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On Normal Integral Bases

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Introduction

Let k be a number field, and K/k a finite Galois extension with Galois group G. Let $\mathfrak{o}_k$ and $\mathfrak{o}_K$ be the rings of integers in k and K. We denote by $\mathfrak{o}_kG$ the group ring of G over $\mathfrak{o}_k$. $\mathfrak{o}_K$ can be regarded as an $\mathfrak{o}_kG$-module by the action $r\cdot\alpha=\sum_{s\in G}a_s s\alpha$ for $\alpha\in\mathfrak{o}_K$, $r=\sum_{s\in G}a_s s\in\mathfrak{o}_kG$. These notations will be used throughout this paper. K/k is said to have a *normal integral basis* (abbr. n.i.b.) when there is an element $\alpha\in\mathfrak{o}_K$ such that $\{s\alpha\}_{s\in G}$ is a relative integral basis of K/k, and α is called a *generator* of this basis. It is known that a finite Galois extension with n.i.b. is tamely ramified ([4], Chapter 9, Theorem (1, 2)).

In case where k is the field $\boldsymbol{Q}$ of rational numbers, every tamely ramified abelian field has an n.i.b. (Hilbert-Speiser), so that when $k=\boldsymbol{Q}$ and G is abelian, K/k has an n.i.b. if and only if K/k is tamely ramified ([4], Chapter 9, Theorem (3, 4)). Furthermore, Fröhlich [2] has given a necessary and sufficient condition for K/k to have an n.i.b., when K/k is a Kummer extension. On the other hand, Okutsu [8] has shown that when $k=\boldsymbol{Q}(\zeta_l)$, $\zeta_l=\exp(2\pi i/l)$, l: odd prime, and $K=k(\sqrt[l]{a})$, $a\in\boldsymbol{Z}$, K/k has always a relative integral basis and given an explicit form of this basis. After preparations in §1, giving in particular a more precise form to the results of [2], we shall apply them in §2 to obtain a necessary and sufficient condition for K/k to have an n.i.b. for the case where k and K are as in [8]. We shall also give explicitly a generator of n.i.b. when this exists. In the final §3, we shall construct many examples of normal extensions K/k with n.i.b.'s where $k\neq\boldsymbol{Q}$, and K/k are tamely ramified. We shall also mention an example of such K/k without n.i.b..

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§1. Preparations.

Let K/k be tamely ramified. For each prime ideal $\mathfrak{p}$ of k, let $k_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of k and $\mathfrak{o}_{\mathfrak{p}}$ be the valuation ring of $k_{\mathfrak{p}}$. $k_{\mathfrak{p}}$-algebra $K_{\mathfrak{p}}$ is defined by $k_{\mathfrak{p}} \otimes_k K$. Then $k_{\mathfrak{p}}$ and K are naturally embedded in $K_{\mathfrak{p}}$. We define $\mathfrak{o}_{\mathfrak{p}}$-algebra $\mathfrak{o}_{K,\mathfrak{p}}$ by $\mathfrak{o}_{\mathfrak{p}} \otimes_{\mathfrak{o}_k} \mathfrak{o}_K$. As $\mathfrak{o}_{\mathfrak{p}}$ is a flat $\mathfrak{o}_k$-module, $\mathfrak{o}_{K,\mathfrak{p}}$ is also naturally embedded in $K_{\mathfrak{p}}$. Consequently $\mathfrak{o}_{\mathfrak{p}}$ and $\mathfrak{o}_K$ are naturally embedded in $\mathfrak{o}_{K,\mathfrak{p}}$. $M_{\mathfrak{p}}$ denotes $\mathfrak{o}_{\mathfrak{p}} \otimes_{\mathfrak{o}_k} M$ for an $\mathfrak{o}_k$-submodule M of K. If M is an $\mathfrak{o}_k$-lattice of the k-vector space K, then

$$M = K \cap (\bigcap_{\mathfrak{p}} M_{\mathfrak{p}}) , \tag{1}$$

where $\mathfrak{p}$ ranges over all prime ideals of k ([10], Theorem (5, 3)). Since a finite Galois extension K/k has a normal basis and $\mathfrak{o}_K$ is a projective $\mathfrak{o}_k G$-module, $\mathfrak{o}_{K,\mathfrak{p}}$ and $\mathfrak{o}_{\mathfrak{p}} G$ are isomorphic as $\mathfrak{o}_{\mathfrak{p}} G$-modules for any prime $\mathfrak{p}$ of k (Swan [9], Corollary 6, 4). Hence there is an element $\beta_{\mathfrak{p}} \in \mathfrak{o}_{K,\mathfrak{p}}$ such that $\{s\beta_{\mathfrak{p}}\}_{s \in G}$ is an $\mathfrak{o}_{\mathfrak{p}}$-basis of $\mathfrak{o}_{K,\mathfrak{p}}$. We call this $\beta_{\mathfrak{p}}$ a *generator* of local normal basis for $\mathfrak{p}$. $b \in K$ is a *generator* of global basis if and only if $\{sb\}_{s \in G}$ is a k-basis of K.

In the remainder of this section, we assume as in [2] K/k to be a finite tamely ramified Kummer extension of exponent n with Galois group G. $\hat{G}$ denotes the character group of G. If A is an abelian group, let $M(\hat{G}, A)$ be the set of maps from $\hat{G}$ into A. If we define the product of maps $f_1, f_2 \colon \hat{G} \to A$ by $f_1 f_2(\chi) = f_1(\chi) f_2(\chi)$ for $\chi \in \hat{G}$, $M(\hat{G}, A)$ becomes an abelian group. For a map $f \colon \hat{G} \to K_{\mathfrak{p}}$, define $f^*(s)$ by $(1/|G|) \sum_{\chi \in \hat{G}} \chi(s) f(x)$ for $s \in G$, where $|G|$ is the order of G. Since k contains a primitive n-th root of unity, f^* is the map from $\hat{G}$ into $K_{\mathfrak{p}}$. Let J_k be the idele group of k and U_k be $\prod_{\mathfrak{p}:\text{ finite}} \mathfrak{o}_{\mathfrak{p}}^{\times} \times \prod_{\mathfrak{p}_\infty} k_{\mathfrak{p}_\infty}^{\times}$, where $\mathfrak{p}$ ranges over all finite primes and $\mathfrak{p}_\infty$ all infinite primes of k. (For a ring R, $R^{\times}$ means the unit group of R.)

