n)\ .$$

Let us denote by $\mathscr{D}'(S^1)$, $\mathscr{D}'(S^{n-1})$ and $\mathscr{D}'(\Sigma^n)$ the spaces of distributions on S^1, S^{n-1} and Σ^n, i.e., the dual spaces of $C^\infty(S^1)$, $C^\infty(S^{n-1})$ and $C^\infty(\Sigma^n)$. We deonte by $\mathscr{B}(S^1)$, $\mathscr{B}(S^{n-1})$ and $\mathscr{B}(\Sigma^n)$ the spaces of hyperfunctions on S^1, S^{n-1} and Σ^n, i.e., the dual spaces of $\mathscr{A}(S^1)$, $\mathscr{A}(S^{n-1})$ and $\mathscr{A}(\Sigma^n)$. The dual spaces of $\mathscr{O}(K^0_{R,R})$, $\mathscr{O}(\tilde{S}^{n-1}(R))$ and $\mathscr{O}(\tilde{V}(R, R; R))$ will be denoted by $\mathscr{O}'(K^0_{R,R})$, $\mathscr{O}'(\tilde{S}^{n-1}(R))$ and $\mathscr{O}'(\tilde{V}(R, R; R))$ respectively, whose elements will be called generally analytic functionals. Then taking the dual sequences of (0.21), (0.22) and (0.23), we get the following inclusion relations:

$$\mathscr{O}'(K^0_{R,R})\supset\mathscr{B}(S^1)\supset\mathscr{D}'(S^1)\supset L^2(S^1)\,, \tag{0.24}$$

$$\mathscr{O}'(\tilde{S}^{n-1}(R))\supset\mathscr{B}(S^{n-1})\supset\mathscr{D}'(S^{n-1})\supset L^2(S^{n-1}) \tag{0.25}$$

and

$$\mathscr{O}'(\tilde{V}(R, R; R))\supset\mathscr{B}(\Sigma^n)\supset\mathscr{D}'(\Sigma^n)\supset L^2(\Sigma^n)\,. \tag{0.26}$$

We will show in this paper that, even for an analytic functional $f\in \mathscr{O}'(\tilde{V}(R, R; R))$, we can define the (m, k)-component $S_{m,k}(f; \omega)$ and that we can characterize the spaces which appear in the sequences (0.23) and (0.26) by the behavior of the (m, k)-components: namely

$$f\in\mathscr{O}(\tilde{V}(R, R; R)) \Longleftrightarrow \limsup_{|m|+k\to\infty} [\|S_{m,k}\|]^{1/(|m|+k)}\leqslant R^{-1}\,, \tag{0.27}$$

$$f\in\mathscr{A}(\Sigma^n) \Longleftrightarrow \limsup_{|m|+k\to\infty} [\|S_{m,k}\|]^{1/(|m|+k)}<1\,, \tag{0.28}$$

$$f\in C^\infty(\Sigma^n) \Longleftrightarrow \|S_{m,k}\| \text{ is rapidly decreasing on } \Lambda\,, \tag{0.29}$$

$$f\in L^2(\Sigma^n) \Longleftrightarrow \|S_{m,k}\|\in \ell^2(\Lambda)\,, \tag{0.30}$$

$$f\in\mathscr{D}'(\Sigma^n) \Longleftrightarrow \|S_{m,k}\| \text{ is slowly increasing on } \Lambda\,. \tag{0.31}$$

$$f\in\mathscr{B}(\Sigma^n) \Longleftrightarrow \limsup_{|m|+k\to\infty} [\|S_{m,k}\|]^{1/(|m|+k)}\leqslant 1\,, \tag{0.32}$$

$$f\in\mathscr{O}'(\tilde{V}(R, R; R)) \Longleftrightarrow \limsup_{|m|+k\to\infty} [\|S_{m,k}\|]^{1/(|m|+k)}<R\,, \tag{0.33}$$

where we put $\|S_{m,k}\|=\|S_{m,k}(f; \omega)\|_{L^2(S^{n-1})}$.

The analogous results for S^1 are classical and recalled in our paper [6]. The case S^{n-1} was treated in [7]. Using the same ideas as in [6] and [7], we will prove the above equivalences.

Let us return to the case of unit circle S^1. We know a holomorphic function $f(z)$ defined in the unit disc $\tilde{B}^1=\{z\in C; |z|<1\}$ has the hyperfunction trace $T(e^{i\theta})\in\mathscr{B}(S^1)$, whose Fourier coefficients

$$c_m=0 \quad\text{for}\quad m<0\,, \tag{0.34}$$

i.e., the hyperfunction $T(e^{i\theta})$ is orthogonal to the spaces $\mathscr{H}^{(m)}(S^1)$ for

$m>0$ with respect to the bilinear form $(\ ,\ )_{S^1}$. Conversely, if a hyperfunction $T(e^{i\theta})$ is given on S^1 and satisfies the condition (0.34), then there exists a unique holomorphic function $\tilde{f}(z)\in\mathscr{O}(\tilde{B}^1)$ in such a way that the hyperfunction trace of $\tilde{f}$ coincides with the given hyperfunction. The holomorphic function $\tilde{f}(z)$ is represented by the Cauchy integral formula:

$$\tilde{f}(z)=(T(e^{i\theta})\,,\quad(1-e^{-i\theta}z)^{-1})\,, \tag{0.35}$$

$(\ ,\ )$ being the canonical bilinear form on $\mathscr{B}(S^1)\times\mathscr{A}(S^1)$. By the trace operator $\rho:\mathscr{O}(\tilde{B}^1)\to\mathscr{B}(S^1)$, we can consider $\mathscr{O}(\tilde{B}^1)$ as a subspace of $\mathscr{B}(S^1)$.

Now let us consider the space $\mathscr{O}(\tilde{B}^1[1])$, where $\tilde{B}^1[1]$ is the closed unit disc: $\tilde{B}^1[1]=\{z\in\boldsymbol{C};\ |z|\leqslant 1\}$. Then the trace operator ρ maps $\mathscr{O}(\tilde{B}^1[1])$ injectively into $\mathscr{A}(S^1)$. The mapping $\gamma: f\mapsto\tilde{f}$ of $\mathscr{A}(S^1)$ onto $\mathscr{O}(\tilde{B}^1[1])$ is a left inverse of the trace operator ρ, where $\tilde{f}$ is defined by the Cauchy integral formula (0.35). By the dual mapping γ^*, we can identify $\mathscr{O}'(\tilde{B}^1[1])$ with the subspace of $\mathscr{B}(S^1)$ of the hyperfunctions on S^1 whose Fourier coefficients c_m vanish for $m>0$.

We will show also in this paper that we can generalize these facts to the Lie sphere case. We define the Lie ball $\tilde{B}=\tilde{B}^n$ as follows:

$$\tilde{B}=\tilde{B}^n=\{z\in\boldsymbol{C}^n;\ L(z)<1\}\,, \tag{0.36}$$

where $L(z)$ is the Lie norm. The Lie ball is E. Cartan's classical domain of type 4. (See Hua [5].) We will prove that every holomorphic function $f(z)$ on the Lie ball $\tilde{B}^n$ has the hyperfunction trace $T(e^{i\theta}\omega)\in\mathscr{B}(\Sigma^n)$, whose (m, k)-coefficients

$$S_{m,k}(T;\omega)=0 \quad\text{for}\quad m<k\,, \tag{0.37}$$

i.e., the hyperfunction $T(e^{i\theta}\omega)$ is orthogonal to the spaces $\mathscr{H}^{m,k}(\Sigma^n)$ with respect to the bilinear form $(\ ,\ )_{\Sigma^n}$ for $m>-k$. Conversely if a hyperfunction $T(e^{i\theta}\omega)$ is given on Σ^n and satisfies the condition (0.37), then there exists a unique holomorphic function $\tilde{f}(z)\in\mathscr{O}(\tilde{B}^n)$, the trace of which coincides with the given hyperfunction $T(e^{i\theta}\omega)$. The holomorphic function $\tilde{f}(z)$ can be represented by the Cauchy-Hua integral formula:

$$\tilde{f}(z)=(T(e^{i\theta}\omega)\,,\quad((\omega-e^{-i\theta}z)^2)^{-n/2})\,, \tag{0.38}$$

where $(\ ,\ )$ denotes the canonical bilinear form on $\mathscr{B}(\Sigma^n)\times\mathscr{A}(\Sigma^n)$. In this way, using the trace operator $\rho:\mathscr{O}(\tilde{B}^n)\to\mathscr{B}(\Sigma^n)$, we can consider $\mathscr{O}(\tilde{B}^n)$ as a subspace of $\mathscr{B}(\Sigma^n)$. Now let us consider the space $\mathscr{O}(\tilde{B}^n[1])$, where $\tilde{B}^n[1]$ is the closed Lie ball: $\tilde{B}^n[1]=\{z\in C^n; L(z)\leqslant 1\}$. Then the trace operator ρ maps $\mathcal{O}(\tilde{B}^n[1])$ injectively into $\mathcal{A}(\Sigma^n)$. The mapping $\gamma: f\mapsto \tilde{f}$ of $\mathcal{A}(\Sigma^n)$ onto $\mathcal{O}(\tilde{B}^n[1])$ is a left inverse of ρ, where $\tilde{f}$ is defined by the Cauchy-Hua integral formula (0.38). The dual mapping γ^* being injective, we can identify, by γ^*, the space $\mathcal{O}'(\tilde{B}^n[1])$ with the subspace of $\mathcal{B}(\Sigma^n)$ of the hyperfunctions on Σ^n whose (m, k)-components $S_{m,k}$ vanish for $m>-k$.

The plan of this paper is as follows. In § 1, we will recall some facts about the Lie norm, spherical harmonics and harmonic polynomials, introduce the Lie sphere Σ^n and define the complex neighborhood $\tilde{V}(A, B; R)$ of Σ^n. In § 2, we will consider the space $L^2(\Sigma^n)$, the space $C^\infty(\Sigma^n)$ and the space $\mathcal{A}(\Sigma^n)$, and prove the equivalences (0.30) and (0.29). As for the equivalence (0.28), we mention it as Theorem 2.3, which will be proved in § 3 as Corollary to Theorem 3.1. In § 3, we will study the space of holomorphic functions $\mathcal{O}(\tilde{V}(A, B; R))$ and prove the equivalence (0.27). We will study also the space of holomorphic functions $\mathcal{O}(\tilde{B})$. In § 4, we mention the results on $\mathcal{D}'(\Sigma^n)$ and $\mathcal{B}(\Sigma^n)$. In § 5, we study the space of analytic functionals $\mathcal{O}'(\tilde{V}(A, B; R))$ and prove the equivalence (0.33). We show also in § 5 that $\mathcal{O}(\tilde{B})$ and $\mathcal{O}'(\tilde{B}[1])$ can be considered as subspaces of $\mathcal{B}(\Sigma^n)$ by the trace operator ρ and the mapping γ^*. We will give a characterization of these subspaces.

§ 1. Preliminaries.

Let $\boldsymbol{Z}=\{0, \pm 1, \pm 2, \cdots\}$, $\boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$ and $\boldsymbol{R}$ be the real number line. Let $\boldsymbol{C}=\boldsymbol{R}+i\boldsymbol{R}$ be the complex number plane and $\boldsymbol{C}^*=\boldsymbol{C}\backslash\{0\}$. For $z=(z_1, z_2, \cdots, z_n)\in \boldsymbol{C}^n$ and $w=(w_1, w_2, \cdots, w_n)\in \boldsymbol{C}^n$, we put

$$\langle z, w\rangle = z_1w_1+z_2w_2+\cdots+z_nw_n \tag{1.1}$$

and

$$z^2=\langle z, z\rangle = z_1^2+z_2^2+\cdots+z_n^2\ . \tag{1.2}$$

$\|z\|=\langle z, \bar{z}\rangle^{1/2}$ is the Euclidean norm of z. For $x\in \boldsymbol{R}^n$, we have $\|x\|=(x^2)^{1/2}$.

The Lie norm $L(z)$ on $\boldsymbol{C}^n$ is defined by the formula:

$$L(z)^2=\|z\|^2+[\|z\|^4-|z^2|^2]^{1/2}\ . \tag{1.3}$$

By the simple calculations, we get

$$(1.4)\qquad L(z)^2 = \sum_{j=1}^{n} |z_j|^2 + \Big[\sum_{j<k} (2\,\mathrm{Im}\, z_j \bar{z}_k)^2\Big]^{1/2} ,$$

and

$$(1.5)\qquad L(z)^2 = \|x\|^2 + \|y\|^2 + 2[\|x\|^2 \|y\|^2 - \langle x, y\rangle^2]^{1/2} ,$$

where $z = x + iy$, $x, y \in \boldsymbol{R}^n$. Therefore we have

$$(1.6)\qquad \|x+iy\| \leqslant L(x+iy) \leqslant \|x\| + |\|y\| ,$$

where the first equality holds if and only if $\|z\|^2 = |z^2|$ and the second equality holds if and only if $\langle x, y\rangle = 0$. It has been proved by Drużkowski [1] that $L(z)$ is the cross norm of the Euclidean norm $\|x\|$ on $\boldsymbol{R}^n$, i.e.,

$$(1.7)\qquad L(z) = \inf \Big\{\sum_{j=1}^{m} |\lambda_j|\, \|x_j\|;\ z = \sum_{j=1}^{m} \lambda_j x_j,\ \lambda_j \in \boldsymbol{C},\ x_j \in \boldsymbol{R}^n,\ m \in \boldsymbol{Z}_+\Big\} .$$

$L(z)$ has been introduced by Harris [2] by a different method.

