TOKYO J. MATH.
VOL. 7, NO. 1, 1984

Uniserial Rings and Skew Polynomial Rings

Takashi MANO

Sophia University

(Communicated by Y. Kawada)

Introduction

The purpose of this paper is to study the structure of uniserial rings and to generalize the results of E.-A. Behrens [1]. A left and right Artinian ring R is called *uniserial* if it is primary decomposable and for each primitive idempotent $e\in R$, Re as well as eR has a unique composition series. Every uniserial ring is Morita equivalent to a finite direct product of local uniserial rings. A local uniserial ring R will be called of *split type* (or cleft) if there exists a subring S of R such that $R=S+\mathrm{Rad}(R)$ and $S\cap \mathrm{Rad}(R)=0$. Let D be a division ring and $\tau\in \mathrm{Aut}(D)$. A factor ring $D[X;\tau]/(X^c)$ of a skew polynomial ring $D[X;\tau]$ is a local uniserial ring of split type, but the converse does not hold in general (cf. Example in §2). In [1], E.-A. Behrens has given a sufficient condition for a local uniserial ring of split type to be a factor ring of an ordinary polynomial ring. Our main theorem states a necessary and sufficient condition for a local uniserial ring to be isomorphic to a factor ring of a skew polynomial ring over a division ring, and the result of Behrens mentioned above is obtained from our theorem as a corollary.

§1. Preliminaries.

Throughout this paper, all rings have identity elements and all subrings have the same identity elements. Let A be a ring. We will denote the Jacobson radical of A, the center of A, and the unit group of A by $\mathrm{Rad}(A)$, $Z(A)$ and $U(A)$, respectively. For a right A-module M_A, $c(M_A)$ denotes the composition length of M_A.

In the latter part of this section, R denotes a local uniserial ring. Let $J=\mathrm{Rad}(R)$ and $c=c(R_R)$. Then

Received April 25, 1983
Revised October 19, 1983

$$R \supset J \supset J^2 \supset \cdots \supset J^{c-1} \supset J^c = 0$$

is the unique composition series of R_R. We assume that R is of split type. Then there exists a subring D of R such that $R = D + J$ and $D \cap J = 0$. Since $D \cong R/J$, D is a division ring. In the case that $J = 0$, we will regard R as a ring of split type.

LEMMA 1. *Let* $w \in J \backslash J^2$. *Then*

(i) $\{1, w, w^2, \cdots, w^{c-1}\}$ *is a right and left linearly independent set over* D.

(ii) *For any* k, *we have*

$$J_D^k = w^k D \oplus w^{k+1} D \oplus \cdots \oplus w^{c-1} D ,$$
$$_D J^k = D w^k \oplus D w^{k+1} \oplus \cdots \oplus D w^{c-1} .$$

PROOF. (i) Let $a_0, \cdots, a_{c-1} \in D$ and assume that $\sum_{i=0}^{c-1} w^i a_i = 0$. If there exists a non-zero coefficient a_i, then there exists an integer k such that $a_0 = \cdots = a_{k-1} = 0$, $a_k \neq 0$ and $k < c-1$. Then we have $w^k(a_k + w a_{k+1} + \cdots + w^{c-k-1} a_{c-1}) = 0$. Since R is local uniserial and $a_k \neq 0$, we have $w^k = 0$. This contradicts to $k < c-1$.

(ii) We shall prove (ii) by the induction on k. In the case that $k = c-1$, we have $J^{c-1} = w^{c-1} R = w^{c-1}(D+J) = w^{c-1} D$. Let $k < c-1$ and assume that $J^{k+1} = w^{k+1} D \oplus \cdots \oplus w^{c-1} D$. Then

$$\begin{aligned} J^k &= w^k R = w^k (D+J) = w^k D + J^{k+1} \\ &= w^k D + w^{k+1} D + \cdots + w^{c-1} D \\ &= w^k D \oplus w^{k+1} D \oplus \cdots \oplus w^{c-1} D \end{aligned}$$

from (i). □

§2. The Main Theorem.

Let A be a ring and $\tau \in \mathrm{Aut}(A)$. By $A[X; \tau]$, we shall denote the skew polynomial ring over A, i.e., $A[X; \tau]$ is the set of all polynomials $\sum X^i a_i$ and the multiplication is defined by the formula $aX = X\tau(a)$ for $a \in A$. For $u \in U(A)$, ι_u denotes the inner automorphism of A by u; $\iota_u(a) = uau^{-1}$ for all $a \in A$.