DEFINITION. For each prime ideal $\mathfrak{p}$ of k, let $M_0(\mathfrak{p})$ be the set of maps $f_{\mathfrak{p}} \colon \hat{G} \to \mathfrak{o}_{\mathfrak{p}}^{\times}$ satisfying $\operatorname{Im} f_{\mathfrak{p}}^* \subset \mathfrak{o}_{\mathfrak{p}}$. We define $M_0(\hat{G}, U_k)$ to be the set of maps $f = (f_{\mathfrak{p}}) \in M(\hat{G}, U_k)$ satisfying $f_{\mathfrak{p}} \in M_0(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$.

It is easily seen that $M_0(\mathfrak{p})$ is a group and consequently $M_0(\hat{G}, U_k)$ is a subgroup of $M(\hat{G}, J_k)$. Let $\beta_{\mathfrak{p}} \in \mathfrak{o}_{K,\mathfrak{p}}$ be a generator of local normal basis for each $\mathfrak{p}$ and $b \in K$ be a generator of global normal basis. For an element $\alpha \in K_{\mathfrak{p}}$ and $\chi \in \hat{G}$, define $(\alpha|\chi) = \sum_{s \in G} \bar{\chi}(s) s\alpha$. Put $\varphi_{\mathfrak{p}}(\chi) = (\beta_{\mathfrak{p}}|\chi)/(b|\chi)$. Then $\varphi_{\mathfrak{p}}(\chi)$ is an element of $k_{\mathfrak{p}}^{\times}$. Putting $\varphi(\chi) = (\cdots, \varphi_{\mathfrak{p}}(\chi), \cdots) \in \prod_{\mathfrak{p}} k_{\mathfrak{p}}^{\times}$ for each $\chi \in \hat{G}$, we have $\varphi \in M(\hat{G}, J_k)$. The residue class of φ in the finite abelian group $M(\hat{G}, J_k)/M(\hat{G}, k^{\times}) M_0(\hat{G}, U_k)$ does not depend upon the choice of generators of global and local normal bases. The following lemma is proved in [2], §7, 7.2.

LEMMA 1. *Suppose that* $\mathfrak{p}$ *is a prime ideal of* k *and* $f_{\mathfrak{p}}$ *is a map from* $\hat{G}$ *into* $k_{\mathfrak{p}}$. *Set* $\alpha_{\mathfrak{p}}=(1/|G|)\sum_{\chi\in\hat{G}} f_{\mathfrak{p}}(\chi)(\beta_{\mathfrak{p}}|\chi)$. *Then* $\alpha_{\mathfrak{p}}$ *is a generator of local normal basis for* $\mathfrak{p}$ *if and only if* $f_{\mathfrak{p}}\in M_0(\mathfrak{p})$.

THEOREM 1. *A necessary and sufficient condition for* K/k *to have an* n.i.b. *is that* φ *lies in* $M(\hat{G}, k^{\times})M_0(\hat{G}, U_k)$. *If* $\varphi=gf$, $f=(f_{\mathfrak{p}})\in M_0(\hat{G}, U_k)$ *and* $g\in M(\hat{G}, k^{\times})$, *then* $(1/|G|)\sum_{\chi\in\hat{G}} g(\chi)(b|\chi)$ *generates an* n.i.b. *of* K/k.

PROOF. If K/k has an n.i.b., it is a local normal basis for each $\mathfrak{p}$ and a global normal basis at the same time. Hence we obtain $\varphi=1$. Conversely, if φ has the above decomposition, we have for all $\mathfrak{p}$ and all $\chi\in\hat{G}$

$$f_{\mathfrak{p}}^{-1}(\chi)(\beta_{\mathfrak{p}}|\chi)=g(\chi)(b|\chi)\ . \tag{2}$$

Let $\alpha_{\mathfrak{p}}$ be $(1/|G|)\sum_{\chi\in\hat{G}} f_{\mathfrak{p}}^{-1}(\chi)(\beta_{\mathfrak{p}}|\chi)$. Since $M_0(\mathfrak{p})$ is a group, we have $f_{\mathfrak{p}}^{-1}\in M_0(\mathfrak{p})$. Therefore $\alpha_{\mathfrak{p}}$ is a local normal basis for $\mathfrak{p}$ by Lemma 1. But $\alpha_{\mathfrak{p}}$ is independent of each $\mathfrak{p}$ by (2). So we may set $\alpha_{\mathfrak{p}}=\alpha=(1/|G|)\sum_{\chi\in\hat{G}} g(\chi)$ $(b|\chi)$. Then for all $\mathfrak{p}$,

$$\mathfrak{o}_{K,\mathfrak{p}}=\bigoplus_{s\in G}\mathfrak{o}_{\mathfrak{p}}s\alpha=(\bigoplus_{s\in G}\mathfrak{o}_k s\alpha)_{\mathfrak{p}}\ .$$

Hence, by (1), we have $\mathfrak{o}_K=\bigoplus_{s\in G}\mathfrak{o}_k s\alpha$. This proves our theorem.

§2. In case $k=\boldsymbol{Q}(\zeta_l)$, $K=k(\sqrt[l]{a})$.