We call *the complex sphere* (of complex dimension $n-1$) the set

$$(1.8)\qquad \begin{aligned} \tilde{S}^{n-1} &= \{z \in \boldsymbol{C}^n;\ z^2 = 1\} \\ &= \{z = x + iy;\ \|x\|^2 - \|y\|^2 = 1,\ \langle x, y\rangle = 0\} . \end{aligned}$$

More generally we put, for $R > 1$,

$$(1.9)\qquad \begin{aligned} \tilde{S}^{n-1}(R) &= \{z \in \tilde{S}^{n-1};\ L(z) < R\} \\ &= \Big\{z = x + iy \in \tilde{S}^{n-1};\ \|y\| < \frac{1}{2}\Big(R - \frac{1}{R}\Big)\Big\} \end{aligned}$$

and, for $R \geqslant 1$,

$$(1.10)\qquad \begin{aligned} \tilde{S}^{n-1}[R] &= \{z \in \tilde{S}^{n-1};\ L(z) \leqslant R\} \\ &= \Big\{z = x + iy \in \tilde{S}^{n-1};\ \|y\| \leqslant \frac{1}{2}\Big(R - \frac{1}{R}\Big)\Big\} . \end{aligned}$$

$S^{n-1} = \tilde{S}^{n-1} \cap \boldsymbol{R}^n = \tilde{S}^{n-1}[1]$ is the (real unit) sphere. We will consider $\tilde{S}^{n-1} = \tilde{S}^{n-1}(\infty)$.

The complex orthogonal group $O(n; \boldsymbol{C})$ is the group of complex non-singular matrices U such that

$$(1.11)\qquad \langle Uz, Uw\rangle = \langle z, w\rangle \quad \text{for every } z \text{ and } w \text{ in } \boldsymbol{C}^n .$$

The *special complex orthogonal group*

$$(1.12)\qquad SO(n; \boldsymbol{C}) = \{U \in O(n; \boldsymbol{C});\ \det U = 1\}$$

is the connected component of the identity of the group $O(n; \boldsymbol{C})$. $O(n) = O(n; \boldsymbol{C}) \cap \mathrm{GL}(n; \boldsymbol{R})$ is *the othogonal group* and $SO(n) = SO(n; \boldsymbol{C}) \cap \mathrm{GL}(n; \boldsymbol{R})$ is *the special orthogonal group*. (See, for example, Helgason [4].)

LEMMA 1.1. (i) *For* $R>1$, $\tilde{S}^{n-1}(R)$ *is connected.* (ii) *For* $R \geqslant 1$, $\tilde{S}^{n-1}[R]$ *is connected.* (iii) *The group* $SO(n; \boldsymbol{C})$ *acts transitively on the complex sphere* $\tilde{S}^{n-1}$.

PROOF. (i) Suppose $z = x + iy \in \tilde{S}^{n-1}(R)$. There exists a matrix $U \in SO(n)$ such that $x' = Ux = \|x\| e_1$, where

$$e_j = {}^t(0, \cdots, 0, 1, 0, \cdots, 0)$$

is the j-th unit vector. Put $Uy = y' = {}^t(y'_1, y'_2, \cdots, y'_n)$. Then $\|y'\| = \|y\|$ and $\|x\| y'_1 = \langle x', y' \rangle = \langle x, y \rangle = 0$. By (1.8), $\|x\| \neq 0$. Therefore $y'_1 = 0$. Now there exists $U' \in SO(n)$ such that $U'e_1 = e_1$ and $U'y' = \|y\| e_2$. Put $x'' = U'Ux = \|x\| e_1$ and $y'' = U'Uy = \|y\| e_2$. As we have $\|x\|^2 - \|y\|^2 = 1$ by (1.8), there exists $s_0 \in \boldsymbol{R}$ such that

$$\|x\| = \mathrm{ch}\, s_0\,, \quad \|y\| = \mathrm{sh}\, s_0\,.$$

Define

$$U_s = \begin{bmatrix} \mathrm{ch}\, s & i\, \mathrm{sh}\, s & 0 \\ -i\, \mathrm{sh}\, s & \mathrm{ch}\, s & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix} \in SO(n; \boldsymbol{C})\,, \tag{1.13}$$

where I_{n-2} is the identity matrix of order $n-2$. Then we have

$$\begin{aligned} U_s U' U z &= U_s {}^t(\|x\|, i\|y\|, 0, \cdots, 0) \\ &= {}^t(\mathrm{ch}(s_0 - s), i\,\mathrm{sh}(s_0 - s), 0, \cdots, 0)\,. \end{aligned}$$

Now the set $\tilde{S}^{n-1}(R)$ is invariant under the action of the group $SO(n)$ and $|\mathrm{sh}(s_0 - s)| \leqslant |\mathrm{sh}\, s_0|$ for $0 \leqslant s \leqslant s_0$. The group $SO(n)$ being connected, we can find an arc joining the vector z and the unit vector e_1 in $\tilde{S}^{n-1}(R)$. (ii) and (iii) are clear by the above proof. q.e.d.

In this paper, we will study the set

$$\tilde{V} = \{z \in \boldsymbol{C}^n;\ z^2 \neq 0\}\,. \tag{1.14}$$

The following lemma is clear:

LEMMA 1.2. *The scalar product* $\mu: (\alpha, z) \mapsto \alpha z$ *is a holomorphic mapping of* $\boldsymbol{C}^* \times \tilde{S}^{n-1}$ *onto* $\tilde{V}$. *We have the diffeomorphism:*

$$(1.15) \qquad (C^* \times \tilde{S}^{n-1})/\sim \,\cong \tilde{V}\,,$$

where $\sim$ is the equivalence relation defined by

$$(1.16) \qquad (\alpha, z) \sim (\alpha', z') \iff \alpha=\alpha',\ z=z' \quad \text{or} \quad \alpha=-\alpha',\ z=-z'\,.$$

We put further, for $A, B>0$ with $AB>1$ and $R>1$,

$$(1.17) \qquad \tilde{V}(A, B; R)=\{z \in \tilde{V};\ B^{-2}<|z^2|<A^2,\ L(z)^2<|z^2|\,R^2\}\,.$$

We will consider $\tilde{V}=\tilde{V}(\infty, \infty; \infty)$. We put, for $A, B>0$ with $AB\geqslant 1$ and $R\geqslant 1$,

$$(1.18) \qquad \tilde{V}[A, B; R]=\{z \in \tilde{V};\ B^{-2}\leqslant|z^2|\leqslant A^2,\ L(z)^2\leqslant|z^2|R^2\}\,.$$

LEMMA 1.3. (i) *$\tilde{V}(A, B; R)$ is the image of the open set $K^0_{A,B}\times\tilde{S}^{n-1}(R)$ under the two-to-one diffeomorphism μ: $C^*\times\tilde{S}^{n-1}\to\tilde{V}$, where*

$$(1.19) \qquad K^0_{A,B}=\{\alpha \in C^*;\ B^{-1}<|\alpha|<A\}$$

and $\tilde{S}^{n-1}(R)$ is defined by (1.9).

(ii) *$\tilde{V}(A, B; R)$ is a domain,* i.e., *a connected open set in C^n.*

(iii) *$\tilde{V}[A, B, R]$ is the image of the compact set $K_{A,B}\times\tilde{S}^{n-1}[R]$ under the two-to-one diffeomorphism μ: $C^*\times\tilde{S}^{n-1}\to\tilde{V}$, where*

$$(1.19') \qquad K_{A,B}=\{\alpha \in C^*;\ B^{-1}\leqslant|\alpha|\leqslant A\}$$

and $\tilde{S}^{n-1}[R]$ is defined by (1.10).

(iv) *$\tilde{V}[A, B, R]$ is a connected compact set in C^n.*

PROOF. (i) Suppose $z \in \tilde{V}$ and put $z=\alpha z'$, $\alpha \in C^*$, $z' \in \tilde{S}^{n-1}$. Then $z^2=\alpha^2$ and $L(z)=|\alpha|\,L(z')$. Therefore, $z \in \tilde{V}(A, B, R)$ if and only if $B^{-1}<|\alpha|<A$ and $L(z')<R$.

(ii) Because the set $\tilde{S}^{n-1}(R)$ is connected (Lemma 1.1), we have (ii). The proof of (iii) and (iv) is similar. q.e.d.

We denote by $d\Omega_n$ the invariant measure on the (real unit) sphere S^{n-1} induced by the Lebesgue measure on R^n. The volume Ω_n of S^{n-1} is given by:

$$(1.20) \qquad \Omega_n=2\pi^{n/2}\Gamma(n/2)^{-1}\,.$$

$\mathscr{H}^k(S^{n-1})$ denotes *the space of spherical harmonics of degree* k and put

$$(1.21) \qquad N(n, k)=\dim \mathscr{H}^k(S^{n-1})=\frac{(2k+n-2)(k+n-3)!}{k!\,(n-2)!}\,.$$

For a spherical harmonic function $S_k(\omega) \in \mathscr{H}^k(S^{n-1})$, there exists a unique harmonic homogeneous polynomial $\tilde{S}_k(z)$ of degree k such that $\tilde{S}_k(\omega) = S_k(\omega)$ for $\omega \in S^{n-1}$. $P_k(n; t)$ denotes *the Legendre polynomial of degree* k *and of dimension* n, namely

$$P_k(2; t) = \cos(k \cos^{-1} t) \quad \text{(the Tchebytchef polynomial)}, \tag{1.22}$$

$$P_k(n; t) = \frac{k!(n-3)!}{(k+n-3)!} C_k^{(n-2)/2}(t) \quad \text{for} \quad n \geqq 3, \tag{1.23}$$

where $C_k^{(n-2)/2}(t)$ is the Gegenbauer polynomial. (See for example C. Müller [8].)

Let us denote by $L^2(S^{n-1})$ the Hilbert space of square integrable functions on S^{n-1}. The inner product of $L^2(S^{n-1})$ is given by

$$(\varphi, \psi)_{L^2(S^{n-1})} = (\varphi, \bar{\psi})_{S^{n-1}}, \tag{1.24}$$

where $(\ ,\ )_{S^{n-1}}$ is the following bilinear form:

$$(f, g)_{S^{n-1}} = \frac{1}{\Omega_n} \int_{S^{n-1}} f(\omega) g(\omega) d\Omega_n(\omega). \tag{1.25}$$

It is classical that the Hilbert space $L^2(S^{n-1})$ can be decomposed into the direct sum:

$$L^2(S^{n-1}) = \bigoplus_{k=0}^{\infty} \mathscr{H}^k(S^{n-1}) \tag{1.26}$$

and the orthogonal projection of $L^2(S^{n-1})$ onto $\mathscr{H}^k(S^{n-1})$: $f(\omega) \mapsto S_k(f; \omega)$ is given by

$$\begin{aligned} S_k(f; \omega) &= \frac{N(n, k)}{\Omega_n} \int_{S^{n-1}} f(\tau) P_k(n; \langle \tau, \omega \rangle) d\Omega_n(\tau) \\ &= N(n, k)(f(\tau), P_k(n; \langle \omega, \tau \rangle))_{S^{n-1}}. \end{aligned} \tag{1.27}$$

Therefore, for a spherical harmonic function $S_k(\omega) \in \mathscr{H}^k(S^{n-1})$, we have the reproducing property:

$$S_k(\omega) = \frac{N(n, k)}{\Omega_n} \int_{S^{n-1}} S_k(\tau) P_k(n; \langle \tau, \omega \rangle) d\Omega_n(\tau). \tag{1.28}$$

From (1.28), we can conclude

LEMMA 1.4. *Suppose* $S_k(\omega) \in \mathscr{H}^k(S^{n-1})$. *Then we have*

$$N(n, k)^{-1/2}\|S_k(\omega)\|_{L^\infty(S^{n-1})} \leqslant \|S_k(\omega)\|_{L^2(S^{n-1})} \leqslant \|S_k(\omega)\|_{L^\infty(S^{n-1})} , \tag{1.29}$$

where $\| \ \|_{L^\infty(S^{n-1})}$ *denotes the supremum norm on* S^{n-1}.

We will put, for $f \in L^2(S^{n-1})$,

$$S_k(f; z) = \frac{N(n, k)}{\Omega_n} \int_{S^{n-1}} f(\tau) P_k(n; \langle \tau, z \rangle) d\Omega_n(\tau) , \tag{1.30}$$

where $\langle \tau, z \rangle = \tau_1 z_1 + \tau_2 z_2 + \cdots + \tau_n z_n$. $S_k(f; z)$ is a polynomial of degree k and its restriction to the sphere S^{n-1} coincides with the spherical harmonic function $S_k(f; \omega)$. Put further

$$\tilde{S}_k(f; z) = (\sqrt{z^2})^k S_k(f; z/\sqrt{z^2}) . \tag{1.31}$$

Then $\tilde{S}_k(f; z)$ is the (unique) harmonic homogeneous polynomial of degree k such that $\tilde{S}_k(f; \omega) = S_k(f; \omega)$ for all $\omega \in S^{n-1}$.

