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A Note on the Differential of the Exponential Map and Jacobi Fields in a Symmetric Space

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It was remarked by Rauch [4] that the differential equation for Jacobi fields in a symmetric space has constant coefficients with respect to suitable moving frames along a geodesic. Using a slightly different method, we give here an explicit form of a Jacobi field in a coordinate free form.

We also give, in the Lemma in §1, a relationship between the differential of the exponential map and Jacobi fields. This provides an alternate proof of the well known formula (Helgason [1], Chap. IV, Theorem 4.1) on the differential of the exponential map in a symmetric space.

§1. Let M be a manifold with a torsion-free affine connection ∇, and consider $\exp=\exp_x\colon T_xM\to M$, $x\in M$, and let $d\exp_X\colon T_xM\to T_{\exp X}M$, $X\in T_xM$, be its differential at X. Furthermore write $\gamma(t)=\exp tX$, and set

$$\mathscr{J}=\{J\colon \text{a Jacobi field along } \gamma,\ J(0)=0\}\ .$$

Then we have the isomorphism:

$$\phi\colon T_xM\to\mathscr{J}$$

where, for $Y\in T_xM$, $\phi(Y)=J$ is defined by $J(0)=0$ and $\nabla J/dt(0)=Y$.

LEMMA. *The following diagram is commutative*:

$$\begin{array}{ccc} T_xM & \xrightarrow{d\exp_{tX}} & T_{\exp tX}M \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle t\cdot} \\ \mathscr{J} & \xrightarrow{ev} & T_{\exp tX}M \end{array}$$

where the "bottom map" is the evaluation map sending J to $J(t)$ and

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the "vertical map" on the right is the scalar multiplication by t.

PROOF. For $Y\in T_xM$, set

$$\alpha(u, t)=\exp t(X+uY)\ .$$

We denote by $\partial\alpha/\partial u$ the tangent vector of "u-curves" and by $\nabla/\partial u$ the covariant differentiation along them, and similarly for "t-curves". Since, for a fixed t, $Y(u)=t(X+uY)$ defines a curve in T_xM such that $Y(0)=tX$, $Y'(0)=tY$, we have

$$\frac{\partial\alpha}{\partial u}(0, t)=td\exp_{tX}(Y)\ .$$

On the other hand, writing

$$\left.\frac{\partial\alpha}{\partial u}\right|_{u=0}=J$$

we get, for the covariant derivative along γ,

$$\begin{aligned}\left.\frac{\nabla}{dt}\right|_{t=0}J&=\left.\frac{\nabla}{\partial t}\frac{\partial\alpha}{\partial u}\right|_{t=0,u=0}\\&=\left.\frac{\nabla}{\partial u}\frac{\partial\alpha}{\partial t}\right|_{t=0,u=0}\\&=\left.\frac{\nabla}{du}(X+uY)\right|_{u=0}\\&=Y\ .\end{aligned}$$

Note that the second equality holds because ∇ is assumed to be torsion free. q.e.d.

§2. Let $M=G/K$ be an affine symmetric space, σ the associated involutive automorphism of G, $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ the decomposition of the Lie algebra into the eigenspaces of σ, $\pi\colon G\to G/K$ the natural projection, and let $\pi(e)=o$ (e is the unit of G). As usual we identify $\mathfrak{m}$ with T_oM via $d\pi\colon\mathfrak{m}\to T_oM$. Furthermore put $x_t=\exp tX$ ($X\in\mathfrak{m}$), and put $\gamma(t)=\tau_{x_t}o=\pi(\exp tX)$, where τ_x is the action of $x\in G$ on M. On the other hand M carries the canonical connection ∇ and it is well-known that the parallel translation $\tau_t^{t'}$ along γ is given by τ_{x_t}, i.e.,

$$\tau_t^{t'}=d\tau_{x_{-(t'-t)}}\colon T_{\gamma(t')}M\to T_{\gamma(t)}M$$

in particular

(*) $$\gamma(t)=\pi(\exp tX) \text{ is a geodesic for } X\in \mathfrak{m}\,.$$

PROPOSITION. *The Jacobi field J along γ with the initial condition $J(0)=0$, $\nabla J/dt(0)=Y\in\mathfrak{m}$ is given by*

$$J(t)=\tau_{x_t}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}T_X{}^n(Y)$$

where $T_X=(\operatorname{ad}X)^2\colon \mathfrak{m}\to\mathfrak{m}$.