In this §, we consider as in [8] the case $k=\boldsymbol{Q}(\zeta_l)$, $K=k(\sqrt[l]{a})$ where l is an odd prime, ζ_l is a primitive l-th root of unity, $a(\neq\pm1)$ is a rational integer without l-th power factor. a has the decomposition $\prod_{i=1}^{l-1}a_i^i$, where the a_i's are square-free integers and $(a_i, a_j)=1$ $(i\neq j)$. Put $\omega=(\sqrt[l]{a}-a)/(1-\zeta_l)$ and $b_m=\prod_{i=1}^{l-1}a_i^{[im/l]}$ $(0\leqq m\leqq l-1)$, where $[x]$ is the greatest integer$\leqq x$ as usual. The following theorem is proved in [8].

OKUTSU'S THEOREM. $(1-\zeta_l)\mathfrak{o}_k$ *is unramified in* K/k *if and only if* $a^{l-1}\equiv1 \bmod l^2$. *Furthermore* $\{\omega^m/b_m\}_{0\leqq m\leqq l-1}$ *is a relative integral basis of* K/k *when* $(1-\zeta_l)\mathfrak{o}_k$ *is unramified. And the discriminant of* K/k *is* $\prod_{i=1}^{l-1}a_i^{l-1}$.

Now assume K/k is tamely ramified extension, i.e. $a^{l-1}\equiv1 \bmod l^2$. Let σ be a fixed generator of G, say $\sigma\sqrt[l]{a}=\sqrt[l]{a}\,\xi_l$ and χ be a fixed generator of $\hat{G}$, say $\chi(\sigma)=\zeta_l^{-1}$. We write $\zeta=\zeta_l$.

LEMMA 2. *Suppose that* α *is an element of* $\mathfrak{o}_K$ *and write* $\alpha=\sum_{m=0}^{l-1} u_m(\omega^m/b_m)$ $(u_m \in \mathfrak{o}_k)$. *Therefore there exists a matrix* A *in* $M_l(\mathfrak{o}_k)$ *such that* $(\alpha, \sigma\alpha, \cdots, \sigma^{l-1}\alpha)=(1, \omega/b_1, \cdots, \omega^{l-1}/b_{l-1})\,A$. *Then*

$$(3)\qquad (\alpha|\chi^j)=\frac{l}{(\zeta-1)^{l-1}}\frac{(-\sqrt[l]{a})^{l-j}}{b_{l-j}}\varepsilon_{l-j} \qquad (1\leqq j\leqq l)$$

and

$$(4)\qquad \det A=\zeta^{l(l-1)(l+1)/6}\cdot\prod_{i=2}^{l-1} t_i^{l-i}\prod_{j=0}^{l-1}\varepsilon_j\ ,$$

where $t_i=(\zeta^i-1)/(\zeta-1)$ *and* $\varepsilon_j=\sum_{m=j}^{l-1}(\zeta-1)^{l-1-m}\binom{m}{j}(a^{m-j}b_j/b_m)u_m$.

REMARK. The t_i's are units of k. Since $b_m|a$ $(0\leqq m\leqq l-1)$, the $a^{m-j}b_j/b_m$'s are rational integers. So we note that the ε_j's are elements of $\mathfrak{o}_k$.

PROOF OF LEMMA 2. We shall calculate $(\alpha|\chi^j)$ in the first place.

$$\begin{aligned}(\alpha|\chi^j)&=\sum_{i=0}^{l-1}\bar{\chi}^j(\sigma^i)\sum_{m=0}^{l-1}\frac{u_m}{b_m(1-\zeta)^m}(\sigma^i\sqrt[l]{a}-a)^m\\&=\sum_{m=0}^{l-1}\frac{u_m}{b_m(1-\zeta)^m}\sum_{i=0}^{l-1}\bar{\chi}^j(\sigma^i)\sum_{p=0}^{m}\binom{m}{p}(\sigma^i\sqrt[l]{a})^p(-a)^{m-p}\\&=\sum_{m=0}^{l-1}\frac{u_m}{b_m(1-\zeta)^m}\sum_{p=0}^{m}\binom{m}{p}(-a)^{m-p}(\sqrt[l]{a}^p|\chi^j)\ .\end{aligned}$$

And

$$\sqrt[l]{a}^p|\chi^j=\sqrt[l]{a}^p\sum_{i=0}^{l-1}\zeta^{i(j+p)}=\begin{cases}l\sqrt[l]{a}^p & \text{if } l\,|\,j+p\\ 0 & \text{if } l\nmid j+p\ .\end{cases}$$

Since $l|j+p$ is equivalent to $j+p=l$, we have

$$\begin{aligned}(\alpha|\chi^j)&=\sum_{m=l-j}^{l-1}\frac{u_m}{b_m(1-\zeta)^m}\binom{m}{l-j}(-a)^{m-(l-j)}l\sqrt[l]{a}^{l-j}\\&=l(-\sqrt[l]{a})^{l-j}\sum_{m=l-j}^{l-1}\frac{u_m}{b_m(\zeta-1)^m}\binom{m}{l-j}a^{m-(l-j)}\\&=\frac{l}{(\zeta-1)^{l-1}}\frac{(-\sqrt[l]{a})^{l-j}}{b_{l-j}}\varepsilon_{l-j}\ .\end{aligned}$$

For $\alpha_0, \cdots, \alpha_{l-1}\in K$, put $\Delta_{K/k}(\alpha_0, \cdots, \alpha_{l-1})=\det(\sigma^i\alpha_j)_{0\leqq i,j\leqq l-1}$. Then

$$(5)\qquad \Delta_{K/k}(\alpha, \sigma\alpha, \cdots, \sigma^{l-1}\alpha)=\Delta_{K/k}\left(1, \frac{\omega}{b_1}, \cdots, \frac{\omega^{l-1}}{b_{l-1}}\right)\det A\ .$$