Now let us define *the Lie sphere* Σ^n to be the set

$$\Sigma^n = \{e^{i\theta}\omega \in \boldsymbol{C}^n; \theta \in \boldsymbol{R}, \omega \in S^{n-1}\} . \tag{1.32}$$

The Lie sphere Σ^n is a compact real-analytic manifold of dimension n.

LEMMA 1.5. *The scalar product* $\mu: (e^{i\theta}, \omega) \mapsto e^{i\theta}\omega$ *is a real-analytic mapping of* $S^1 \times S^{n-1}$ *onto the Lie sphere* Σ^n. *We have the real-analytic diffeomorphism*:

$$(S^1 \times S^{n-1})/\sim\, = \Sigma^n , \tag{1.33}$$

where $\sim$ *is the equivalence relation defined by*

$$(\theta, \omega) \sim (\theta', \omega') \Longleftrightarrow \theta = \theta' (\operatorname{mod} 2\pi) , \quad \omega = \omega' \tag{1.34}$$

$$\text{or}$$

$$\theta = \theta' + \pi (\operatorname{mod} 2\pi) , \quad \omega = -\omega' .$$

Proof is the same as that of Lemma 1.3.

The Laplace-Beltrami operator Δ_Σ on Σ^n is given by

$$\Delta_\Sigma = \frac{\partial^2}{\partial \theta^2} + \Delta_S , \tag{1.35}$$

where Δ_S is the Laplace-Beltrami operator on S^{n-1}. The measure $d\theta d\Omega_n$ is the invariant measure on Σ^n and the volume of Σ^n is $\pi\Omega_n$.

LEMMA 1.6. (i) *We have the identity*

$$\Sigma^n = \tilde{V}[1, 1; 1] . \tag{1.36}$$

(ii) *$\{\tilde{V}(A, B; R);\ A>1, B>1, R>1\}$ is a fundamental system of complex neighborhoods of the Lie sphere Σ^n.*

PROOF. If $z \in \Sigma^n$, then $z=e^{i\theta}x$, $\theta \in \boldsymbol{R}$, $x \in S^{n-1}$. Therefore $|z^2|=|e^{i2\theta}|=1$ and $L(z)=|e^{i\theta}|L(x)=\|x\|=1$, i.e., $z \in \tilde{V}[1, 1; 1]$.

Conversely, if $z \in \tilde{V}[1, 1; 1]$ and put $z=\alpha z'$, $\alpha \in \boldsymbol{C}^*$, $z' \in \tilde{S}^{n-1}$. Then $|z^2|=|\alpha^2|=1$ and $|\alpha|=1$. If we put $z'=x'+iy'$, $x', y' \in \boldsymbol{R}^n$, the condition $z' \in \tilde{S}^{n-1}$ is equivalent to

$$\|x'\|^2-\|y'\|^2=1 \quad \text{and} \quad \langle x', y'\rangle=0 .$$

Therefore we have by (1.6)

$$1=L(z)=|\alpha|L(z')=L(z')=\|x'\|+\|y'\|=\|x'\|+[\|x'\|^2-1]^{1/2} ,$$

from which results $\|x'\|=1$ and $y'=0$. This proves z' belongs to the real sphere S^{n-1} and $z=\alpha z' \in \Sigma^n$. The second part of the lemma is clear by Lemma 1.3. q.e.d.

Suppose $r>0$. *The Lie ball $\tilde{B}(r)$* of radius r is defined as follows:

$$\tilde{B}(r)=\{z \in \boldsymbol{C}^n;\ L(z)<r\} . \tag{1.37}$$

We denote $\tilde{B}=\tilde{B}(1)$.

We will use the following facts in the sequel:

PROPOSITION 1.1. *Suppose $r>0$. $r\Sigma^n$ is the Šilov boundary of the Lie ball $\tilde{B}(r)$.*

For the proof of the proposition, see L. Hua [5], where $\tilde{B}$ is called E. Cartan's classical domain of type 4.

COROLLARY 1. *If $f \in \mathcal{O}(\boldsymbol{C}^n)$ is a homogeneous polynomial of degree k, then we have*

$$|f(z)| \leqslant L(z)^k \sup\{|f(\omega)|;\ \omega \in S^{n-1}\} . \tag{1.38}$$

PROOF. By Proposition 1.1,

$$\sup\{|f(z)|;\ z \in \tilde{B}\}=\sup\{|f(e^{i\theta}\omega)|;\ \theta \in \boldsymbol{R},\ \omega \in S^{n-1}\} .$$

By the homogeneity of f, the right hand side is equal to $\sup\{|f(\omega)|;\ \omega \in S^{n-1}\}$. (1.38) is now clear. q.e.d.

COROLLARY 2. *$P_k(n, t)$ denotes the Legendre polynomial of degree k and of dimension n. Then*

$$F(z, w)=(\sqrt{z^2})^k(\sqrt{w^2})^k P_k(n;\ \langle z/\sqrt{z^2},\ w/\sqrt{w^2}\rangle)$$

is a polynomial of z and w and we have the estimate

(1.39) $|(\sqrt{z^2})^k)(\sqrt{w^2})^k P_k(n; \langle z/\sqrt{z^2}, w/\sqrt{w^2}\rangle)| \leqslant L(z)^k L(w)^k$ *for* $z, w \in \boldsymbol{C}^n$.

PROOF. $P_k(n; t)$ is a polynomial of degree k and we have

$$P_k(n; -t)=(-1)^k P_k(n; t) .$$

Therefore $F(z, w)$ is a polynomial of z and w. If we fix $w \in \boldsymbol{C}^n$, then $F(z, w)$ is a homogeneous polynomial of degree k in z. By Corollary 1,

$$|F(z, w)| \leqslant L(z)^k \sup\{|F(\omega, w)|; \omega \in S^{n-1}\} .$$

If we fix $\omega \in S^{n-1}$, then $F(\omega, w)$ is a homogeneous polynomial of degree k in w. Again by Corollary 1,

$$|F(\omega, w)| \leqslant L(w)^k \sup\{|F(\omega, \tau)|; \tau \in S^{n-1}\} .$$

But, for $\omega, \tau \in S^{n-1}$,

$$F(\omega, \tau)=P_k(n; \langle\omega, \tau\rangle)$$

and we know

$$-1 \leqslant P_k(n; t) \leqslant 1 \quad \text{for} \quad -1 \leqslant t \leqslant 1 .$$

(See for example Müller [8].) Therefore we have (1.39). q.e.d.

§2. Function spaces on the Lie sphere Σ^n.

Let $L^2(\Sigma^n)$ be the space of square integrable functions on the Lie sphere Σ^n. The inner product of the Hilbert space $L^2(\Sigma^n)$ is given by

(2.1) $$(\varphi, \psi)_{L^2(\Sigma^n)}=(\varphi, \bar{\psi})_{\Sigma^n} ,$$

where $(\;, \;)_{\Sigma^n}$ is the following bilinear form:

(2.2) $$(f, g)_{\Sigma^n}=\frac{1}{\pi\Omega_n}\int_0^\pi\int_{S^{n-1}} f(e^{i\theta}\omega)g(e^{i\theta}\omega)d\theta d\Omega_n(\omega) .$$

For $S_k(\omega) \in \mathscr{H}^k(S^{n-1})$, the function $e^{im\theta}S_k(\omega)$ is defined on the Lie sphere Σ^n if and only if $m \equiv k \pmod 2$. Put

(2.3) $$\Lambda=\{(m, k); m \in \boldsymbol{Z}, k \in \boldsymbol{Z}_+, m \equiv k (\mathrm{mod}\, 2)\} .$$

For $(m, k) \in \Lambda$, we define

(2.4) $$\mathscr{H}^{m,k}(\Sigma^n)=\{e^{im\theta}S_k(\omega); S_k \in \mathscr{H}^k(S^{n-1})\} .$$

LEMMA 2.1. *The spaces $\mathscr{H}^{m,k}(\Sigma^n)$ are mutually orthogonal with respect to the scalar product* (2.1). *The space $\mathscr{H}^{m,k}(\Sigma^n)$ is the eigen-space of the Laplace-Beltrami operator Δ_Σ of eigen value $-[m^2+k(k+n-2)]$.*

PROOF. The orthogonality of the spaces $\mathscr{H}^{m,k}(\Sigma^n)$ results from the orthogonality of trigonometric functions and spherical harmonic functions. In fact, we have

$$(e^{im\theta}S_k(\omega),\ e^{im'\theta}S_{k'}(\omega))_{L^2(\Sigma^n)}=\delta_{m,m'}\delta_{k,k'}\|S_k(\omega)\|^2_{L^2(S^{n-1})} \tag{2.5}$$

for $e^{im\theta}S_k(\omega)\in\mathscr{H}^{m,k}(\Sigma^n)$ and $e^{im'\theta}S_{k'}(\omega)\in\mathscr{H}^{m',k'}(\Sigma^n)$, where $\delta_{m,m'}$ and $\delta_{k,k'}$ are Kronecker's symbols.

It is known that

$$\begin{cases}\dfrac{\partial^2}{\partial\theta^2}e^{im\theta}=-m^2e^{im\theta}\,, \\ \Delta_S S_k(\omega)=-k(k+n-2)S_k(\omega)\end{cases} \tag{2.6}$$

for $S_k(\omega)\in\mathscr{H}^k(S^{n-1})$. Therefore we have the second part of the lemma.

q.e.d.

For $f\in L^2(\Sigma^n)$, we will define the (m, k)-component $S_{m,k}(f;\omega)$ by the following formula:

$$\begin{aligned}S_{m,k}(f;\omega)&=\frac{1}{2\pi}\frac{N(n, k)}{\Omega_n}\int_0^{2\pi}\int_{S^{n-1}}f(e^{i\theta}\tau)e^{-im\theta}P_k(n;\langle\omega,\tau\rangle)d\theta d\Omega_n(\tau)\\ &=\frac{N(n, k)}{\pi\Omega_n}\int_0^{\pi}\int_{S^{n-1}}f(e^{i\theta}\tau)e^{-im\theta}P_k(n;\langle\omega,\tau\rangle)d\theta d\Omega_n(\tau)\,.\end{aligned} \tag{2.7}$$

$S_{m,k}(f;\omega)$ is a spherical harmonic function of degree k, i.e., $S_{m,k}(f;\omega)\in\mathscr{H}^k(S^{n-1})$. Remark that

$$S_{m,k}(f;-\omega)=(-1)^kS_{m,k}(f;\omega)$$

and that, by the definition formula (2.7),

$$S_{m,k}(f;-\omega)=(-1)^mS_{m,k}(f;\omega)\,.$$

Therefore we must have

$$S_{m,k}(f;\omega)=0 \quad \text{if} \quad m\not\equiv k\ (\mathrm{mod}\ 2)\,. \tag{2.8}$$

Similarly to (1.30) and (1.31), we define the polynomials

$$S_{m,k}(f;z)=\frac{N(n, k)}{\pi\Omega_n}\int_0^{\pi}\int_{S^{n-1}}f(e^{i\theta}\tau)e^{-im\theta}P_k(n;\langle z,\tau\rangle)d\theta d\Omega_n(\tau) \tag{2.9}$$

and

$$\tilde{S}_{m,k}(f; z) = (\sqrt{z^2})^k S_{m,k}(f; z/\sqrt{z^2}) . \tag{2.10}$$

The polynomial $\tilde{S}_{m,k}(f; z)$ is the (unique) harmonic homogeneous polynomial of degree k such that

$$\tilde{S}_{m,k}(f; \omega) = S_{m,k}(f; \omega) \quad \text{for every} \quad \omega \in S^{n-1} .$$

THEOREM 2.1. *We have the direct sum decomposition of the Hilbert space* $L^2(\Sigma^n)$:

$$L^2(\Sigma^n) = \bigoplus_{(m,k) \in \Lambda} \mathscr{H}^{m,k}(\Sigma^n) . \tag{2.11}$$

The orthogonal projection of $L^2(\Sigma^n)$ *onto* $\mathscr{H}^{m,k}(\Sigma^n)$ *is given by* $f \mapsto e^{im\theta} S_{m,k}(f; \omega)$. *We have the following identity in the* L^2 *sense for* $f \in L^2(\Sigma^n)$:

$$f(e^{i\theta}\omega) = \sum_{(m,k) \in \Lambda} e^{im\theta} S_{m,k}(f; \omega) . \tag{2.12}$$

PROOF. By Lemma 1.4, the space $L^2(\Sigma^n)$ can be identified with the subspace of $L^2(S^1 \times S^{n-1})$ of the functions satisfying the condition $f(\theta + \pi, -\omega) = f(\theta, \omega)$. Therefore it is clear by (1.26) that every function $f(e^{i\theta}\omega) \in L^2(\Sigma^n)$ can be expanded in the L^2 sense:

$$f(e^{i\theta}\omega) = \sum_{(m,k) \in \Lambda} e^{im\theta} S_k(\omega) \tag{2.13}$$

with some $S_k \in \mathscr{H}^k(S^{n-1})$. By the orthogonality of trigonometric functions and the reproducing property (1.28) of the Legendre polynomial, $S_k(\omega)$ must be equal to $S_{m,k}(f; \omega)$. q.e.d.

