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Calculation of Discriminants of High Degree Equations

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ABSTRACT. Discriminants of equations up to the ninth degree are calculated by using a computer. The numbers of terms included in the discriminants are 2, 5, 16, 59, 246, 1103, 5247, and 26059 for equations of degree two, three, four, five, six, seven, eight, and nine, respectively. Expressions of discriminants up to the fifth degree are included in this paper.

Introduction

In various fields of sciences and engineerings, computers are often used to perform laborious algebraic calculations. Many of such calculations are performed by formula manipulation systems developed by computer scientists. We believe the systems are also useful for mathematicians.

In this short article, we would like to show the usefulness of formula manipulation systems to mathematicians through calculating discriminants of high degree equations. Let $f(x)$ be a polynomial of degree n:

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0,\ a_n\neq 0\ .$$

The equation $f(x)=0$ has multiple roots if and only if its discriminant $D(f)$ is equal to zero. Expressions of $D(f)$ are well-known for equations of degree two and three:

$$n=2 : D(f)=a_1^2-4\cdot a_2a_0\ ,$$
$$n=3 : D(f)=a_2^2a_1^2+18\cdot a_3a_2a_1a_0-4\cdot a_3a_1^3-4\cdot a_2^3a_0-27\cdot a_3^2a_0^2\ .$$

The expression of $D(f)$ becomes very large as n increases, and paper-and-pencil calculation of such a large expression is quite laborious.

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Our calculation is based on Sylvester's determinant. We reduce the $2n$-th order Sylvester's determinant for two polynomials of degree n to a determinant of order n. The resulting determinant is similar to Bezout's determinant of order n, but the process of forming our determinant is simpler and easier than that of Bezout's determinant.

§ 1. Method of calculation.

Let $g(x)$ be a polynomial of degree m:

$$g(x)=b_m x^m+b_{m-1}x^{m-1}+\cdots+b_1 x+b_0,\ b_m\neq 0 .$$

The equations $f(x)=0$ and $g(x)=0$ have common roots if and only if their resultant $R(f, g)$ is equal to zero. The $R(f, g)$ is defined by the following determinant of order $n+m$ (Sylvester's determinant [1]):

$$(1)\qquad R(f,g)=\left|\begin{array}{cccccccccc} a_n & a_{n-1} & \cdot & \cdot & a_1 & a_0 & & & & \\ & a_n & a_{n-1} & \cdot & \cdot & a_1 & a_0 & & & \\ & & \cdot & & & & & \cdot & & \\ & & & \cdot & & & & & \cdot & \\ & & & & a_n & a_{n-1} & \cdot & \cdot & a_1 & a_0 \\ b_m & b_{m-1} & \cdot & \cdot & b_1 & b_0 & & & & \\ & b_m & b_{m-1} & \cdot & \cdot & b_1 & b_0 & & & \\ & & \cdot & & & & & \cdot & & \\ & & & \cdot & & & & & \cdot & \\ & & & & \cdot & & & & & \cdot \\ & & & & b_m & b_{m-1} & \cdot & \cdot & b_1 & b_0 \end{array}\right| \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} m \text{ rows} \\ \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} n \text{ rows} \end{array}$$

The $D(f)$ is closely related to the resultant of f and $f'\equiv df/dx$:

$$(2)\qquad R(f, f')=(-1)^{n(n-1)/2}a_n D(f) .$$

Since $R(f, f')$ is a homogeneous polynomial of degree $2n-1$ with respect to a_i and b_i, $D(f)$ is a homogeneous polynomial of degree $2n-2$. The coefficient of the term $a_n^n a_0^{n-1}$ in $R(f, f')$ is $(-1)^{n(n-1)}n^n$. Hence, $D(f)$ contains the term $(-1)^{n(n-1)/2}n^n a_n^{n-1}a_0^{n-1}$.

The most preferable method for calculating determinants of multivariate polynomial entries is a successive expansion by minors. Since a combinatorial number of terms are handled in the minor expansion method, we reduce the order of determinants to be calculated as far as possible.