Put $\theta=\sqrt[l]{a}-a$ and $\Delta=(-1)^{l(l-1)/2}\cdot\prod_{1\leq i<j\leq l}(\zeta^i-\zeta^j)$. By $\sigma^i\theta-\sigma^j\theta=\sqrt[l]{a}(\zeta^i-\zeta^j)$, we have

$$(6)\qquad \Delta_{K/k}\left(1, \frac{\omega}{b_1}, \cdots, \frac{\omega^{l-1}}{b_{l-1}}\right)=\left\{(1-\zeta)^{l(l-1)/2}\prod_{m=1}^{l-1}b_m\right\}^{-1}\Delta_{K/k}(1, \theta, \cdots, \theta^{l-1})$$
$$=\left\{(1-\zeta)^{l(l-1)/2}\prod_{m=1}^{l-1}b_m\right\}^{-1}a^{(l-1)/2}\Delta\ .$$

By using orthogonality relations of the character group of a finite abelian group, we obtain ([2], §7, (7, 2))

$$\prod_{j=1}^{l}(\alpha|\chi^j)=\det(\sigma^i\sigma^{-j}\alpha)_{0\leq i,j\leq l-1}=(-1)^{(l-1)/2}\Delta_{K/k}(\alpha, \sigma\alpha, \cdots, \sigma^{l-1}\alpha)\ .$$

Therefore by (3),

$$(7)\qquad \Delta_{K/k}(\alpha, \sigma\alpha, \cdots, \sigma^{l-1}\alpha)=\left\{(\zeta-1)^{l(l-1)}\prod_{j=0}^{l-1}b_j\right\}^{-1}l^l a^{(l-1)/2}\prod_{j=0}^{l-1}\varepsilon_j\ .$$

By (5), (6), (7), we have

$$\det A=(-1)^{l(l-1)/2}l^l(\zeta-1)^{-(l(l-1)/2)}\Delta^{-1}\prod_{j=0}^{l-1}\varepsilon_j\ .$$

Since $\Delta^2=(-1)^{l(l-1)/2}\prod_{i=1}^{l}f'(\zeta^i)=(-1)^{l(l-1)/2}l^l$ ($f(x)=x^l-1$), we have $\det A=\zeta^{l(l-1)(l+1)/6}\cdot\prod_{i=2}^{l-1}t_i^{l-i}\prod_{j=0}^{l-1}\varepsilon_j$. This proves our lemma.

THEOREM 2. *Suppose that l is an odd prime and $a(\neq\pm1)$ is a rational integer without l-th power factor such that $a^{l-1}\equiv 1 \bmod l^2$. Then a necessary and sufficient condition for $\boldsymbol{Q}(\zeta_l, \sqrt[l]{a})/\boldsymbol{Q}(\zeta_l)$ to have an* n.i.b. *is that there are units u_j ($j=0, \cdots, l-1$) of $\boldsymbol{Q}(\zeta_l)$ such that*

$$(8)\qquad \sum_{j=0}^{l-1}\binom{l-1}{j}\zeta_l^{ij}u_j a^{l-1-j}b_j\equiv 0 \bmod l$$

for any $i=0, \cdots, l-1$.

Furthermore, if there are such u_j's, then $(1/l)\sum_{i=0}^{l-1}u_j^{-1}((-\sqrt[l]{a})^j/b_j)$ *generates an* n.i.b. *of $\boldsymbol{Q}(\zeta_l, \sqrt[l]{a})/\boldsymbol{Q}(\zeta_l)$.*

PROOF. As we are used to in this section, we write $k=\boldsymbol{Q}(\zeta)$ and $K=\boldsymbol{Q}(\zeta, \sqrt[l]{a})$. Let $\beta_\mathfrak{p}\in\mathfrak{o}_{K,\mathfrak{p}}$ be a generator of local normal basis for each prime ideal $\mathfrak{p}$ of k and $b\in\mathfrak{o}_K$ be a generator of global normal basis of K/k. We write $b=\sum_{m=0}^{l-1}u_m(\omega^m/b_m)(u_m\in\mathfrak{o}_k)$. We note that $\{\omega^m/b_m\}_{0\leq m\leq l-1}$ is also an $\mathfrak{o}_{\mathfrak{p}}$-basis of $\mathfrak{o}_{K,\mathfrak{p}}$. Hence we can write $\beta_{\mathfrak{p}}=\sum_{m=0}^{l-1} u_{m,\mathfrak{p}}(\omega^m/b_m)(u_{m,\mathfrak{p}}\in\mathfrak{o}_{\mathfrak{p}})$. Then we can hold the results for $\beta_{\mathfrak{p}}$ similar to the calculations of Lemma 2. Therefore, if we put $\varepsilon_{j,\mathfrak{p}}=\sum_{m=j}^{l-1}(\zeta-1)^{l-1-m}\binom{m}{j}(a^{m-j}b_j/b_m)u_{m,\mathfrak{p}}$, we obtain for each $\mathfrak{p}$ and $j=1,\cdots,l$, by (3),

$$\varphi_{\mathfrak{p}}(\chi^j)=\frac{(\beta_{\mathfrak{p}}|\chi^j)}{(b|\chi^j)}=\frac{\varepsilon_{l-j,\mathfrak{p}}}{\varepsilon_{l-j}}.$$