COROLLARY 1 (Bessel's equality). *For* $f \in L^2(\Sigma^n)$, *we have*

$$\|f\|^2_{L^2(\Sigma^n)} = \sum_{(m,k) \in \Lambda} \|S_{m,k}(f; \omega)\|^2_{L^2(S^{n-1})} . \tag{2.14}$$

COROLLARY 2. *If we have a sequence* $\{S_{m,k}; (m, k) \in \Lambda\}$, $S_{m,k} \in \mathscr{H}^k(S^{n-1})$ *such that* $\|S_{m,k}\|_{L^2(S^{n-1})} \in \ell^2(\Lambda)$, *then the function* $f(e^{i\theta}w) = \sum_{(m,k) \in \Lambda} e^{im\theta} S_{m,k}(\omega)$ *belongs to* $L^2(\Sigma^n)$ *and* $S_{m,k}(f; \omega) = S_{m,k}(\omega)$ *for every* $(m, k) \in \Lambda$.

Let us denote by $C^\infty(\Sigma^n)$ (resp. $\mathscr{A}(\Sigma^n)$) the space of C^∞ (resp. real analytic) functions on the compact real analytic manifold Σ^n. We endow $C^\infty(\Sigma^n)$ and $\mathscr{A}(\Sigma^n)$ with the usual locally convex linear topology.

THEOREM 2.2. *A function*

$$f(e^{i\theta}\omega)=\sum_{(m,k)\in\Lambda} e^{im\theta}S_{m,k}(f;\omega)\in L^2(\Sigma^n) \tag{2.12}$$

belongs to $C^\infty(\Sigma^n)$ *if and only if the sequence* $\{\|S_{m,k}(f;\omega)\|_{L^2(S^{n-1})};(m,k)\in\Lambda\}$ *is rapidly decreasing on* Λ, i.e., $\{(|m|+k)^p\|S_{m,k}(f;\omega)\|_{L^2(S^{n-1})}\}$ *is bounded on* Λ *for every* $p\in Z_+$. *For* $f\in C^\infty(\Sigma^n)$, *the convergence of the series* (2.12) *is in the topology of* $C^\infty(\Sigma^n)$.

PROOF. It is known that $f\in C^\infty(\Sigma^n)$ if and only if $(\Delta_\Sigma)^p f\in L^2(\Sigma^n)$ for every $p\in Z_+$. Because of Lemma 2.1,

$$(-\Delta_\Sigma)^p f(e^{i\theta}\omega)=\sum_{(m,k)\in\Lambda}(m^2+k(k+n-2))^p e^{im\theta}S_{m,k}(f;\omega)\ , \tag{2.15}$$

from which results the theorem. q.e.d.

THEOREM 2.3. *A function*

$$f(e^{i\theta}\omega)=\sum_{(m,k)\in\Lambda} e^{im\theta}S_{m,k}(f;\omega)\in L^2(\Sigma^n) \tag{2.12}$$

belongs to $\mathscr{A}(\Sigma^n)$ *if and only if*

$$\limsup_{|m|+k\to\infty}[\|S_{m,k}(f;\omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)}<1\ . \tag{2.16}$$

For $f\in\mathscr{A}(\Sigma^n)$, *the series* (2.12) *converges in the topology of* $\mathscr{A}(\Sigma^n)$.

REMARK. Hashizume-Minemura-Okamoto [3] has proved this theorem in more general context using the following fact: $f\in\mathscr{A}(\Sigma^n)$ if and only if

$$\sup\left\{\frac{1}{p!\,h^p}\|(-\Delta_\Sigma)^{p/2}f\|_{L^2(\Sigma^n)};\ p\in Z_+\right\}<\infty \tag{2.17}$$

for some $h>0$, where

$$(-\Delta_\Sigma)^{p/2}f(e^{i\theta}\omega)=\sum_{(m,k)\in\Lambda}(m^2+k(k+n-2))^{p/2}e^{im\theta}S_{m,k}(f;\omega)\ . \tag{2.18}$$

We do not reproduce here their proof. We will give a new proof to Theorem 2.3 in the following section (Corollary 2 (ii) to Theorem 3.1).

§3. Some spaces of holomorphic functions.

For an open set Ω of C^n, we denote by $\mathscr{O}(\Omega)$ the space of holomorphic functions on Ω. We endow $\mathscr{O}(\Omega)$ with the topology of uniform convergence on every compact set of Ω. $\mathscr{O}(\Omega)$ is a Fréchet space.

LEMMA 3.1. *Suppose* $A>1$, $B>1$ *and* $R>1$ *and define the open sets*

$\tilde{V}$ and $\tilde{V}(A, B; R)$ by (1.14) *and* (1.17). *Then we have the following continuous inclusions:*

$$\mathcal{O}(\tilde{V}) \hookrightarrow \mathcal{O}(\tilde{V}(A, B; R)) \hookrightarrow \mathscr{A}(\Sigma^n) \hookrightarrow C^\infty(\Sigma^n) \hookrightarrow L^2(\Sigma^n)\,, \tag{3.1}$$

where the first and the second mappings are restrictions.

PROOF. The continuity of the mappings is clear. The injectivity of the first mapping results from the uniqueness of analytic continuation. Suppose $F \in \mathcal{O}(\tilde{V}(A, B; R))$ vanishes on Σ^n. The Lie sphere Σ^n being totally real in $\tilde{V}(A, B; R)$, f vanishes in a complex neighborhood of Σ^n. The set $\tilde{V}(A, B; R)$ being connected (Lemma 1.3), f vanishes identically on $\tilde{V}(A, B; R)$ by the uniqueness of analytic continuation. q.e.d.

LEMMA 3.2. *$\mathcal{O}(\tilde{V})$ is dense in $\mathscr{A}(\Sigma^n)$, $C^\infty(\Sigma^n)$ and $L^2(\Sigma^n)$ respectively.*

PROOF. By Theorems 2.1, 2.2 and 2.3, the linear space $\sum_{(m,k)\in\Lambda}\mathscr{H}^{m,k}(\Sigma^n)$ is dense in $L^2(\Sigma^n)$, $C^\infty(\Sigma^n)$ and $\mathscr{A}(\Sigma^n)$. As the function $e^{im\theta}S_{m,k}(\omega)$ is the restriction onto Σ^n of the holomorphic function on $\tilde{V}$, $(\sqrt{z^2})^{m-k}\tilde{S}_{m,k}(z)$, the linear space $\sum_{(m,k)\in\Lambda}\mathscr{H}^{m,k}(\Sigma^n)$ is a subspace of $\mathcal{O}(\tilde{V})$. q.e.d.

For a set $K\subset \boldsymbol{C}^n$, we put

$$\sigma(K)=\{e^{i\theta}z;\ z\in K,\ \theta\in \boldsymbol{R}\}\,. \tag{3.2}$$

For example, $\sigma(S^{n-1})=\Sigma^n$. For a set $K\subset \boldsymbol{C}^n$ and $a\in \boldsymbol{C}$, we will put

$$aK=\{az;\ z\in K\}\,. \tag{3.3}$$

LEMMA 3.3. *Suppose $A>1$, $B>1$ and $R>1$. For $f\in\mathcal{O}(\tilde{V}(A, B; R))$, we define, for $r>0$ and $z\in\tilde{V}(A/r, rB; R)$,*

$$f_m(r; z)=\frac{1}{2\pi i}\oint_{|t|=r} f(tz)t^{-(m+1)}dt\,. \tag{3.4}$$

(i) *We can define $f_m(z)\in\mathcal{O}(\tilde{V}(\infty, \infty; R))$ by the following formula:*

$$f_m(z)=f_m(r; z)\quad \text{for}\quad z\in\tilde{V}(A/r, rB; R)\,. \tag{3.5}$$

(ii) *For any $\alpha\in\boldsymbol{C}^*$ and $z\in\tilde{V}(\infty, \infty; R)$, we have*

$$f_m(\alpha z)=\alpha^m f_m(z)\,. \tag{3.6}$$

(iii) *For every compact set $K\subset\tilde{S}^{n-1}(R)$ and every r, $B^{-1}<r<A$, we have*

$$\sup\{|f_m(z)|;\, z\in K\}\leqslant r^{-m}\sup\{|f(z)|;\, z\in r\sigma(K)\}\,. \tag{3.7}$$

(iv) *We have the development*

$$f(z)=\sum_{m=-\infty}^{\infty} f_m(z) \quad \textit{for} \quad z\in\tilde{V}(A, B; R)\,, \tag{3.8}$$

the convergence being uniform on every compact set of $\tilde{V}(A, B; R)$.

PROOF. By Lemma 1.3, there exists a function $F(\alpha', z')\in\mathcal{O}(K^0_{A,B}\times\tilde{S}^{n-1}(R))$ *such that*

$$F(-\alpha', -z')=F(\alpha', z')$$

and that $F(\alpha', z')=f(\alpha'z')$. Remark that, for $z=\alpha'z'$, $\alpha'\in C^*$, $z'\in\tilde{S}^{n-1}$, we have $z^2=\alpha'^2$ and we have

$$\begin{aligned} f_m(r;\alpha'z') &= \frac{1}{2\pi i}\oint_{|t|=r}\frac{f(t\alpha'z')}{t^{m+1}}\,dt \\ &= \frac{1}{2\pi i}\oint_{|t|=r}\frac{F(t\alpha', z')}{t^{m+1}}\,dt \end{aligned}$$

for $B^{-1}<r|\alpha'|<A$. By the Cauchy integral theorem, the function

$$F_m(\alpha', z')=\frac{1}{2\pi i}\oint_{|t|=r}\frac{F(t\alpha', z')}{t^{m+1}}\,dt \tag{3.9}$$

is defined independently of r, holomorphic on $C^*\times\tilde{S}^{n-1}(R)$ and satisfies

$$F_m(\alpha', z')=\alpha'^m F_m(1, z')\,.$$

Therefore, the function

$$f_m(z)=F_m(\alpha', z')\,,\quad z=\alpha'z'\,,$$

is defined on $\tilde{V}(\infty, \infty; R)$ and satisfies (3.6).

(iii) For $z\in\tilde{S}^{n-1}(R)$ and $B^{-1}<r<A$, we have the integral formula:

$$f_m(z)=\frac{1}{2\pi i}\oint_{|t|=r}\frac{f(tz)}{t^{m+1}}\,dt\,. \tag{3.10}$$

Therefore we get (3.7).

(iv) The Laurent development

$$F(\alpha', z')=\sum_{m=-\infty}^{\infty}\alpha'^m F_m(1, z')=\sum_{m=-\infty}^{\infty}F_m(\alpha', z')$$

converges uniformly on every compact set of $K^0_{A,B}\times\tilde{S}^{n-1}(R)$. Therefore we have (3.8). q.e.d.

LEMMA 3.4. *Suppose* $A>1$, $B>1$ *and* $R>1$. *Let* $f\in\mathcal{O}(\tilde{V}(A, B; R))$. *We will denote the* (m, k)*-component of* $f|_{\Sigma^n}$ *by* $S_{m,k}(f; \omega)$. *The polynomials* $S_{m,k}(f; z)$ *and* $\tilde{S}_{m,k}(f; z)$ *are defined by* (2.9) *and* (2.10). *Then we have*

$$S_{m,k}(f; z)=\frac{N(n, k)}{\Omega_n}\int_{S^{n-1}}f_m(\tau)P_k(n; \langle\tau, z\rangle)d\Omega_n(\tau)\,, \tag{3.11}$$

where f_m *is defined in Lemma* 3.3.

$$S_{m,k}(f; z)=\tilde{S}_{m,k}(f; z)=0 \tag{3.12}$$

for $m\not\equiv k(\mathrm{mod}\,2)$.

$$f_m(z)=\sum_{k=0}^{\infty}(\sqrt{z^2})^{m-k}\tilde{S}_{m,k}(f; z) \tag{3.13}$$

for $z\in\tilde{V}(\infty, \infty; R)$, *the convergence being uniform on every compact set of* $\tilde{V}(\infty, \infty; R)$.

PROOF. Put $r=1$ in (3.10) and (3.11) coincides with (2.9). (3.12) and (3.13) are special cases of (2.8) and (2.10) respectively. We have, by Corollary to Theorem 5.2 in M. Morimoto [7],

$$f_m(z)=\sum_{k=0}^{\infty}\tilde{S}_{m,k}(f; z) \quad \text{for} \quad z\in\tilde{S}^{n-1}(R)\,, \tag{3.14}$$

the convergence being uniform on every compact set of $\tilde{S}^{n-1}(R)$. Both sides of (3.13) are homogeneous of degree m and if $z\in\tilde{S}^{n-1}(R)$, (3.13) coincides with (3.14). Therefore the equality in (3.13) holds also for $z\in V(\infty, \infty; R)$ (Lemma 2.1). q.e.d.