If f and f' have common factors then so are $\tilde{f}\equiv nf-xf'$ and f', and vice versa. That is

$$R(f, f')=0 \leftrightarrow R(\tilde{f}, f')=0 .$$

Thus, we may use $R(\tilde{f}, f')$ instead of $R(f, f')$ to calculate $D(f)$. The $R(\tilde{f}, f')$ is given by the following determinant:

$$(3)\qquad \begin{vmatrix} a_{n-1} & 2a_{n-2} & \cdots & (n-1)a_1 & na_0 & & & & \\ & a_{n-1} & \cdots & (n-2)a_2 & (n-1)a_1 & na_0 & & & \\ & & \ddots & & & & \ddots & & \\ & & & a_{n-1} & 2a_{n-2} & 3a_{n-3} & \cdots & & na_0 \\ na_n & (n-1)a_{n-1} & \cdots & 2a_2 & a_1 & & & & \\ & na_n & \cdots & 3a_3 & 2a_2 & a_1 & & & \\ & & \ddots & & & & \ddots & & \\ & & & na_n & (n-1)a_{n-1} & \cdots & & & a_1 \end{vmatrix}$$

$$\equiv \begin{vmatrix} A & B \\ C & D \end{vmatrix} .$$

Here, A, B, C, and D are matrices of order $n-1$ such that

$$A_{ij}=A_{1,j-i+1}=\begin{cases}(j-i+1)a_{n-(j-i+1)} & \text{if } i\leqq j ,\\ 0 & \text{if } i>j ,\end{cases}$$

$$C_{ij}=C_{1,j-i+1}=\begin{cases}(n-j+i)a_{n-j+i} & \text{if } i\leqq j ,\\ 0 & \text{if } i>j ,\end{cases}$$

and so on.

LEMMA 1. *The matrices A and C, or B and D, are commutative, i.e., $AC=CA$ and $BD=DB$.*

PROOF. We first consider A and C. The kl elements of AC and CA are zero if $k>l$. For $k\leqq l<n$, we have

$$(CA)_{kl}=\sum_{i=k}^{l} C_{ki}A_{il}=\sum_{i=k}^{l} C_{1,i-k+1}A_{1,l-i+1} .$$

Replacing i by $l+k-j$ in the right-hand side, we have

$$(CA)_{kl}=\sum_{j=k}^{l} C_{1,l-j+1}A_{1,j-k+1}=\sum_{j=k}^{l} C_{jl}A_{kj} .$$

The right-hand side of this equation is the kl element of AC. This proves the commutativity of A and C. Since the mapping $a_i \to a_{n-i}$, $i=0, 1, \cdots, n$, transforms A and C into D^t and B^t, respectively, the relation $BD=DB$ is obtained from the commutativity of A and C.

We use Schur's formula for reducing the determinant (3).

THEOREM (Schur). *Let P, Q, R, and S be square matrices of order m such that $|P| \neq 0$. Let Δ be the determinant of the matrix*

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix} .$$

Then, $\Delta = |PS - PRP^{-1}Q|$, and $\Delta = |PS - RQ|$ if $PR = RP$.

This theorem is easily proved by the following decomposition:

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ RP^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} P & 0 \\ 0 & S-RP^{-1}Q \end{bmatrix} \cdot \begin{bmatrix} I & P^{-1}Q \\ 0 & I \end{bmatrix} .$$

Schur's formula and Lemma 1 allow us to reduce the determinant (3) as follows:

$$R(\tilde{f}, f') = |AD - CB| . \tag{4}$$

This is our required formula. The $R(\tilde{f}, f')$ is a homogeneous polynomial of degree $2n-2$, and the coefficient of the term $a_n^{n-1} a_0^{n-1}$ in $R(\tilde{f}, f')$ is $(-1)^{n-1} n^{2(n-1)}$. Hence,

$$D(f) = (-1)^{(n-1)(n-2)/2} R(\tilde{f}, f')/n^{n-2} . \tag{5}$$

Our method is easily generalized to the calculation of $R(f, g)$. Suppose $n > m$. Let $\hat{f}$ be the pseudo-remainder of f and xg:

$$b_m^{n-m} f = q \cdot xg + \hat{f}, \ \deg(\hat{f}) \leqq m .$$

We may consider the degree of $\hat{f}$ to be m: if $\deg(\hat{f}) < m$, we have only to add terms of coefficients zero. We first calculate the resultant of $\hat{f}$ and g. The $R(\hat{f}, g)$ is given by a Sylvester's determinant of order $2m$:

$$R(\hat{f}, g) = \begin{vmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{vmatrix} ,$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are matrices of order m. We can easily prove the commutativity of $\hat{A}$ and $\hat{C}$, and $\hat{B}$ and $\hat{D}$, and the determinant of order $2m$ is readily reduced to a determinant of order m by Schur's formula. Since $R(\hat{f}, g)$ is related to $R(f, g)$ by the formula

$$R(\hat{f}, g)=(-1)^{(n-m)m}b_m^{(n-m)(m-1)}R(f, g) ,$$

we can calculate $R(f, g)$ from $R(\hat{f}, g)$.