Now we put $f_{\mathfrak{p}}(\chi^j)=\varepsilon_{l-j,\mathfrak{p}}$ and $g(\chi^j)=\varepsilon_{l-j}^{-1}$. Since $\beta_{\mathfrak{p}}$ and b are local and global normal bases, we have $f=(f_{\mathfrak{p}})\in M(\hat{G},U_k)$ and $g\in M(\hat{G},k^\times)$ by (4). Let $\varphi=g'f'$, $f'\in M(\hat{G},U_k)$ and $g'\in M(\hat{G},k^\times)$ be another decomposition of φ. Then it is easy to see that there is $u\in M(\hat{G},\mathfrak{o}_k^\times)$ such that $f'=uf$ and $g'=u^{-1}g$. Hence, by Theorem 1, K/k has an n.i.b. if and only if there is $u\in M(\hat{G},\mathfrak{o}_k^\times)$ such that $uf_{\mathfrak{p}}\in M_0(\mathfrak{p})$ for every prime ideal $\mathfrak{p}$ of k. Since $(uf_{\mathfrak{p}})^*(\sigma^i)=(1/l)\sum_{j=0}^{l-1}\zeta^{ij}u(\chi^{l-j})\varepsilon_{j,\mathfrak{p}}$ $(0\leqq i\leqq l-1)$, it is sufficient to show $uf_{\mathfrak{p}}\in M_0(\mathfrak{p})$ only for a prime ideal of k dividing l for proving that $uf_{\mathfrak{p}}\in M_0(\mathfrak{p})$ takes place for all $\mathfrak{p}$'s. Now let $\mathfrak{p}|l$. Putting $u_{0,\mathfrak{p}}=\cdots=u_{l-2,\mathfrak{p}}=0$ and $u_{l-1,\mathfrak{p}}=b_{l-1}$, by $l\nmid a$, we have $\varepsilon_{j,\mathfrak{p}}=\binom{l-1}{j}a^{l-1-j}b_j\in\mathfrak{o}_{\mathfrak{p}}^\times$ $(0\leqq j\leqq l-1)$. Therefore $\beta_{\mathfrak{p}}=\omega^{l-1}$ generates a local normal basis for $\mathfrak{p}$ by (4). Then

$$(uf_{\mathfrak{p}})^*(\sigma^i)=\frac{1}{l}\sum_{j=0}^{l-1}\binom{l-1}{j}\zeta^{ij}u(\chi^{l-j})a^{l-1-j}b_j \qquad (0\leqq i\leqq l-1).$$

Setting $u_j=u(\chi^{l-j})$, the first part of the theorem is established. By (3),

$$\frac{1}{|G|}\sum_{j=1}^{l}u^{-1}g(\chi^j)(b|\chi^j)=\frac{1}{(\zeta-1)^{l-1}}\sum_{j=0}^{l-1}u_j^{-1}\frac{(-\sqrt[l]{a})^j}{b_j}.$$

This gives a generator of the n.i.b. by Theorem 1. Since $(\zeta-1)^{l-1}/l\in\mathfrak{o}_k^\times$, $(1/l)\sum_{j=0}^{l-1}u_j^{-1}((-\sqrt[l]{a})^j/b_j)$ is also a generator and the proof is completed.

Now we examine the case in which (8) holds for $u_j=b_j=1$ $(j=0,\cdots,l-1)$. Let $\mathfrak{p}=(\zeta-1)\mathfrak{o}_k$. Since $\sum_{j=0}^{l-1}\binom{l-1}{j}\zeta^{ij}a^{l-1-j}=(a+\zeta^i)^{l-1}=(a+1+\zeta^i-1)^{l-1}$ and $l=\mathfrak{p}^{l-1}$, (8) implies $a\equiv -1 \bmod l$. By the definition, $b_j=1$ $(j=0,\cdots,l-1)$ means that a is a square-free integer. Furthermore $a^{l-1}\equiv 1 \bmod l^2$ and $a\equiv -1 \bmod l$ mean $a\equiv -1 \bmod l^2$, and since l is an odd prime, we have $k(\sqrt[l]{a})=k(\sqrt[l]{-a})$. By Theorem 2, we obtain the following theorem.

THEOREM 3. *Suppose that l is odd prime and a $(\neq\pm1)$ is square-free rational integer such that $a\equiv\pm1 \bmod l^2$. Then $\alpha=(1/l)\sum_{j=0}^{l-1}(-\sqrt[l]{\varepsilon a})^j$ generates an* n.i.b. *of $\boldsymbol{Q}(\zeta_l,\sqrt[l]{a})/\boldsymbol{Q}(\zeta_l)$, where*

$$\varepsilon=\begin{cases} 1 & if \quad a\equiv -1 \bmod l^2 \\ -1 & if \quad a\equiv 1 \bmod l^2 . \end{cases}$$

COROLLARY. *Let l, a and α be as in Theorem 3. Then $\zeta_l\alpha$ generates an* n.i.b. *of the non-abelian extension* $\boldsymbol{Q}(\zeta_l, \sqrt[l]{a})/\boldsymbol{Q}$.

PROOF. Since $\boldsymbol{Q}(\zeta_l, \sqrt[l]{a})=\boldsymbol{Q}(\zeta_l, \sqrt[l]{-a})$, we may prove in case where $a\equiv -1 \bmod l^2$. Put $\Gamma=\mathrm{Gal}(K/\boldsymbol{Q})$. Let σ, τ be fixed elements of Γ, say $\sigma\zeta=\zeta$, $\sigma\sqrt[l]{a}=\sqrt[l]{a}\zeta$, $\tau\zeta=\zeta^g$ and $\tau\sqrt[l]{a}=\sqrt[l]{a}$, where g is a primitive root mod l. Then we have $\Gamma=\{\sigma^i\tau^j|i=0, \cdots, l-1, \ j=1, \cdots, l-1\}$. By Theorem 3, we obtain $\mathfrak{o}_K=\bigoplus_{i=0}^{l-1}\mathfrak{o}_k\sigma^i\alpha$ and also $\mathfrak{o}_k=\bigoplus_{j=1}^{l-1}\boldsymbol{Z}\tau^j\zeta$. Consequently, we have $\mathfrak{o}_K=\bigoplus_{i=0}^{l-1}\bigoplus_{j=1}^{l-1}\boldsymbol{Z}\sigma^i\alpha\tau^j\zeta$. Since α has the explicit form given above, we have $\sigma^i\tau^j(\zeta\alpha)=\sigma^i\alpha\tau^j\zeta$. Hence we have $\mathfrak{o}_K=\bigoplus_{i=0}^{l-1}\bigoplus_{j=1}^{l-1}\boldsymbol{Z}\sigma^i\tau^j(\zeta\alpha)$ and this proves our corollary.