THEOREM 3.1. *Let* $f\in\mathcal{O}(\tilde{V}(A, B; R))$ *and put* $S_{m,k}(\omega)=S_{m,k}(f; \omega)$ *for* $(m, k)\in\Lambda$. *Then we have*

$$\limsup_{m+k\to\infty,\, m\geq 0}[A^m R^k||S_{m,k}(\omega)||_{L^2(S^{n-1})}]^{1/(m+k)}\leqslant 1 \tag{3.15}$$

and

$$\limsup_{|m|+k\to 0,\, m<0}[B^{|m|}R^k||S_{m,k}(\omega)||_{L^2(S^{n-1})}]^{1/(|m|+k)}\leqslant 1\,. \tag{3.16}$$

The following series

$$f(z)=\sum_{(m,k)\in\Lambda}(\sqrt{z^2})^{m-k}\tilde{S}_{m,k}(z) \tag{3.17}$$

converges uniformly on every compact set of $\tilde{V}(A, B; R)$.

Conversely, if we have a sequence $\{S_{m,k}; (m, k) \in \Lambda\}$, $S_{m,k} \in \mathscr{H}^k(S^{n-1})$ *satisfying the conditions* (3.15) *and* (3.16), *then the right hand side of* (3.17) *converges uniformly on every compact set of* $\tilde{V}(A, B; R)$ *and defines a function* $f \in \mathscr{O}(\tilde{V}(A, B; R))$, *whose* (m, k)*-components* $S_{m,k}(f; \omega)$ *coincides with the given functions* $S_{m,k}(\omega)$.

PROOF. The restriction ρ: $\mathscr{O}(\tilde{B}(R)) \to \mathscr{O}(\tilde{S}^{n-1}(R))$ being a continuous linear mapping of the Fréchet space $\mathscr{O}(\tilde{B}(R))$ onto the Fréchet space $\mathscr{O}(\tilde{S}^{n-1}(R))$, ρ is a homomorphism. Therefore, for every R_1, $1<R_1<R$. There exists a compact set $K(R_1)$ of $\tilde{S}^{n-1}(R)$ and a constant $C(R_1) \geqslant 0$ such that, for every $g \in \mathscr{O}(\tilde{S}^{n-1}(R))$, we can find $G \in \mathscr{O}(\tilde{B}(R))$ for which

$$G(z)=g(z) \quad \text{for} \quad z \in \tilde{S}^{n-1}(R)$$

and

$$\sup\{|G(z)|; z \in R_1\Sigma^n\} \leqslant C(R_1)\sup\{|g(z)|; z \in K(R_1)\} . \tag{3.18}$$

Now given $f \in \mathscr{O}(\tilde{V}(A, B; R))$, we define $f_m(z) \in \mathscr{O}(\tilde{V}(\infty, \infty; R))$ for every $m \in Z$ as in Lemma 3.3. Then by the above remark, we can find entire functions $F_m \in \mathscr{O}(\tilde{B}(R))$ such that

$$F_m(z)=f_m(z) \quad \text{for} \quad z \in \tilde{S}^{n-1}(R) \tag{3.19}$$

and that

$$\sup\{|F_m(z)|; z \in R_1\Sigma^n\} \leqslant C(R_1)\sup\{|f_m(z)|; z \in K(R_1)\} . \tag{3.20}$$

Put, for $j \in Z_+$ and $0<R_1<R$,

$$F_{m,j}(z)=\frac{1}{2\pi i}\oint_{|t|=R_1} F_m(tz)t^{-(j+1)}dt . \tag{3.21}$$

Then $F_{m,j}(z)$ is a homogeneous polynomial of degree j and we have

$$F_m(z)=\sum_{j=0}^{\infty} F_{m,j}(z) \quad \text{for} \quad z \in \tilde{B}(R) , \tag{3.22}$$

the convergence being uniform on every compact set of $\tilde{B}(R)$. We have by (3.21), for every R_1, $0<R_1<R$, and every $\tau \in S^{n-1}$,

$$|F_{m,j}(\tau)| \leqslant R_1^{-j} \sup\{|F_m(z)|; z \in R_1\Sigma^n\} . \tag{3.23}$$

Now by (3.19), (3.22) and the orthogonality of spherical harmonics, we get

$$\begin{aligned} S_{m,k}(f;\omega) &= \frac{N(n,k)}{\Omega_n}\int_{S^{n-1}} f_m(\tau)P_k(n;\langle\omega,\tau\rangle)d\Omega_n(\tau) \\ &= \frac{N(n,k)}{\Omega_n}\int_{S^{n-1}} F_m(\tau)P_k(n;\langle\omega,\tau\rangle)d\Omega_n(\tau) \\ &= \frac{N(n,k)}{\Omega_n}\sum_{j\geq k}\int_{S^{n-1}} F_{m,j}(\tau)P_k(n;\langle\omega,\tau\rangle)d\Omega_n(\tau) . \end{aligned}$$

Therefore, for every R_1, $1<R_1<R$, we have by (3.23) and (3.20)

$$\begin{aligned} |S_{m,k}(f;\omega)| &\leqslant N(n,k)\sum_{j\geq k} R_1^{-j}\sup\{|F_m(z)|;\, z\in R_1\Sigma^n\} \\ &\leqslant N(n,k)R_1^{-k}(1-R_1^{-1})^{-1}C(R_1)\sup\{|f_m(z)|;\, z\in K(R_1)\} . \end{aligned}$$

Applying Lemma 3.3, (iii) for the compact set $K(R_1)$ of $\tilde{S}^{n-1}(R)$, we get, for every r, $B^{-1}<r<A$,

$$|S_{m,k}(f;\omega)|\leqslant N(n,k)R_1^{-k}C'(R_1)r^{-m}\sup\{|f(z)|;\, z\in\tilde{K}(r)\} ,$$

where $C'(R_1)=(1-R_1^{-1})^{-1}C(R_1)$ is a constant independent of k and m and $\tilde{K}(r)=r\sigma(K(R_1))$ is a compact set of $\tilde{V}(A,B;R)$. Therefore we have, for every $1<A_1<A$ and $1<B_1<B$,

$$(3.24)\qquad |S_{m,k}(f;\omega)|\leqslant N(n,k)R_1^{-k}A_1^{-m}C'(R_1)\sup\{|f(z)|;\, z\in\tilde{K}\}$$

for $m\geqslant 0$ and

$$(3.24')\qquad |S_{m,k}(f;\omega)|\leqslant N(n,k)R_1^{-k}B_1^{-|m|}C'(R_1)\sup\{|f(z)|;\, z\in\tilde{K}\}$$

for $m<0$, where $\tilde{K}=\cup\{\tilde{K}(r);\, B_1^{-1}\leqslant r\leqslant A_1\}$ is a compact set of $\tilde{V}(A,B;R)$. As we have, by (1.21), $N(n,k)=O(k^{n-2})$, we get (3.15) and (3.16).

Now let us suppose that we have (3.15) and (3.16). Suppose $0<\theta<1$ and put $A_1=A\theta^2$, $B_1=B\theta^2$ and $R_1=R\theta^2$. Then, from Lemma 1.4, we can conclude that there exists $M\geqslant 0$ such that $|m|+k\geqslant M$ implies

$$A_1^m R_1^k\|S_{m,k}(\omega)\|_{L^\infty(S^{n-1})}\leqslant\theta^{m+k}\quad\text{for}\quad m\geqslant 0$$

and

$$B_1^{|m|}R_1^k\|S_{m,k}(\omega)\|_{L^\infty(S^{n-1})}\leqslant\theta^{|m|+k}\quad\text{for}\quad m<0 .$$

Let us denote by $\tilde{S}_{m,k}(z)$ the harmonic homogeneous polynomial of degree k such that $\tilde{S}_{m,k}(\omega)=S_{m,k}(\omega)$ for $\omega\in S^{n-1}$. Then by Corollary 1 to Proposition 1.1, we have, for $|m|+k\geqslant M$,

$$|\tilde{S}_{m,k}(z)|\leqslant A_1^{-m}R_1^{-k}L(z)^k\theta^{m+k}\quad\text{for}\quad m\geqslant 0$$

and

$$|\tilde{S}_{m,k}(z)|\leqslant B_1^{-|m|}R_1^{-k}L(z)^k\theta^{|m|+k} \quad \text{for} \quad m<0 .$$

If $|m|+k\geqslant M$ and $m\geqslant 0$, we have, for $z\in\tilde{V}[A_1, B_1; R_1]$,

$$|(\sqrt{z^2})^{m-k}\tilde{S}_{m,k}(z)|\leqslant A_1^m|\sqrt{z^2}|^{-k}A_1^{-m}R_1^{-k}L(z)^k\theta^{m+k}\leqslant\theta^{m+k} .$$

If $|m|+k\geqslant M$ and $m<0$, we have, for $z\in\tilde{V}[A_1, B_1; R_1]$,

$$|(\sqrt{z^2})^{m-k}\tilde{S}_{m,k}(z)|\leqslant B_1^{|m|}|\sqrt{z^2}|^{-k}B_1^{-|m|}R_1^{-k}L(z)^k\theta^{|m|+k}\leqslant\theta^{|m|+k} .$$

Therefore the series (3.17) converges uniformly on the set $\tilde{V}[A_1, B_1; R_1]$. θ being arbitrary with $0<\theta<1$, the series (3.17) defines a holomorphic function on the open set $\tilde{V}(A, B; R)$. If $S_{m,k}(\omega)$ is the (m, k)-components of the function $f\in\mathcal{O}(\tilde{V}(A, B; R))$, the formula (3.17) results from (3.8) and (3.13).

Conversely, if we are given a sequence $\{S_{m,k}; (m, k)\in\Lambda\}$, satisfying the conditions (3.15) and (3.16), then we can define a function $f(z)$ by the formula (3.17). By the uniform convergence of the series, we can easily show that the (m, k)-components of the function $f(z)$ thus defined coincide with the given functions $S_{m,k}(\omega)$. q.e.d.

COROLLARY 1. *Suppose $A, B>0$ with $AB\geqslant 1$ and $R\geqslant 1$. Then f belongs to $\mathcal{O}(\tilde{V}[A, B; R])$, if and only if*

$$\limsup_{\substack{m+k\to\infty\\ m\geq 0}} [A^m R^k \| S_{m,k}(f; \omega)\|_{L^2(S^{n-1})}]^{1/(m+k)}<1 \tag{3.15'}$$

and

$$\limsup_{\substack{|m|+k\to\infty\\ m<0}} [B^{|m|} R^k \| S_{m,k}(f; \omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)}<1 . \tag{3.16'}$$

PROOF. As we have

$$\mathcal{O}(\tilde{V}[A, B; R])=\liminf_{a>1} \mathcal{O}(\tilde{V}(aA, aB; aR)) ,$$

Corollary results from the theorem. q.e.d.

It is worth while to mention the following special cases.

COROLLARY 2. (i) *For $f(z)\in\mathcal{O}(\tilde{V})$, if and only if*

$$\limsup_{|m|+k\to\infty} [\| S_{m,k}(f; \omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)}=0 . \tag{3.25}$$

(ii) *For $f\in\mathscr{A}(\Sigma^n)$, if and only if*

$$\limsup_{|m|+k\to\infty} [\| S_{m,k}(f; \omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)}<1 . \tag{3.26}$$

PROOF. We have

$$\mathcal{O}(\tilde{V}) = \lim_{R>1} \operatorname{proj} \mathcal{O}(\tilde{V}(R, R; R)) \tag{3.27}$$

and, by Lemma 1.5,

$$\mathscr{A}(\Sigma^n) = \lim_{R>1} \operatorname{ind} \mathcal{O}(\tilde{V}(R, R; R)) . \tag{3.28}$$

But, by Theorem 3.1, for $f \in \mathcal{O}(\tilde{V}(R, R; R))$ if and only if

$$\limsup_{|m|+k\to\infty} [\| S_{m,k}(f; \omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)} \leqslant R^{-1} .$$

Therefore we have (3.25) and (3.26). q.e.d.

REMARK. With a holomorphic function $f(z) \in \mathcal{O}(\tilde{V}(A, B; R))$, we associated a holomorphic function $F(\alpha', z')$ on $K^0_{A,B} \times \tilde{S}^{n-1}(R)$, for which $F(-\alpha', -z') = F(\alpha', z')$. If the function F can be continued analytically to the set $\{\alpha' \in C; |\alpha'| < A\} \times \tilde{S}^{n-1}(R)$, then the (m, k)-components $S_{m,k}(f; \omega)$ of f vanish for $m < 0$. If the function F can be analytically continued to the set $(\{\alpha' \in C; B^{-1} < |\alpha'|\} \cup \{\infty\}) \times \tilde{S}^{n-1}(R)$, then $S_{m,k}(f; \omega)$ vanish for $m > 0$.