PROOF. For a vector field J along γ set

$$\bar J(t)=\tau_0^t J(t)\in\mathfrak{m}$$

so that $\bar J$ is an $\mathfrak{m}$-valued function in t, and we have

$$\frac{\nabla J}{dt}=\tau_t^0\frac{d\bar J}{dt}\,.$$

Now the Jacobi equation for J is given by

$$\frac{\nabla^2 J}{dt^2}+R\Big(J,\frac{d\gamma}{dt}\Big)\frac{d\gamma}{dt}=0\,.$$

Since R is parallel, applying τ_0^t to the above equation we get

$$\frac{d^2\bar J}{dt^2}+R_o(\bar J,X)X=0\,.$$

But it is well-known that

$$R_o(X,Y)=-\operatorname{ad}[X,Y] \qquad X,Y\in\mathfrak{m}\,.$$

Hence

(#) $$\frac{d^2\bar J}{dt^2}-T_X(\bar J)=0 \quad\text{or}$$

$$\frac{d}{dt}\begin{pmatrix}\bar J\\ \bar J'\end{pmatrix}=\begin{pmatrix}0 & 1\\ T_X & 0\end{pmatrix}\begin{pmatrix}\bar J\\ \bar J'\end{pmatrix} \quad\text{with}\quad \bar J'=\frac{d\bar J}{dt}\,.$$

Thus, setting $S_X=\begin{pmatrix}0 & 1\\ T_X & 0\end{pmatrix}$ we get

$$\begin{pmatrix}\bar J\\ \bar J'\end{pmatrix}=\exp tS_X\begin{pmatrix}0\\ Y\end{pmatrix}\,. \qquad\text{q.e.d.}$$

In view of the Lemma, the following theorem is an immediate consequence of the Proposition:

Theorem. *For* $X \in T_o(M) = \mathfrak{m}$ *and* $\exp: \mathfrak{m} \to G/K$ *we have*

$$d\exp_X = \tau_{\exp X} \circ \sum_{n=0}^{\infty} \frac{T_X{}^n}{(2n+1)!}$$

Corollary. *For a Lie group* G, $X \in \mathfrak{g}$, *and* $\exp: \mathfrak{g} \to G$, *we have*

$$d\exp_X = dL_{\exp X} \circ \frac{1 - e^{-\mathrm{ad} X}}{\mathrm{ad}\, X}$$

where L_x *is the left translation by* $x \in G$.

Proof. We put, as usual, $G^* = G \times G$, $\sigma(x, y) = (y, x)$, $K^* = \{(x, x): x \in G\}$ and apply the Theorem to the pair (G^*, K^*). We have the following identifications:

$$\theta: M^* = G^*/K^* \longrightarrow G \qquad (x, y)K^* \longmapsto xy^{-1}$$

and setting $\mathfrak{m}^* = \{(X, -X): X \in \mathfrak{g}\}$

$$\theta: T_o M^* = \mathfrak{m}^* \longrightarrow \mathfrak{g} \qquad X^* = (X, -X) \longmapsto 2X\,.$$

The diagram below is commutative because of (*),

$$\begin{array}{ccc} T_o M^* = \mathfrak{m}^* & \xrightarrow{\exp} & G^*/K^* \\ \theta \downarrow & & \downarrow \theta \\ \mathfrak{g} & \xrightarrow{\exp} & G \end{array}$$

thus, so is the left square of the following diagram.

$$\begin{array}{ccccc} \mathfrak{m}^* & \xrightarrow{d\exp_{X^*}} & T_{\exp X^*}(G^*/K^*) & \xleftarrow{d\tau_{\exp X^*}} & \mathfrak{m}^* \\ \theta \downarrow & & \theta \downarrow & & \theta \downarrow \\ \mathfrak{g} & \xrightarrow{d\exp_{2X}} & T_{\exp 2X} G & \xleftarrow{dL_{\exp X} \circ dR_{\exp X}} & \mathfrak{g} \end{array}$$

The right half is also commutative, because the action of $(a, b) \in G^*$ on G^*/K^* corresponds, via θ, to $L_a \circ R_b^{-1}$. (R_x is the right multiplication by x.) Writing $\Phi = d\tau_{\exp X^*}{}^{-1} \circ d\exp_{X^*}$ and $\Psi = (dL_{\exp X} \circ dR_{\exp X})^{-1} \circ d\exp_{2X} = e^{\mathrm{ad} X} \circ dL_{\exp 2X}{}^{-1} \circ d\exp_{2X}$, we have shown that, for $Y^* = (Y, -Y) \in \mathfrak{m}^*$,

$$\Phi(Y^*) = \sum_{n=0}^{\infty} \frac{\mathrm{ad}(X, -X)^{2n}}{(2n+1)!}(Y, -Y)\,.$$

Hence,

$$\begin{aligned} dL_{\exp(-2X)} \circ d\exp_{2X}(2Y) &= e^{-\operatorname{ad}X} \circ \Psi(2Y) \\ &= e^{-\operatorname{ad}X}\left(2\sum_{n=0}^{\infty} \frac{(\operatorname{ad}X)^{2n}}{(2n+1)!}(Y)\right) \\ &= e^{-\operatorname{ad}X} \frac{e^{\operatorname{ad}X} - e^{-\operatorname{ad}X}}{\operatorname{ad}X}(Y) \\ &= \frac{1}{\operatorname{ad}(2X)}(1 - e^{-\operatorname{ad}(2X)})(2Y) \,. \end{aligned}$$

q.e.d.

References

[1] S. HELGASON, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.
[2] S. KOBAYASHI and K. NOMIZU, Foundations of Differential Geometry, Interscience, New York, 1963.
[3] J. MILNOR, Morse Theory, Ann. Math. Studies 51, Princeton University Press, 1963.
[4] H. RAUCH, The global study of geodesics in symmetric and nealy symmetric Riemannian manifolds, Proc. U.S.-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965, 115–127.

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