§3. Examples of K/k with or without n.i.b..

We can construct many examples of normal extensions K/k with n.i.b., $k\neq\boldsymbol{Q}$, using our theorem 3, its corollary and Hilbert-Speiser's theorem in the abelian extensions of $\boldsymbol{Q}$ on ground of the following lemma 3.

NOTATIONS. For an extension K/k, $d_{K/k}$, $D_{K/k}$ mean the discriminant and the different of K/k, respectively. Let K/k be of degree n. If $\alpha_1, \cdots, \alpha_n\in K$, $d_{K/k}(\alpha_1, \cdots, \alpha_n)$ denotes the discriminant of $\alpha_1, \cdots, \alpha_n$.

LEMMA 3. *Suppose that K_1/k is a Galois extension of degree n and K_2/k is an extension of degree m, where $K_1\cap K_2=k$. Let L be the composite field of K_1 and K_2. Suppose $(d_{K_1/k}, d_{K_2/k})=1$.*

(i) *If $\{\alpha_i\}_{i=1,\cdots,n}$ is a relative (normal) integral basis of K_1/k, then it is also a relative (normal) integral basis of L/K_2.*

(ii) *If $\{\alpha_i\}_{i=1,\cdots,n}$ and $\{\beta_j\}_{j=1,\cdots,m}$ are relative integral bases of K_1/k and K_2/k, then $\{\alpha_i\beta_j\}_{i=1,\cdots,n,j=1,\cdots,m}$ is a relative integral basis of L/k.*

PROOF. (ii) is well-known (Cf. Lang [3], Chapter III, Proposition 17). Through (i) seems also known, a proof of (i) will be given here, as no reference for it is known to the author.

As $(d_{K_1/k}, d_{K_2/k})=1$, we have $D_{K_1/k}=D_{L/K_2}$ (Cf. Lang [3], Chapter III, Proposition 17). Since K_1/k and L/K_2 are Galois extensions of degree n, $d_{K_1/k}=D^n_{K_1/k}$ and $d_{L/K_2}=D^n_{L/K_2}$. Hence $d_{K_1/k}=d_{L/K_2}$. By the hypothesis, $d_{K_1/k}=d_{K_1/k}(\alpha_1, \cdots, \alpha_n)$ (Mann [5], Theorem 1). Therefore $d_{L/K_2}=d_{L/K_2}(\alpha_1, \cdots, \alpha_n)$. Consequently $\{\alpha_i\}_{i=1,\cdots,n}$ is a relative integral basis of L/K_2 (Mann [5], Theorem 1 Corollary).

In the following proposition, suppose that l_i is an odd prime and $a_i(\neq\pm1)$ is a square-free rational integer such that $a_i\equiv\pm1 \bmod l_i^2$ and put $\alpha_i=(1/l_i)\sum_{j=0}^{l_i-1}(-\sqrt[l_i]{\varepsilon_i a_i})^j$, where

$$\varepsilon_i=\begin{cases}-1 & \text{if } a_i\equiv 1 \bmod l_i^2 \\ 1 & \text{if } a_i\equiv -1 \bmod l_i^2\end{cases} \quad (1\leqq i\leqq s)\,.$$

PROPOSITION 1. (I) *Let k be an abelian extension of $\boldsymbol{Q}$ whose conductor n is odd and square-free (i.e. $k/\boldsymbol{Q}$ is tamely ramified). Let K be a number field such that $(d_{K/\boldsymbol{Q}}, n)=1$. Then* $\mathrm{Tr}_{\boldsymbol{Q}(\zeta_n)/k}(\zeta_n)$ *generates an* n.i.b. *of the abelian extension kK/K.* ($\mathrm{Tr}_{\boldsymbol{Q}(\zeta_n)/k}(\zeta_n)$ *denotes the trace of ζ_n in $\boldsymbol{Q}(\zeta_n)/k$ and ζ_n is a primitive n-th root of unity.*)

(II) *Let k be as in* (I) *and $a_1, \cdots, a_s, l_1, \cdots, l_s$ be pairwise prime and suppose $(n, \prod_{i=1}^s a_i l_i)=1$. Then $\prod_{i=1}^s \alpha_i\cdot\zeta_{l_1\cdots l_s}\cdot\mathrm{Tr}_{\boldsymbol{Q}(\zeta_n)/k}(\zeta_n)$ generates an* n.i.b. *of the non-abelian extension $k(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{l_1\cdots l_s})/\boldsymbol{Q}$.*

(III) *Let k be a number field and $a_1, \cdots, a_s, l_1, \cdots, l_s$ be pairwise prime and suppose $(d_{k/\boldsymbol{Q}}, \prod_{i=1}^s a_i l_i)=1$. Then $\prod_{i=1}^s \alpha_i\cdot\zeta_{l_1\cdots l_s}$ generates an* n.i.b. *of the non-abelian extension $k(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{l_1\cdots l_s})/k$.*

(IV) *Let n be the product of all the distinct primes among $l_1, \cdots, l_s$. Let k be a number field which contains ζ_n and m be an odd and square-free integer. Suppose $a_1, \cdots, a_s$ are pairwise prime and $(l_i, a_j)=1$ $(1\leqq i, j\leqq s)$. Suppose $(m, n\prod_{i=1}^s a_i)=1$, $(d_{k/\boldsymbol{Q}(\zeta_n)}, m\prod_{i=1}^s a_i)=1$ and $\boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{mn})\cap k=\boldsymbol{Q}(\zeta_n)$. Then $\prod_{i=1}^s \alpha_i\zeta_m$ generates an* n.i.b. *of the abelian extension $k(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{mn})/k$.*