Now recall $\tilde{B}(r)$ is the Lie ball of radius $r > 0$:

$$\tilde{B}(r) = r\tilde{B} = \{z \in C^n; L(z) < r\} . \tag{3.29}$$

LEMMA 3.5. *Let $f \in \mathcal{O}(\tilde{B}(r))$. Define f_m as in Lemma 3.3. Then $f_m(z)$ is a homogeneous polynomial of degree m for $m \geqslant 0$ and $f_m = 0$ for $m < 0$. We have*

$$\limsup_{m\to\infty} [\| f_m(\omega)\|_{L^\infty(S^{n-1})}]^{1/m} \leqslant r^{-1} \tag{3.30}$$

and

$$f(z) = \sum_{m=0}^{\infty} f_m(z) \quad \textit{for} \quad z \in \tilde{B}(r) , \tag{3.31}$$

the convergence being uniform on every compact subset of $\tilde{B}(r)$.

Conversely if we have a sequence $\{f_m(z); m \in Z_+\}$ of homogeneous polynomials $f_m(z)$ of degree m and if we have the condition (3.30), *then the right hand side of* (3.31) *converges uniformly on every compact subset of $\tilde{B}(r)$ and is holomorphic there.*

PROOF. The function $f \in \mathcal{O}(\tilde{B}(r))$ can be expanded into the Taylor series in a neighborhood of 0. Therefore f_m is a homogeneous polynomial of degree m for $m \geqslant 0$. Other statements results from the integral formula (3.10) of $f_m(z)$ and Corollary to Proposition 1.1. q.e.d.

THEOREM 3.2. *Let* $f(z) \in \mathcal{O}(\tilde{B}(r))$. *Define* f_m *and* $\tilde{S}_{m,k}(f; z)$ *as in Lemmas* 3.3 *and* 3.4. *Let us denote* $\tilde{S}_{m,k}(z) = \tilde{S}_{m,k}(f; z)$. (i) *We have*

$$\tilde{S}_{m,k}(z) = 0 \quad \textit{for} \quad k > m \tag{3.32}$$

and

$$\limsup_{m+k\to\infty} [r^m \| S_{m,k}(\omega) \|_{L^2(S^{n-1})}]^{1/(m+k)} \leqslant 1 . \tag{3.33}$$

(ii) *Let us denote*

$$\Lambda_+ = \{(m, k) \in \Lambda; k \leqslant m\} .$$

Then we have

$$f(z) = \sum_{(m,k) \in \Lambda_+} (\sqrt{z^2})^{m-k} \tilde{S}_{m,k}(f; z) \tag{3.34}$$

for $z \in \tilde{B}(r)$, *the convergence being uniform on every compact set of* $\tilde{B}(r)$.

Conversely, if we are given a sequence of spherical harmonic functions $\{S_{m,k}(\omega); (m, k) \in \Lambda\}$ *satisfying* (3.32) *and* (3.33), *then the right hand side of* (3.34) *converges to a holomorphic function* $f(z)$ *uniformly on every compact set of* $\tilde{B}(r)$ *and we have*

$$\tilde{S}_{m,k}(z) = \tilde{S}_{m,k}(f : z) \quad \textit{for} \quad (m, k) \in \Lambda .$$

PROOF. (3.32) results from the orthogonality of spherical harmonics. By Lemma 3.5, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon \geqslant 0$ such that

$$\| f_m(\omega) \|_{L^\infty(S^{n-1})} \leqslant C_\varepsilon r^{-m} (1+\varepsilon)^m .$$

Therefore, by (3.11), we have

$$| S_{m,k}(\omega) | \leqslant N(n, k) C_\varepsilon r^{-m} (1+\varepsilon)^m .$$

By Lemma 1.4 and (1.21), we get

$$\limsup_{m+k\to\infty} [r^m \| S_{m,k}(\omega) \|_{L^2(S^{n-1})}]^{1/(m+k)} \leqslant 1 + \varepsilon .$$

$\varepsilon > 0$ being arbitrary, we get (3.34).

Now suppose we have (3.32) and (3.33). Then, by Lemma 1.4, we can conclude from (3.33) that, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon \geqslant 0$ such that

$$r^m \| S_{m,k}(\omega) \|_{L^\infty(S^{n-1})} \leqslant C_\varepsilon (1+\varepsilon)^{m+k} .$$

If we have $m \geqslant k$, we have, by Corollary to Proposition 1.1,

$$\begin{aligned}(\sqrt{z^2})^{m-k} \tilde{S}_{m,k}(z) \\ &\leqslant L(z)^{m-k} L(z)^k C_\varepsilon (1+\varepsilon)^{m+k} r^{-m} \\ &\leqslant L(z)^m C_\varepsilon (1+\varepsilon)^{2m} r^{-m} .\end{aligned}$$

Therefore, the right hand side of (3.34) converges uniformly on the set

$$\{z \in C^n ; L(z) \leqslant r(1+2\varepsilon)^{-2}\} .$$

If $S_{m,k}(\omega) = S_{m,k}(f; \omega)$ for a function $f \in \mathcal{O}(\tilde{B}(r))$, we have, by Theorem 3.1, the identity (3.34).

Conversely, if we are given a sequence $\{S_{m,k}\}$ satisfying (3.32) and (3.33), we can define by the right hand side of (3.34) a function $f \in \mathcal{O}(\tilde{B}(r))$. It is clear that we have $\tilde{S}_{m,k}(f; z) = \tilde{S}_{m,k}(z)$, because (3.34) converges uniformly on the set $r_1 \Sigma^n$ for every r_1, $0 < r_1 < r$. q.e.d.

Let us denote by $\mathcal{O}_\Delta(\tilde{B}(r))$ the space of holomorphic functions f on $\tilde{B}(r)$ which satisfy the differential equation:

$$\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \cdots + \frac{\partial^2}{\partial z_n^2} \right) f(z) = 0 . \tag{3.35}$$

$\mathcal{O}_\Delta(\tilde{B}(r))$ is a closed subspace of $\mathcal{O}(\tilde{B}(r))$.

THEOREM 3.3. *Let $f(z) \in \mathcal{O}_\Delta(\tilde{B}(r))$ and define f_m and $\tilde{S}_{m,k}(f; z)$ as in Lemmas 3.3 and 3.4.*

(i) *f_m is a homogeneous harmonic polynomial of degree m for $m \geqslant 0$* and *$f_m = 0$ for $m < 0$.*

(ii) *$\tilde{S}_{m,k}(f; z) = f_m(z)$ for $k = m$ and $\tilde{S}_{m,k}(f; z) = 0$ for $k \neq m$.*

(iii) *$f(z) = \sum_{m=0}^\infty \tilde{S}_{m,m}(f; z)$ for $z \in \tilde{B}(r)$,*

the convergence being uniform on every compact subset of $\tilde{B}(r)$.

This theorem is due to Siciak [9]. See also Theorem 5.2, (i) in M. Morimoto [7].

§ 4. Spaces of functionals on Σ^n.

Let us denote by $\mathscr{D}'(\Sigma^n)$ the space of distributions on the Lie sphere Σ^n, i.e., $\mathscr{D}'(\Sigma^n)$ is the dual space of $C^\infty(\Sigma^n)$. $\mathscr{B}(\Sigma^n)$ denotes the space of hyperfunctions on Σ^n, i.e., $\mathscr{B}(\Sigma^n)$ is the dual space of $\mathscr{A}(\Sigma^n)$. The dual spaces of $\mathcal{O}(\tilde{V})$ and $\mathcal{O}(\tilde{V}(A, B; R))$ will be denoted by $\mathcal{O}'(\tilde{V})$ and $\mathcal{O}'(\tilde{V}(A, B; R))$ respectively.

LEMMA 4.1. *We have the following inclusions:*

$$\mathcal{O}'(\tilde{V}) \hookrightarrow \mathcal{O}'(\tilde{V}(A, B; R)) \hookrightarrow \mathscr{B}(\Sigma^n) \hookrightarrow \mathscr{D}'(\Sigma^n) \hookrightarrow L^2(\Sigma^n) , \tag{4.1}$$

where $A>1$, $B>1$ *and* $R>1$.

PROOF. Because of Lemma 3.2, we have only to take the dual sequence of the sequence (3.1). q.e.d.

Recall $(\ , \ )_{\Sigma^n}$ is the bilinear form defined by (2.2). For $f, g \in L^2(\Sigma^n)$, we have

$$\begin{aligned}(f(e^{i\theta}\omega), g(e^{i\theta}\omega))_{\Sigma^n} \\ = \sum_{(m,k)\in\Lambda} (S_{m,k}(f; \omega), S_{-m,k}(g; \omega))_{S^{n-1}} ,\end{aligned} \tag{4.2}$$

where $(\ , \ )_{S^{n-1}}$ is the bilinear form defined by the formula (1.25).

Suppose $T \in \mathscr{D}'(\Sigma^n)$ (resp. $T \in \mathscr{B}(\Sigma^n)$). We will write by (T, f) the value of the functional T at a testing function $f \in C^\infty(\Sigma^n)$ (resp. $f \in \mathscr{A}(\Sigma^n)$). By Theorem 2.2 (resp. Theorem 2.3), we have

$$\begin{aligned}(T, f) &= (T, \sum_{(m,k)\in\Lambda} e^{im\theta} S_{m,k}(f; \omega)) \\ &= \sum_{(m,k)\in\Lambda} (T, e^{im\theta} S_{m,k}(f; \omega)) .\end{aligned} \tag{4.3}$$

By (2.7), we have

$$e^{im\theta} S_{m,k}(f; \omega) = N(n, k)(f(e^{i\varphi}\tau), e^{im(\theta-\varphi)} P_k(n; \langle\omega, \tau\rangle))_{\Sigma^n} . \tag{4.4}$$

Let us define *the* $(-m, k)$*-component* of the functional T by the following formula:

$$S_{-m,k}(T; \tau) = N(n, k)(T(\theta, \omega), e^{im\theta} P_k(n; \langle\omega, \tau\rangle)) . \tag{4.5}$$

Then we have, by Theorem 2.1,

$$\begin{aligned}&(T, e^{im\theta} S_{m,k}(f; \omega)) \\ &= (f(e^{i\varphi}\tau), e^{-im\varphi} S_{-m,k}(T; \tau))_{\Sigma^n} \\ &= (\sum_{(m',k')\in\Lambda} e^{im'\varphi} S_{m',k'}(f; \tau), e^{-im\varphi} S_{-m,k}(T, \tau))_{\Sigma^n} \\ &= (S_{m,k}(f; \tau), S_{-m,k}(T; \tau))_{S^{n-1}} .\end{aligned}$$

Therefore we have proved the formula:

$$(T, f) = \sum_{(m,k)\in\Lambda} (S_{-m,k}(T; \omega), S_{m,k}(f; \omega))_{S^{n-1}} . \tag{4.6}$$

THEOREM 4.1. *If* $T \in \mathscr{D}'(\Sigma^n)$, *then the* (m, k)*-component of* T,

$S_{m,k}(T;\omega)$ *belongs to* $\mathscr{H}^k(S^{n-1})$ *and the sequence* $\{\|S_{m,k}(T;\omega)\|_{L^2(S^{n-1})}\}$ *is slowly increasing on the set* Λ, i.e., *there exist* $p\in \boldsymbol{Z}_+$ *and* $C\geqslant 0$ *such that*

$$\|S_{-m,k}(T;\omega)\|_{L^2(S^{n-1})}\leqslant C(|m|+k)^p \tag{4.7}$$

for every $(m, k)\in\Lambda$.

Conversely if we have a sequence $\{S_{-m,k}(\omega);(m, k)\in\Lambda\}$ *with* $S_{-m,k}(\omega)\in \mathscr{H}^k(S^{n-1})$ *and the sequence* $\{\|S_{-m,k}(\omega)\|_{L^2(S^{n-1})}\}$ *is slowly increasing on the set* Λ, *then the formula*

$$\begin{aligned}(T, f)&=\sum_{(m,k)\in\Lambda}(e^{-im\theta}S_{-m,k}(\omega),\ f(e^{i\theta}\omega))_{\Sigma^n}\\ &=\sum_{(m,k)\in\Lambda}(S_{-m,k}(\omega),\ S_{m,k}(f;\omega))_{S^{n-1}}\end{aligned} \tag{4.8}$$

defines a distribution $T\in\mathscr{D}'(\Sigma^n)$ *and we have* $S_{-m,k}(T;\omega)=S_{-m,k}(\omega)$ *for every* $(m, k)\in\Lambda$.

PROOF. The theorem is dual to Theorem 2.2 and the proof is similar to that of Theorem 3.1 in M. Morimoto [7]. q.e.d.