A less complicated method is to reduce Sylvester's determinant of order $2n$ to a determinant of order n. Let us regard $g(x)$ as a polynomial of degree n and denote the formally n-th degree polynomial by $\hat{g}(x)$. Then, $R(f, \hat{g})$ is represented by Sylvester's determinant of order $2n$, and the determinant is readily reduced to a determinant of order n by Schur's formula. Since $R(f, \hat{g})$ is related to $R(f, g)$ by the formula

$$R(f, \hat{g})=a_n^{n-m}R(f, g) ,$$

we can obtain $R(f, g)$ easily from $R(f, \hat{g})$.

Regarding the calculation of the resultant, Bezout presented an efficient way [2]. In his method, $R(f, g)$ is represented by a determinant of order $\max[n, m]$. The order of our determinant is either m or n. Note that, even for the case $n=m$, our handling is different from Bezout's. For example, for $n=m=3$, our determinant is

$$\left|\begin{bmatrix} a_3b_0+a_2b_1+a_1b_2 & a_2b_0+a_1b_1 & a_1b_0 \\ a_3b_1+a_2b_2 & a_3b_0+a_2b_1 & a_2b_0 \\ a_3b_2 & a_3b_1 & a_3b_0 \end{bmatrix} - \begin{bmatrix} \\ a_i \leftrightarrow b_i \\ \\ \end{bmatrix}\right| ,$$

while Bezout's determinant is

$$\left|\begin{bmatrix} a_3b_0 & a_2b_0 & a_1b_0 \\ a_3b_1 & a_3b_0+a_2b_1 & a_2b_0 \\ a_3b_2 & a_3b_1 & a_3b_0 \end{bmatrix} - \begin{bmatrix} \\ a_i \leftrightarrow b_i \\ \\ \end{bmatrix}\right| .$$

§ 2. Results of calculations.

Our final work is to calculate the determinant (4) for various values of n. We used the formula manipulation system REDUCE-2 [3] for this calculation, since the evaluation of non-numeric determinants is quite laborious. The REDUCE-2 is equipped with, like most of the formula manipulation systems, an efficient determinant manipulation routine, and we are quite easy to calculate discriminants so far as n is not so large. In Figure 1, we show a REDUCE program for calculating discriminants. The meaning of the program will be almost apparent to the reader. We found, however, that even a powerful computer at present is hard to calculate discriminants for $n\geqq10$. The largest limi-

```
COMMENT, definition of coefficients;
   ARRAY U(9);
      U(0):=A0;
      U(1):=A1;
      . . . . . .
      . . . . . .
      U(9):=A9;
COMMENT, definition of matrices A, B, C, and D;
   ALGEBRAIC PROCEDURE ABCD (N);
      BEGIN CLEAR A, B, C, D;
         MATRIX A(N-1, N-1), B(N-1, N-1), C(N-1, N-1), D(N-1, N-1);
         FOR I:=1: (N-1) DO BEGIN
            FOR J:=1: (N-1) DO BEGIN
               A(I, J):=IF I>J THEN 0 ELSE (1+(J-I))*U(N-(J-I+1));
               B(I, J):=IF I<J THEN 0 ELSE (N-(I-J))*U(I-J);
               C(I, J):=IF I>J THEN 0 ELSE (N-(J-I))*U(N-(J-I));
               D(I, J):=IF I<J THEN 0 ELSE (1+(I-J))*U(1+(I-J));
            END;
         END;
      END;
COMMENT, discriminant for equation of degree n;
   ALGEBRAIC PROCEDURE DISCR(N);
      BEGIN CLEAR X;
         MATRIX X(N-1, N-1);
         ABCD(N);
         X:=A*D-C*B;
         RETURN (-1)**((N-1)*(N-2)/2) * DET(X)/N**(N-2);
      END;
```

FIGURE 1. REDUCE program for calculating the discriminant of equation of degree n. Inputting DISCR(5);, for example, gives the discriminant for fifth degree equation.

n	No. of terms	time (sec)
2	2	0.079
3	5	0.107
4	16	0.182
5	59	0.414
6	246	1.565
7	1103	9.91
8	5247	78.5
9	26059	1932.

TABLE 1. The number of terms in $D(f)$ and the computation time.

tation is the memory size: we have used 11 mega-byte memory ($11\times 8\times 2^{20}$ bits), and even this huge memory was rather tight for $n=9$.

Table 1 shows the numbers of terms contained in discriminants we have calculated. We also show computation times on a FACOM M-200, one of the Japanese representative computers at present. The reader will recognize the ability of present computers to perform algebraic computations from these results. We show expressions of discriminants for $n=4$ and 5 in appendices. Expressions for $n>5$ are too large to write up in this paper. These expressions are stored in a magnetic tape which will be available from the authers upon request.