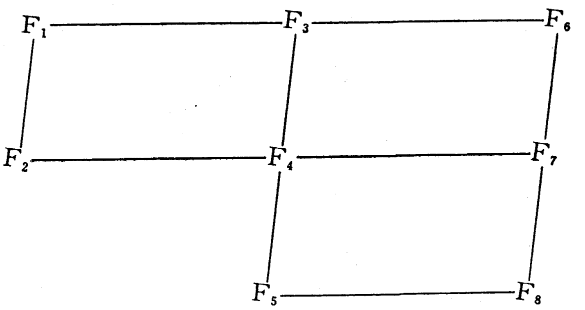
PROOF. (I) Since ζ_l generates an n.i.b. of $\boldsymbol{Q}(\zeta_l)/\boldsymbol{Q}$ (l: odd prime), ζ_n generates, by Lemma 3 (ii), an n.i.b. of $\boldsymbol{Q}(\zeta_n)/\boldsymbol{Q}$. Hence $\mathrm{Tr}_{\boldsymbol{Q}(\zeta_n)/k}(\zeta_n)$ generates an n.i.b. of $k/\boldsymbol{Q}$ ([4], Chapter 9, Theorem (3, 4)). As $(d_{K/\boldsymbol{Q}}, n)=1$, we have $K\cap k=\boldsymbol{Q}$. Therefore, by Lemma 3 (i), $\mathrm{Tr}_{\boldsymbol{Q}(\zeta_n)/k}(\zeta_n)$ generates an n.i.b. of kK/K.

(II) We note $d_{\boldsymbol{Q}(\zeta_{l_i}, \sqrt[l_i]{a_i})/\boldsymbol{Q}(\zeta_{l_i})}=(a_i^{l_i-1})$ by Okutsu's theorem. Hence only prime divisors of $a_i l_i$ ramify in $\boldsymbol{Q}(\zeta_{l_i}, \sqrt[l_i]{a_i})/\boldsymbol{Q}$. Therefore, since $a_1, \cdots, a_s, l_1, \cdots, l_s$ are pairwise prime, $\prod_{i=1}^s \alpha_i\zeta_{l_1\cdots l_s}$ generates, by Corollary of Theorem 3 and Lemma 3 (ii), an n.i.b. of $\boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{l_1\cdots l_s})/\boldsymbol{Q}$. As $(n, \prod_{i=1}^s a_i l_i)=1$, we have $k\cap\boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{l_1\cdots l_s})=\boldsymbol{Q}$. In (I), we have seen that $\mathrm{Tr}_{\boldsymbol{Q}(\zeta_n)/k}(\zeta_n)$ generates an n.i.b. of $k/\boldsymbol{Q}$. Consequently, by Lemma 3 (ii), $\prod_{i=1}^s \alpha_i\cdot\zeta_{l_1\cdots l_s}\cdot\mathrm{Tr}_{\boldsymbol{Q}(\zeta_n)/k}(\zeta_n)$ generates an n.i.b. of $k(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{l_1\cdots l_s})/\boldsymbol{Q}$.

(III) As $(d_{k/\boldsymbol{Q}}, \prod_{i=1}^s a_i l_i)=1$, we have $k\cap\boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{l_1\cdots l_s})=\boldsymbol{Q}$. Hence, using Lemma 3 (i) in place of Lemma 3 (ii) which is used in (II), we can show that $\prod_{i=1}^s \alpha_i\cdot\zeta_{l_1\cdots l_s}$ generates an n.i.b. of $k(\sqrt[l_1]{a_1}, \cdots,$

$\sqrt[l_s]{a_s}, \zeta_{l_1\cdots l_s})/k$.

(IV) In the first place, we shall show by induction in s that $\prod_{i=1}^{s} \alpha_i$ generates an n.i.b. of $\boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_n)/\boldsymbol{Q}(\zeta_n)$. Let n_r be the product of all the distinct primes among $l_1, \cdots, l_r$ $(1 \leqq r \leqq s)$. The case $s=1$ is just Theorem 3 $(n=n_s=l_1)$. To prove that $\prod_{i=1}^{s} \alpha_i$ generates an n.i.b. of $\boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{n_s})/\boldsymbol{Q}(\zeta_{n_s})$ for $s=r+1$ assuming it true for $s=r$, we put $F_1=\boldsymbol{Q}(\zeta_{n_r}, \sqrt[l_1]{a_1}, \cdots, \sqrt[l_r]{a_r})$, $F_2=\boldsymbol{Q}(\zeta_{n_r})$, $F_3=\boldsymbol{Q}(\zeta_{n_{r+1}}, \sqrt[l_1]{a_1}, \cdots, \sqrt[l_r]{a_r})$,