THEOREM 4.2. *If* $T\in\mathscr{B}(\Sigma^n)$, *then* $S_{-m,k}(T;\omega)$ *belongs to* $\mathscr{H}^k(S^{n-1})$. *If we put* $S_{-m,k}(\omega)=S_{-m,k}(T;\omega)$, *we have the following estimate*:

$$\limsup_{|m|+k\to\infty}[\|S_{-m,k}(\omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)}\leqslant 1\ . \tag{4.9}$$

Conversely, if we have a sequence $\{S_{-m,k}(\omega);(m, k)\in\Lambda\}$ *with* $S_{-m,k}(\omega)\in \mathscr{H}^k(S^{n-1})$ *satisfying* (4.9), *then the formula* (4.8) *defines a hyperfunction* $T\in\mathscr{B}(\Sigma^n)$ *and we have* $S_{-m,k}(T;\omega)=S_{-m,k}(\omega)$ *for every* $(m, k)\in\Lambda$.

REMARK. The theorem is dual to Theorem 2.3 and the proof is similar to that of Theorem 1.8 in Hashizume-Minemura-Okamoto [3]. We will give a new proof in the following section (Corollary 2 (ii) to Theorem 5.1).

§5. Spaces of analytic functionals.

Suppose T is an analytic functional belonging to $\mathscr{O}'(\tilde{V}(A, B; R))$ and $f(z)$ is a holomorphic function on $\tilde{V}(A, B; R)$. Then, by Theorem 3.1, $f(z)$ can be expanded as

$$f(z)=\sum_{(m,k)\in\Lambda}(\sqrt{z^2})^{m-k}\tilde{S}_{m,k}(f;z)\ , \tag{5.1}$$

where the convergence is uniform on every compact set of $\tilde{V}(A, B; R)$.

Therefore by the continuity of T, we have

$$(5.2) \qquad (T, f) = \sum_{(m,k)\in\Lambda} (T_z, (\sqrt{z^2})^{m-k}\tilde{S}_{m,k}(f; z)) .$$

As we have, by (3.11) and (2.10),

$$(5.3) \quad \tilde{S}_{m,k}(f; z)$$
$$= (\sqrt{z^2})^k \frac{N(n, k)}{\pi\Omega_n} \int_0^\pi \int_{S^{n-1}} f(e^{i\theta}\tau)e^{-im\theta}P_k(n; \langle z/\sqrt{z^2}, \tau\rangle)d\theta d\Omega_n(\tau) ,$$

we get

$$(5.4) \qquad (T, f) = \sum_{(m,k)\in\Lambda} (e^{-im\theta}S_{-m,k}(T; \tau), f(e^{i\theta}\tau))_{\Sigma^n}$$
$$= \sum_{(m,k)\in\Lambda} (S_{-m,k}(T; \tau), S_{m,k}(f; \tau))_{S^{n-1}} ,$$

where we put

$$(5.5) \qquad S_{-m,k}(T; \tau) = N(n, k)(T_z, (\sqrt{z^2})^m P_k(n; \langle z/\sqrt{z^2}, \tau\rangle)) ,$$

which we call *the* $(-m, k)$*-component* of T.

THEOREM 5.1. *For an analytic functional* $T \in \mathcal{O}'(\tilde{V}(A, B; R))$, *the* $(-m, k)$*-component* $S_{-m,k}(T; \omega)$ *is a spherical harmonic function of degree* k *and we have*

$$(5.6) \qquad \limsup_{\substack{m+k\to\infty \\ m\geq 0}} [A^{-m}R^{-k} \| S_{-m,k}(\omega)\|_{L^2(S^{n-1})}]^{1/(m+k)} < 1$$

and

$$(5.7) \qquad \limsup_{\substack{|m|+k\to\infty \\ m<0}} [B^{-|m|}R^{-k} \| S_{-m,k}(\omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)} < 1 ,$$

where we put $S_{-m,k}(\omega) = S_{-m,k}(T; \omega)$.

Conversely, if a sequence $\{S_{-m,k}(\omega); (m, k) \in \Lambda\}$ *of spherical harmonics* $S_{-m,k}$ *of degree* k *satisfies the conditions* (5.6) *and* (5.7), *we can define an analytic functional* $T \in \mathcal{O}'(\tilde{V}(A, B; R))$ *by the formula*

$$(5.8) \qquad (T, f) = \sum_{(m,k)\in\Lambda} (S_{-m,k}(\omega), S_{m,k}(f; \omega))_{S^{n-1}}$$

for $f \in \mathcal{O}(\tilde{V}(A, B; R))$. *The* $(-m, k)$*-components of the functional* T *coincide with the given spherical harmonics* $S_{-m,k}$.

PROOF. By the coninuity of the analytic functional T, there exist A_1, B_1 and R_1 with $0<A_1<A$, $0<B_1<B$, $1\leqslant A_1B_1$ and $1<R_1<R$, and a constant $C\geqslant 0$ such that

$$(5.9)\qquad |(T, f)| \leqslant C \sup\{|f(z)|;\ z \in \tilde{V}[A_1, B_1; R_1]\}\ .$$

As $(\sqrt{z^2})^k P_k(n; \langle z/\sqrt{z^2}, \tau\rangle)$ is a harmonic homogeneous polynomial of degree k, we have, by Corollaries to Proposition 1.1,

$$|\sqrt{z^2}|^k |P_k(n; \langle z/\sqrt{z^2}, \tau\rangle)| \leqslant L(z)^k \sup\{|P_k(n; \langle \omega, \tau\rangle)|;\ \omega \in S^{n-1}\} = L(z)^k\ .$$

Therefore we have

$$\begin{aligned}
&\sup\{|\sqrt{z^2}|^m\, |P_k(n; \langle z/\sqrt{z^2}, \tau\rangle)|;\ z \in \tilde{V}[A_1, B_1; R_1]\}\\
&\quad = \sup\{|\sqrt{z^2}|^{m-k} |\sqrt{z^2}|^k\, |P_k(n; \langle z/\sqrt{z^2}, \tau\rangle)|;\ z \in \tilde{V}[A_1, B_1; R_1]\}\\
&\quad \leqslant \sup\{|\sqrt{z^2}|^{m-k} L(z)^k;\ z \in \tilde{V}[A_1, B_1; R_1]\}\\
&\quad \leqslant \sup\{|\sqrt{z^2}|^m R_1^k;\ B_1^{-2} \leqslant |z^2| \leqslant A_1^2\}\\
&\quad = \begin{cases} A_1^m R_1^k & \text{for}\quad m \geqslant 0\\ B_1^{|m|} R_1^k & \text{for}\quad m < 0\ . \end{cases}
\end{aligned}$$

We get, by (5.5) and (5.9),

$$|S_{-m,k}(T; \tau)| \leqslant C\, N(n, k) A_1^m R_1^k \qquad \text{for}\quad m \geqslant 0$$

and

$$|S_{-m,k}(T; \tau)| \leqslant C\, N(n, k) B_1^{|m|} R_1^k \qquad \text{for}\quad m < 0\ ,$$

from which result (5.6) and (5.7) thanks to Lemma 1.4.

Conversely, suppose we have a sequence $\{S_{-m,k}(\omega)\}$. Denote by μ_0 the maximum of the left hand sides of (5.6) and (5.7). We have $\mu_0 < 1$ by the assumption. For every μ with $\mu_0 < \mu < 1$, there exists a constant $C_\mu \geqslant 0$ such that

$$\|S_{-m,k}(\omega)\|_{L^2(S^{n-1})} \leqslant C_\mu \mu^{m+k} A^m R^k \qquad \text{for}\quad m \geqslant 0$$

and

$$\|S_{-m,k}(\omega)\|_{L^2(S^{n-1})} \leqslant C_\mu \mu^{|m|+k} B^{|m|} R^k \qquad \text{for}\quad m < 0\ .$$

Suppose a holomorphic function $f \in \mathcal{O}(\tilde{V}(A, B; R))$ is given. For every A_1, B_1 and R_1 with $0 < A_1 < A$, $0 < B_1 < B$, $1 < A_1 B_1$ and $1 < R_1 < R$, we have, by Lemma 1.4, (3.24) and (3.24′),

$$\begin{aligned}
\|S_{m,k}(f; \omega)\|_{L^2(S^{n-1})} &\leqslant \|S_{m,k}(f; \omega)\|_{L^\infty(S^{n-1})}\\
&\leqslant N(n, k) R_1^{-k} A_1^{-m} C'(R_1) \sup\{|f(z)|;\ z \in \tilde{K}\}
\end{aligned}$$

for $m \geqslant 0$ and

$$\begin{aligned}
\|S_{m,k}(f; \omega)\|_{L^2(S^{n-1})} &\leqslant \|S_{m,k}(f; \omega)\|_{L^\infty(S^{n-1})}\\
&\leqslant N(n, k) R_1^{-k} B_1^{-|m|} C'(R_1) \sup\{|f(z)|;\ z \in \tilde{K}\}
\end{aligned}$$

for $m<0$, where we recall $C'(R_1)=(1-R_1^{-1})^{-1}C(R_1)$ is a constant and $\tilde{K}$ is a compact set of $\tilde{V}(A, B; R)$. Now we have

$$(5.10)\quad \begin{aligned}|(S_{-m,k}(\omega), S_{m,k}(f;\omega))_{S^{n-1}}| &\leqslant \|S_{-m,k}(\omega)\|_{L^2(S^{n-1})}\|S_{m,k}(f;\omega)\|_{L^2(S^{n-1})}\\ &\leqslant C_\mu \mu^{m+k}A^m R^k N(n,k) R_1^{-k} A_1^{-m} C'(R_1)\sup\{|f(z)|; z\in\tilde{K}\}\end{aligned}$$

for $m\geqslant 0$ and

$$(5.11)\quad \begin{aligned}&|(S_{-m,k}(\omega), S_{m,k}(f;\omega))_{S^{n-1}}|\\ &\leqslant C_\mu \mu^{|m|+k}B^{|m|}R^k N(n,k) R_1^{-k} B_1^{-|m|} C'(R_1)\sup\{|f(z)|; z\in\tilde{K}\}\end{aligned}$$

for $m<0$. If we fix A_1, B_1 and R_1 for which $\mu A<A_1<A$, $\mu B<B_1<B$, $A_1B_1\geqslant 1$ and $\max\{1, \mu R\}<R_1<R$, then by (5.10) and (5.11) and thanks to (1.21), we can find a constant $C\geqslant 0$ such that

$$\sum_{(m,k)\in\Lambda}|(S_{-m,k}(\omega), S_{m,k}(f;\omega))_{S^{n-1}}|\leqslant C\sup\{|f(z)|; z\in\tilde{K}\},$$

which means that we can define an analytic functional $T\in\mathcal{O}'(\tilde{V}(A, B; R))$ by (5.8). It is clear by the orthogonality of spherical harmonics, that the $(-m, k)$-components of the functional T defined above coincide with the given $S_{-m,k}$. q.e.d.

COROLLARY 1. *Suppose $A, B>0$ satisfy $AB\geqslant 1$ and $R\geqslant 1$. Then an analytic functional T belongs to $\mathcal{O}'(\tilde{V}[A, B; R])$ if and only if*

$$(5.12)\quad \limsup_{\substack{m+k\to\infty\\ m\geqq 0}}[A^{-m}R^{-k}\|S_{-m,k}(T;\omega)\|_{L^2(S^{n-1})}]^{1/(m+k)}\leqslant 1$$

and

$$(5.13)\quad \limsup_{\substack{|m|+k\to\infty\\ m<0}}[B^{-|m|}R^{-k}\|S_{-m,k}(T;\omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)}\leqslant 1\,.$$

The following special cases are worth while to mention.

COROLLARY 2. (i) *T belongs to $\mathcal{O}'(\tilde{V})$, if and only if*

$$(5.14)\quad \limsup_{|m|+k\to\infty}[\|S_{-m,k}(T;\omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)}<\infty\,.$$

(ii) *T belongs to $\mathscr{B}(\Sigma^n)$, if and only if*

$$(5.15)\quad \limsup_{|m|+k\to\infty}[\|S_{-m,k}(T;\omega)\|_{L^2(S^{n-1})}]^{1/(|m|+k)}\leqslant 1\,.$$

Now suppose a holomorphic function $f(z)$ is given on the Lie ball $\tilde{B}$. Then, by Theorem 3.2, $f(z)$ can be expanded as follows:

$$f(z)=\sum_{(m,k)\in\Lambda_+}(\sqrt{z^2})^{m-k}\tilde{S}_{m,k}(f;z) \tag{5.16}$$

for $z\in\tilde{B}$, the convergence being uniform on every compact set of $\tilde{B}$. Thanks to (3.33) with $r=1$, we can define, by Corollary to Theorem 5.1, a hyperfunction $T(e^{i\theta}\omega)$ on the Lie sphere Σ^n as follows:

$$T(e^{i\theta}\omega)=\sum_{(m,k)\in\Lambda_+}e^{im\theta}S_{m,k}(f;\omega)\,. \tag{5.17}$$

This hyperfunction will be called *the trace* of the holomorphic function $f\in\mathcal{O}(\tilde{B})$ on the Lie sphere Σ^n.