Appendix 1. The discriminant of equation of degree four.

$$\begin{aligned}
&256\cdot a_4^3a_0^3-192\cdot a_4^2a_3a_1a_0^2-128\cdot a_4^2a_2^2a_0^2+144\cdot a_4^2a_2a_1^2a_0-27\cdot a_4^2a_1^4+144\cdot a_4a_3^2a_2a_0^2\\
&\quad-6\cdot a_4a_3^2a_1^2a_0-80\cdot a_4a_3a_2^2a_1a_0+18\cdot a_4a_3a_2a_1^3+16\cdot a_4a_2^4a_0-4\cdot a_4a_2^3a_1^2-27\cdot a_3^4a_0^2\\
&\quad+18\cdot a_3^3a_2a_1a_0-4\cdot a_3^3a_1^3-4\cdot a_3^2a_2^3a_0+a_3^2a_2^2a_1^2\ .
\end{aligned}$$

Appendix 2. The discriminant of equation of degree five.

$$\begin{aligned}
&3125\cdot a_5^4a_0^4-2500\cdot a_5^3a_4a_1a_0^3-3750\cdot a_5^3a_3a_2a_0^3+2000\cdot a_5^3a_3a_1^2a_0^2+2250\cdot a_5^3a_2^2a_1a_0^2\\
&\quad-1600\cdot a_5^3a_2a_1^3a_0+256\cdot a_5^3a_1^5+2000\cdot a_5^2a_4^2a_2a_0^3-50\cdot a_5^2a_4^2a_1^2a_0^2+2250\cdot a_5^2a_4a_3^2a_0^3\\
&\quad-2050\cdot a_5^2a_4a_3a_2a_1a_0^2+160\cdot a_5^2a_4a_3a_1^3a_0-900\cdot a_5^2a_4a_2^3a_0^2+1020\cdot a_5^2a_4a_2^2a_1^2a_0\\
&\quad-192\cdot a_5^2a_4a_2a_1^4-900\cdot a_5^2a_3^3a_1a_0^2+825\cdot a_5^2a_3^2a_2^2a_0^2+560\cdot a_5^2a_3^2a_2a_1^2a_0-128\cdot a_5^2a_3^2a_1^4\\
&\quad-630\cdot a_5^2a_3a_2^3a_1a_0+144\cdot a_5^2a_3a_2^2a_1^3+108\cdot a_5^2a_2^5a_0-27\cdot a_5^2a_2^4a_1^2-1600\cdot a_5a_4^3a_3a_0^3\\
&\quad+160\cdot a_5a_4^3a_2a_1a_0^2-36\cdot a_5a_4^3a_1^3a_0+1020\cdot a_5a_4^2a_3^2a_1a_0^2+560\cdot a_5a_4^2a_3a_2^2a_0^2\\
&\quad-746\cdot a_5a_4^2a_3a_2a_1^2a_0+144\cdot a_5a_4^2a_3a_1^4+24\cdot a_5a_4^2a_2^3a_1a_0-6\cdot a_5a_4^2a_2^2a_1^3\\
&\quad-630\cdot a_5a_4a_3^3a_2a_0^2+24\cdot a_5a_4a_3^3a_1^2a_0+356\cdot a_5a_4a_3^2a_2^2a_1a_0-80\cdot a_5a_4a_3^2a_2a_1^3\\
&\quad-72\cdot a_5a_4a_3a_2^4a_0+18\cdot a_5a_4a_3a_2^3a_1^2+108\cdot a_5a_3^5a_0^2-72\cdot a_5a_3^4a_2a_1a_0+16\cdot a_5a_3^4a_1^3\\
&\quad+16\cdot a_5a_3^3a_2^3a_0-4\cdot a_5a_3^3a_2^2a_1^2+256\cdot a_4^5a_0^3-192\cdot a_4^4a_3a_1a_0^2-128\cdot a_4^4a_2^2a_0^2\\
&\quad+144\cdot a_4^4a_2a_1^2a_0-27\cdot a_4^4a_1^4+144\cdot a_4^3a_3^2a_2a_0^2-6\cdot a_4^3a_3^2a_1^2a_0-80\cdot a_4^3a_3a_2^2a_1a_0\\
&\quad+18\cdot a_4^3a_3a_2a_1^3+16\cdot a_4^3a_2^4a_0-4\cdot a_4^3a_2^3a_1^2-27\cdot a_4^2a_3^4a_0^2+18\cdot a_4^2a_3^3a_2a_1a_0-4\cdot a_4^2a_3^3a_1^3\\
&\quad-4\cdot a_4^2a_3^2a_2^3a_0+a_4^2a_3^2a_2^2a_1^2\ .
\end{aligned}$$

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