$F_4=\boldsymbol{Q}(\zeta_{n_{r+1}})$, $F_5=\boldsymbol{Q}(\zeta_{l_{r+1}})$, $F_6=\boldsymbol{Q}(\zeta_{n_{r+1}}, \sqrt[l_1]{a_1}, \cdots, \sqrt[l_{r+1}]{a_{r+1}})$, $F_7=\boldsymbol{Q}(\zeta_{n_{r+1}}, \sqrt[l_{r+1}]{a_{r+1}})$ and $F_8=\boldsymbol{Q}(\zeta_{l_{r+1}}, \sqrt[l_{r+1}]{a_{r+1}})$. If $l_{r+1}|n_r$, we have $n_{r+1}=n_r$, $F_3=F_1$ and $F_4=F_2$. Then, by the hypothesis of induction, $\prod_{i=1}^{s} \alpha_i$ generates an n.i.b. of F_3/F_4. If $l_{r+1}\nmid n_r$, we have $n_{r+1}=n_r l_{r+1}$. By Okutsu's theorem, prime ideals ramified in F_1/F_2 divide $\prod_{i=1}^{r} a_i$. And only prime divisors of l_{r+1} ramify in F_4/F_2. As $(l_{r+1}, \prod_{i=1}^{r} a_i)=1$ we have $(d_{F_1/F_2}, d_{F_4/F_2})=1$ and $F_1 \cap F_4=F_2$. Hence, by the hypothesis of induction and Lemma 3 (i), $\prod_{i=1}^{r} \alpha_i$ generates an n.i.b. of F_3/F_4. As $(n_{r+1}/l_{r+1}, a_{r+1})=1$, we have $(d_{F_4/F_5}, d_{F_8/F_5})=1$ and $F_4 \cap F_8=F_5$. Consequently, by Lemma 3(i), α_{r+1} generates an n.i.b. of F_7/F_4. Prime ideals ramified in F_3/F_4 divide $\prod_{i=1}^{r} a_i$ and prime ideals ramified in F_7/F_4 divide a_{r+1}. As $(\prod_{i=1}^{r} a_i, a_{r+1})=1$, we have $(d_{F_3/F_4}, d_{F_7/F_4})=1$ and $F_3 \cap F_7=F_4$. By Lemma 3 (ii), $\prod_{i=1}^{r+1} \alpha_i$ generates an n.i.b. of F_6/F_4. Thus, we have proved that $\prod_{i=1}^{s} \alpha_i$ generates an n.i.b. of $\boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_n)/\boldsymbol{Q}(\zeta_n)$. As $(m, n)=1$, ζ_m generates, by Lemma 3 (i), an n.i.b. of $\boldsymbol{Q}(\zeta_{mn})/\boldsymbol{Q}(\zeta_n)$. As $(\prod_{i=1}^{s} a_i, m)=1$, we have $\boldsymbol{Q}(\zeta_{mn}) \cap L=\boldsymbol{Q}(\zeta_n)$ and $(d_{\boldsymbol{Q}(\zeta_{mn})/\boldsymbol{Q}(\zeta_n)}, d_{L/\boldsymbol{Q}(\zeta_n)})=1$, where we put $L=\boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_n)$. Consequently, by Lemma 3 (ii), $\prod_{i=1}^{s} \alpha_i \zeta_m$ generates an n.i.b. of $\boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{mn})/\boldsymbol{Q}(\zeta_n)$. Since $(d_{k/\boldsymbol{Q}(\zeta_n)}, m\prod_{i=1}^{s} a_i)=1$ and $k \cap \boldsymbol{Q}(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{mn})=\boldsymbol{Q}(\zeta_n)$, $\prod_{i=1}^{s} \alpha_i \zeta_m$ generates, by Lemma 3 (i), an n.i.b. of $k(\sqrt[l_1]{a_1}, \cdots, \sqrt[l_s]{a_s}, \zeta_{mn})/k$. This proves our proposition.

In general, it is not easy to construct a Galois extension without n.i.b. by applying Theorem 2. For $l=3$, the unit group of quadratic field k is $\langle -1, \zeta_3 \rangle$ and no distinct elements of this group are pairwise congruent modulo 3. Consequently, we can check that $\boldsymbol{Q}(\zeta_3, \sqrt[3]{a})/\boldsymbol{Q}(\zeta_3)$ always has an n.i.b. ($a^2 \equiv 1 \bmod 9$).

The following proposition shows on the other hand that there are infinitely many tamely ramified extensions K/k ($k \neq \boldsymbol{Q}$) without n.i.b..

PROPOSITION 2. *Let m, n be square-free rational integers. Suppose that $m, n \equiv 3 \bmod 4$, $m < -1$, $n < 0$ and $(m, n) = 1$. Then $\boldsymbol{Q}(\sqrt{m}, \sqrt{n})/\boldsymbol{Q}(\sqrt{m})$ is a tamely ramified quadratic extension without n.i.b..*

PROOF. Put $K = \boldsymbol{Q}(\sqrt{m}, \sqrt{n})$ and $k = \boldsymbol{Q}(\sqrt{m})$. By the hypothesis, $\{1, (\sqrt{m} + \sqrt{n})/2\}$ is an $\mathfrak{o}_k$-basis of $\mathfrak{o}_K$ (Bird and Parry [1], Theorem I) and $\{1, \sqrt{m}, (\sqrt{m} + \sqrt{n})/2, (1 + \sqrt{mn})/2\}$ is $\boldsymbol{Z}$-basis of $\mathfrak{o}_K$ (Williams [11]). Let α be an element of $\mathfrak{o}_K$ and $\alpha = a + b\sqrt{m} + c(\sqrt{m} + \sqrt{n})/2 + d(1 + \sqrt{mn})/2$ ($a, b, c, d \in \boldsymbol{Z}$). Noting $\sqrt{m}\sqrt{n} = -\sqrt{mn}$, we obtain

$$\begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} = A \begin{pmatrix} 1 \\ (\sqrt{m} + \sqrt{n})/2 \end{pmatrix}, \quad A = \begin{pmatrix} a + b\sqrt{m} + d(1+m)/2 & c - d\sqrt{m} \\ a + (b+c)\sqrt{m} + d(1-m)/2 & -(c - d\sqrt{m}) \end{pmatrix},$$

where α' is the conjugate element of α in K/k. Hence, we have

$$\det A = -(c - d\sqrt{m})\{(2a+d) + (2b+c)\sqrt{m}\}.$$

α generates an n.i.b. of K/k if and only if $\det A \in \mathfrak{o}_k^{\times}$, i.e. if and only if there exist $a, b, c, d \in \boldsymbol{Z}$ such that

$$(9) \qquad (2a+d)^2 - m(2b+c)^2 = \pm 1$$

$$(10) \qquad c^2 - md^2 = \pm 1 .$$

Since $-m > 1$, we have $2a + d = \pm 1$, $2b + c = 0$, $c = \pm 1$ and $d = 0$ from (9), (10). Therefore we obtain $2a = \pm 1$. So the simultaneous Diophantine equation (9), (10) has no solution and K/k has no n.i.b.. Since 2 is unramified in $\boldsymbol{Q}(\sqrt{mn})/\boldsymbol{Q}$, K/k is tamely ramified. This proves our assertion.

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