Suppose $g\in\mathscr{A}(\Sigma^n)$. Then g is a holomorphic function in a complex neighborhood $\tilde{V}(A, B; R)$ of Σ^n. Consider the integral

$$B_r(f;g)=\frac{1}{2\pi i}\oint_{|\alpha|=r}\int_{S^{n-1}}f(\alpha\omega)g(\alpha\omega)\frac{d\alpha}{\alpha}d\Omega_n(\omega)\,. \tag{5.18}$$

$B_r(f, g)$ is defined for r with $B^{-1}<r<1$ and is independent of such r. We will write $B(f, g)=B_r(f, g)$. Then it is clear that $g\mapsto B(f, g)$ is a continuous linear functional on $\mathscr{A}(\Sigma^n)$ and this functional is, by the definition, the trace T of $f\in\mathcal{O}(\tilde{B})$:

$$T\colon g\longmapsto B(f, g)\,.$$

THEOREM 5.2. *For every holomorphic function $f\in\mathcal{O}(\tilde{B})$, we can define the trace $T(e^{i\theta}\omega)\in\mathscr{B}(\Sigma^n)$. The $(-m, k)$-components $S_{-m,k}(T;\omega)$ vanish for $-m<k$.*

Conversely if we have a hyperfunction $T\in\mathscr{B}(\Sigma^n)$ and its $(-m, k)$-components $S_{-m,k}(T;\omega)$ vanish for $-m<k$, then there exists a holomorphic function $\tilde{f}(z)\in\mathcal{O}(\tilde{B})$, the trace of which coincides with the given hyperfunction T. The holomorphic function $\tilde{f}(z)$ is represented by the Cauchy-Hua formula:

$$\tilde{f}(z)=(T(e^{i\theta}\omega), ((\omega-e^{i\theta}z)^2)^{-n/2}) \tag{5.19}$$

for $z\in\tilde{B}$, where $(\ ,\ )$ is the canonical bilinear form on $\mathscr{B}(\Sigma^n)\times\mathscr{A}(\Sigma^n)$.

PROOF. We prove the second part of the theorem. Expand the hyperfunction $T(e^{i\varphi}\tau)$:

$$T(e^{i\varphi}\tau)=\sum_{(-m,k)\in\Lambda_+}e^{-im\varphi}S_{-m,k}(T;\tau)\,, \tag{5.20}$$

where the $(-m, k)$-component $S_{-m,k}(T;\tau)$ is defined as follow (c.f.(4.5)):

$$S_{-m,k}(T;\tau)=N(n, k)(T(e^{i\theta}\omega), e^{im\theta}P_k(n;\langle\tau,\omega\rangle))\,. \tag{5.21}$$

The holomorphic function $\tilde{f}(z)$ is defined as follows:

$$\tilde{f}(z)=\sum_{(m,k)\in\Lambda_+}(\sqrt{z^2})^{m-k}\tilde{S}_{m,k}(T;z)\ , \tag{5.22}$$

where $\tilde{S}_{m,k}(T;z)$ is the harmonic homogeneous polynomial of degree k such that $\tilde{S}_{m,k}(T;\tau)=S_{m,k}(T;\tau)$ for $\tau\in S^{n-1}$. By Theorem 3.2 and Theorem 4.2, (5.22) converges uniformly on every compact set of $\tilde{B}$ and define a holomorphic function $\tilde{f}$ there. It is known that the polynomial $\tilde{S}_{m,k}(f;z)$ is given by the following formula:

$$\tilde{S}_{m,k}(T;z)=N(n,k)(\sqrt{z^2})^k(T(e^{i\theta}\omega),\ e^{-im\theta}P_k(n;\langle z/\sqrt{z^2},\omega\rangle))\ . \tag{5.23}$$

Therefore we have

$$\tilde{f}(z)=\sum_{(m,k)\in\Lambda_+}N(n,k)(\sqrt{z^2})^m(T(e^{i\theta}\omega),\ e^{-im\theta}P_k(n;\langle z/\sqrt{z^2},\omega\rangle))\ . \tag{5.24}$$

Let us consider the following series

$$\begin{aligned}&\sum_{(m,k)\in\Lambda_+}(\sqrt{z^2})^m N(n,k)e^{-im\theta}P_k(n;\langle z/\sqrt{z^2},\omega\rangle)\\&=\sum_{m=0}^{\infty}\sum_{l=0}^{[m/2]}(\sqrt{z^2})^m N(n,m-2l)e^{-im\theta}P_{m-2l}(n;\langle z/\sqrt{z^2},\omega\rangle)\ .\end{aligned} \tag{5.25}$$

Recall first the following classical formula for the Gegenbauer polynomials: for $\nu>\lambda$,

$$C_m^{\nu}(t)=\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{l=0}^{[m/2]}c_l C_{m-2l}^{\lambda}(t)\ , \tag{5.26}$$

where

$$c_l=\frac{(m-2l+\lambda)\Gamma(l+\nu-\lambda)\Gamma(\nu+m-l)}{l!\,\Gamma(\nu-\lambda)\Gamma(m-l+\lambda+1)}\ . \tag{5.27}$$

(See Hua [5], p. 141.) Putting $\nu=n/2$, $\lambda=(n-2)/2$, we have the following formula:

$$C_m^{n/2}(t)=\sum_{l=0}^{[m/2]}\frac{(2m-4l+n-2)}{n-2}C_{m-2l}^{(n-2)/2}(t)\ . \tag{5.28}$$

On the other hand, we have, by (1.21) and (1.23),

$$N(n,k)P_k(n;t)=\frac{2k+n-2}{n-2}C_k^{(n-2)/2}(t)\ . \tag{5.29}$$

Therefore we have

$$\sum_{l=0}^{[m/2]} N(n,\ m-2l)P_{m-2l}(n;\ t) = C_m^{n/2}(t)\ .$$

Now by (5.29) and Corollary 2 to Proposition 1.1,

$$(\sqrt{z^2}\,)^m(\sqrt{w^2})^m C_m^{n/2}(\langle z/\sqrt{z^2}\,,\ w/\sqrt{w^2}\rangle)$$

is a polynomial in z and w and we have the following estimate:

$$|(\sqrt{z^2}\,)^m(\sqrt{w^2})^m C_m^{n/2}(\langle z/\sqrt{z^2}\,,\ w/\sqrt{w^2}\rangle)| \\ \leqslant L(z)^m L(w)^m n(2m+n)^{-1}N(n+2,\ m)\ .$$

Therefore the series

$$\sum_{m=0}^{\infty} (\sqrt{z^2}\,)^m\ (\sqrt{w^2})^{-m}\ C_m^{n/2}(\langle z/\sqrt{z^2}\,,\ w/\ \sqrt{w^2}\rangle)$$

is uniformly and absolutely convergent for $L(z)\leqslant r$, $|\sqrt{w^2}|\geqslant B^{-1}$ and $L(w)\leqslant R|\sqrt{w^2}|$, provided $rBR<1$.

On the other hand we have the generating formula of the Gegenbauer polynomials:

$$\sum_{m=0}^{\infty} s^m C_m^{\lambda}(t) = (1-2st+s^2)^{-\lambda}\ . \tag{5.30}$$

Therefore we have, for $L(z)\leqslant r$, $|\sqrt{w^2}|\geqslant B^{-1}$ and $L(w)\leqslant R|\sqrt{w^2}|$,

$$\begin{aligned} &\sum_{m=0}^{\infty} (\sqrt{z^2}\,)^m\ (\sqrt{w^2})^{-m}\ C_m^{n/2}(\langle z/\sqrt{z^2}\,,\ w/\ \sqrt{w^2}\rangle) \\ &\quad = (1-2\sqrt{z^2}\ /\sqrt{w^2}\langle z/\sqrt{z^2}\,,\ w/\sqrt{w^2}\rangle + (\sqrt{z^2}\ /\sqrt{w^2})^2)^{-n/2} \\ &\quad = (\langle w/\sqrt{w^2} - z/\sqrt{w^2},\ w/\sqrt{w^2} - z/\sqrt{w^2}\rangle)^{-n/2}\ . \end{aligned} \tag{5.31}$$

We put $w = e^{i\theta}\omega$, $\theta\in R$, $\omega\in S^{n-1}$ and get

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{l=0}^{[m/2]} (\sqrt{z^2})^m N(n,\ m-2l)e^{-im\theta}P_{m-2l}(n;\ \langle z/\sqrt{z^2},\ \omega\rangle) \\ &\quad = \sum_{m=0}^{\infty} (\sqrt{z^2}\,)^m e^{-im\theta}\ C_m^{n/2}(\langle z/\sqrt{z^2}\,,\ \omega\rangle) \\ &\quad = (\langle \omega - e^{-i\theta}z,\ \omega - e^{-i\theta}z\rangle)^{-n/2}\ . \end{aligned} \tag{5.32}$$

If we fix $r<1$, then we can find $A>1$, $B>1$ and $R>1$ such that $rBR<1$. $\tilde{V}(A,\ B;\ R)$ is a complex neighborhood of the Lie sphere Σ^n. The series (5.31) is uniformly and absolutely convergent for $z\in\tilde{B}(r)$ and $w\in\tilde{V}(A,\ B;\ R)$, which implies that, for fixed z with $L(z)<1$, the series (5.32) converges in the topology of $\mathscr{A}(\Sigma^n)$. As T is a continuous linear functional on $\mathscr{A}(\Sigma^n)$, we can conclude from (5.24), (5.25) and (5.32) the Cauchy-Hua integral formula (5.19). q.e.d.

Recall $\tilde{B}[1]$ is the closed Lie ball:

$$\tilde{B}[1]=\{z\in C^n;\ L(z)\leqslant 1\}\ . \tag{5.33}$$

The trace operator ρ: $\mathscr{O}(\tilde{B}[1])\to\mathscr{A}(\Sigma^n)$ being injective, we can consider $\mathscr{O}(\tilde{B}[1])$ as a subspace of $\mathscr{A}(\Sigma^n)$. By Theorems 2.3 and 3.3, the following lemma is clear.

LEMMA 5.1. *Suppose $f\in\mathscr{A}(\Sigma^n)$. For f to belong to the subspace $\rho\mathscr{O}(\tilde{B}[1])$, it is necessary and sufficient that f satisfies the following condition*:

$$S_{m,k}(f;\omega)=0 \quad \textit{for} \quad m<k\ . \tag{5.34}$$

Because the Lie sphere Σ^n is the Šilov boundary of $\tilde{B}$ (Proposition 1.1), $\rho\mathscr{O}(\tilde{B}[1])$ is a closed subspace of $\mathscr{A}(\Sigma^n)$. With a real analytic function $f(e^{i\theta}\omega)\in\mathscr{A}(\Sigma^n)$, we associate the function $\tilde{f}(z)=\gamma(f)(z)$, which is defined by the Cauchy-Hua integral formula:

$$\tilde{f}(z)=\gamma(f)(z)=(f(e^{i\theta}\omega),\ ((\omega-e^{i\theta}z)^2)^{-n/2})_{\Sigma^n}\ . \tag{5.19'}$$

Then the mapping γ is the left inverse of the trace operator ρ, i.e., $\gamma\circ\rho=\mathrm{id}$:

$$\mathscr{O}(\tilde{B}[1]) \underset{\gamma}{\overset{\rho}{\rightleftarrows}} \mathscr{A}(\Sigma^n)\ . \tag{5.35}$$

Take the dual of (5.35):

$$\mathscr{O}'(\tilde{B}[1]) \underset{\gamma^*}{\overset{\rho^*}{\leftrightarrows}} \mathscr{B}(\Sigma^n)\ . \tag{5.36}$$

The dual mapping γ^* being injective, we can consider, by the mapping γ^*, the dual space $\mathscr{O}'(\tilde{B}[1])$ of $\mathscr{O}(\tilde{B}[1])$ as a subspace of $\mathscr{B}(\Sigma^n)$.

For an analytic functional $T\in\mathscr{O}'(\tilde{B}[1])$, we define the $(-m,k)$-component $S_{-m,k}(T;\omega)$ by the following formula:

$$S_{-m,k}(T;\omega)=N(n,k)(T_z,\ (\sqrt{z^2})^{m-k}P_k(n;\ \langle z/\sqrt{z^2},\ \omega\rangle)) \tag{5.37}$$

for $m\geqslant k$, i.e., for $(m,k)\in\Lambda_+$. We put

$$S_{-m,k}(T;\omega)=0 \quad \text{for} \quad m<k\ .$$

For $T\in\mathscr{O}'(\tilde{B}[1])$ and $f\in\mathscr{O}(\tilde{B}[1])$, we have

$$(T, f)=\sum_{(m,k)\in\Lambda_+}(S_{-m,k}(T;\omega), S_{m,k}(f;\omega))_{S^{n-1}} . \tag{5.38}$$

By Lemma 5.1 and Theorem 4.2, we can characterize the subspace $\gamma^*\mathcal{O}'(\tilde{B}[1])$ of the space $\mathscr{B}(\Sigma^n)$ of hyperfunctions on Σ^n as follows:

THEOREM 5.3. *A hyperfunction* $T\in\mathscr{B}(\Sigma^n)$ *belongs to the subspace* $\gamma^*\mathcal{O}'(\tilde{B}[1])$ *if and only if*

$$S_{-m,k}(T;\omega)=0 \quad \textit{for} \quad (m, k)\notin\Lambda_+ . \tag{5.39}$$

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