Throughout this section, the following notation will be fixed. Let R be a local uniserial ring with the radical J. Let $c = c(R_R)$ and $w \in J \backslash J^2$. Then we have $J = wR = Rw$. Hence for each $r \in R$, there exists $\sigma(r) \in R$ such that

(1) $$rw = w\sigma(r) .$$

$\sigma(r)$ is not uniquely determined by r, but we shall fix one element of R satisfying (1). Let $\pi: R \to R/J^{c-1}$ be the natural ring homomorphism. Since $J \cdot J^{c-1} = J^{c-1} \cdot J = 0$, it is easy to prove that the function $\pi \circ \sigma: R \to R/J^{c-1}$ is an onto ring homomorphism with the kernel J^{c-1}. Hence σ defines the automorphism $\bar{\sigma}$ of R/J^{c-1}. For each $r \in R$, we shall denote $\bar{r} = \pi(r) \in R/J^{c-1}$.

The following theorem is the main result of this paper.

THEOREM 2. *The following conditions for a local uniserial ring R are equivalent*:

(a) *There exists $\tau \in \mathrm{Aut}(R/J)$ such that*

$$R \cong (R/J)[X; \tau]/(X^c) .$$

(b) *There exist a subring D of R and $u \in U(R)$ satisfying the following conditions*;

$$\text{(i)} \qquad R = D + J \quad \textit{and} \quad D \cap J = 0 ,$$

$$\text{(ii)} \qquad \bar{u}^{-1} \bar{D} \bar{u} = \bar{\sigma}(\bar{D}) .$$

PROOF. We have only to prove (b)$\Rightarrow$(a). Assume (b). Let us put $w_1 = wu^{-1}$ and $\tau_1 = \iota_u \circ \sigma: R \to R$. Then

$$rw_1 = rwu^{-1} = w\sigma(r)u^{-1} = w_1\tau_1(r) \quad \text{for all} \quad r \in R$$

and $\overline{\tau_1(D)} = \bar{D}$. Since $\pi \circ \tau_1 = \iota_{\bar{u}} \circ \pi \circ \sigma$, $\pi \circ \tau_1: R \to R/J^{c-1}$ is an onto ring homomorphism. Hence it induces the ring automorphism $\bar{\tau}_1$ of R/J^{c-1}. Put $\tau = \bar{\tau}_1|_{\bar{D}}$. Then τ is a ring automorphism of $\bar{D}$. Put $S = \bar{D}[X; \tau]/(X^c)$. Since

$$R_D = D \oplus w_1 D \oplus w_1^2 D \oplus \cdots \oplus w_1^{c-1} D$$

by Lemma 1 and

$$S_{\bar{D}} = \bar{D} \oplus x\bar{D} \oplus x^2\bar{D} \oplus \cdots \oplus x^{c-1}\bar{D}$$

where $x = X + (X^c) \in S$, we can define a map $\Phi: R \to S$ by

$$\Phi: R \ni \sum_{i=0}^{c-1} w_1^i a_i \longmapsto \sum_{i=0}^{c-1} x^i \bar{a}_i \in S .$$

Since $\pi|_D: D \to \bar{D}$ is a ring isomorphism, it is easy to prove that Φ is an additive isomorphism. Let $w_1^i a, w_1^j b \in R$. Since $\overline{\tau_1^j(a)} \in \bar{D}$, there uniquely exists $a' \in D$ such that $\overline{a'} = \overline{\tau_1^j(a)}$. Then we have

$$\Phi(w_1^i a w_1^j b)=\Phi(w_1^{i+j}\tau_1^j(a)b)=\Phi(w_1^{i+j}a'b)=x^{i+j}\overline{a'b}$$
$$=x^{i+j}\overline{\tau_1^j(a)b}=x^{i+j}\tau^j(\bar{a})\bar{b}=\Phi(w_1^i a)\cdot\Phi(w_1^j b)\ .$$

Thus Φ is a ring isomorphism. □

Let us proceed the applications of Theorem 2. Several known results will be obtained as the corollaries of Theorem 2 (cf. Corollaries 4, 5 and 6). The notations and the assumptions are as above. Furthermore, we shall assume that R is of split type. Then there exists a division subring D of R such that $R=D+J$ and $D\cap J=0$. Let us put $Z=D\cap Z(R)$.

COROLLARY 3. *If D is a separable Z-algebra, then $R\cong(R/J)[X;\tau]/(X^c)$ for some $\tau\in\mathrm{Aut}(R/J)$.* (As for separable algebras, cf. [3, §71].)

PROOF. From Wedderburn-Malcev Theorem (cf. [3, Theorem 72.19]), the condition (b) in Theorem 2 is satisfied. Thus Corollary 3 holds. □

The following Corollary 4 is immediately obtained from Corollary 3 since a skew polynomial ring $A[X;\tau]$ is an ordinary polynomial ring $A[Y]$ if τ is an inner automorphism.

COROLLARY 4 (E.-A. Behrens [1]). *If D is a separable Z-algebra and if any Z-automorphism of D is inner, then $R\cong(R/J)[X]/(X^c)$.* □

COROLLARY 5 (I. S. Cohen). *If R is commutative, then $R=D[X]/(X^c)$.*

PROOF. It is obvious since σ is taken to be the identity map on R. □

COROLLARY 6 (W. A. Clark and D. A. Drake [2]). *If R is a finite ring, then $R\cong\boldsymbol{F}_q[X;\tau]/(X^c)$ for some $\tau\in\mathrm{Aut}(\boldsymbol{F}_q)$, where $q=\sharp(R/J)$ and $\boldsymbol{F}_q$ is the finite field with q elements.*

PROOF. Since $R/J\cong\boldsymbol{F}_q$ and $\boldsymbol{F}_q$ is a separable algebra over its prime subfield, the assertion is directly proved from Corollary 3. □

The following result is a generalization of a result of E.-A. Behrens [1].

COROLLARY 7. *If $J^2=0$, then $R\cong(R/J)[X;\tau]/(X^c)$ for some $\tau\in\mathrm{Aut}(R/J)$.*

PROOF. Assume $J\neq0$. Since $c-1=1$, we have $\bar{\sigma}\in\mathrm{Aut}(R/J)$. Moreover $\bar{D}=\bar{\sigma}(\bar{D})$. Thus the assertion is directly proved from Theorem 2. □

In the case that $J^3=0$, the following example which is given by

E.-A. Behrens [1] shows that there exists a local uniserial ring of split type which is not isomorphic to a factor ring of any skew polynomial ring over a division ring.

EXAMPLE. Let D be a division ring with a derivation $\alpha: D\to D$ which is not inner. Put $R=D\oplus D\oplus D$. Then R is an additive group. For $(a_0, a_1, a_2), (b_0, b_1, b_2)\in R$, define

$$(a_0, a_1, a_2)\cdot(b_0, b_1, b_2) = (a_0b_0,\ a_1b_0+a_0b_1,\ a_2b_0+a_1b_1+a_0b_2+\alpha(a_0)b_1)\ .$$

Then R is a local uniserial ring of split type. Moreover, it is not difficult to prove that R does not satisfy the condition (b) in Theorem 2. Thus R is not isomorphic to a factor ring of a skew polynomial ring over a division ring.

References

[1] E.-A. BEHRENS, Einreihige Ringe, Math. Z., **77** (1961), 207-211.
[2] W. E. CLARK and D. A. DRAKE, Finite chain rings, Abh. Math. Sem. Univ. Hamburg, **39** (1973), 147-163.
[3] C. W. CURTIS and I. REINER, Representation Theory of Finite Groups and Associative Algebras, Interscience, New York, 1962.
[4] G. KÖTHE, Verallgemeinerte Abelsche Gruppen mit Hyperkomplexem Operatorenring, Math. Z., **39** (1935), 31-44.
[5] K. MORITA, Duality for modules and its applications to the theory of rings with minimum condition, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, **6** (1958), 83-142.
[6] T. NAKAYAMA, Note on uni-serial and generalized uni-serial rings, Proc. Imp. Acad. Japan, **16** (1940), 285-289.

Present Address:
FUJISAWA DEVELOPMENT LABORATORY
IBH JAPAN
KIRIHARA-CHO, FUJISAWA-SHI,
KANAGAWA 252