TOKYO J. MATH.
VOL. 7, NO. 1, 1984

Boundary Regularity for Minima of Certain Variational Integrals

Atsushi TACHIKAWA

Keio University
(Communicated by T. Saito)

Introduction

Let Ω be a bounded open domain of $\boldsymbol{R}^n$, $n\geqq 2$, with boundary $\partial\Omega$ of class C^2, Γ_1 a relatively open subset, of $\partial\Omega$, and $\Gamma_0=\partial\Omega-\Gamma_1$. We consider the variational integral

$$F(u):=\int_\Omega f(x, u, Du)dx ,$$

for a function $u: \Omega\to\boldsymbol{R}^N$, where $Du=(\partial u^i/\partial x^\alpha)_{1\leq i\leq N,1\leq\alpha\leq n}$, and $f(x, u, \xi)$: $\Omega\times\boldsymbol{R}^N\times\boldsymbol{R}^{nN}\to\boldsymbol{R}$ is a Carathéodory function; i.e. measurable in x for each $(u, \xi)\in\boldsymbol{R}^N\times\boldsymbol{R}^{nN}$, and continuous in (u, ξ) for almost every $x\in\Omega$.

In this paper we consider the following variational problem with mixed boundary condition:

$$(*)\qquad\begin{cases}\text{Find a minimizing function } u: \Omega\to\boldsymbol{R}^N \text{ of } F(u) \text{ which maps } \Gamma_1\\ \text{into some hyperplane } \Sigma:=\{v\in\boldsymbol{R}^N: v^{s+1}=\cdots=v^N=0\} \text{ and has}\\ \text{prescribed Dirichlet data } \phi \text{ on } \Gamma_0, \text{ where } \phi(\Gamma_0\cap\bar{\Gamma}_1)\subset\Sigma.\end{cases}$$

(See [1] for the mixed boundary problem for harmonic maps.)

In the paper [4], M. Giaquinta and E. Giusti prove interior regularity of minima of variational integrals (see also [5]). On boundary regularity for Dirichlet problem a result due to J. Jost and M. Meier [8] is known.

In this paper we investigate the behavior of the solution of (*) near Γ_1.

§1. L^p-estimate for the gradient.

We suppose that the function f satisfies the growth condition:

$$(1.1)\qquad a|\xi|^m-k\leqq f(x, u, \xi)\leqq b|\xi|^m+k ,$$

Received April 18, 1983
Revised November 21, 1983

in $\Omega\times\boldsymbol{R}^N\times\boldsymbol{R}^{nN}$, for some $m\geqq 2$, $k\geqq 0$ and $b\geqq a>0$.

For convenience we define the following function class:

$$\begin{aligned}V^m&=V^m(\Omega,\Gamma_1)\\&=\{v\in H^{1,m}(\Omega,\boldsymbol{R});\ v=0 \text{ on } \Gamma_0,\ v(\Gamma_1)\subset\Sigma\}\,.\end{aligned}$$

Then the problem (*) can be rewritten in the following way:

$$(**)\qquad\begin{cases}\text{Find a function } u\in H^{1,m}(\Omega,\boldsymbol{R}^N) \text{ such that}\\ \quad u=\phi \text{ on } \Gamma_0,\\ \quad u(\Gamma_1)\subset\Sigma,\\ \quad F(u)\leqq F(u+v) \text{ for every } v\in V^m.\end{cases}$$

THEOREM 1.1. *Let f satisfy* (1.1) *and let $u\in H^{1,m}(\Omega,\boldsymbol{R}^N)$ be a solution of* (**). *Then there exists an exponent $p>m$ such that $u\in H^{1,p}(\Omega,\boldsymbol{R}^N)$. Moreover for every $x_0\in\Omega\cup\Gamma_1$, $R<\operatorname{dist}(x_0,\Gamma_0)$, writing $⨍_D g\,dx=(1/|D|)\int_D g\,dx$ the following inequarity holds*:

$$\left(⨍_{B_{R/2}(x_0)\cap\Omega}(1+|Du|)^p dx\right)^{1/p}\leqq C_1\left(⨍_{B_R(x_0)\cap\Omega}(1+|Du|)^m dx\right)^{1/m},$$

C_1 being a constant depending only on a, b, k, N, m and n.

PROOF. Let $x_0\in\Omega\cup\Gamma_1$, $0<R<\operatorname{dist}(x_0,\Gamma_0)$. For convenience we extend functions u and Du to the whole $\boldsymbol{R}^n$ in such a way that they are zero outside Ω, and we write B_R for $B_R(x_0)$.

Let us treat two cases, 1) $\operatorname{dist}(x_0,\Gamma_1)>3R/4$ and 2) $\operatorname{dist}(x_0,\Gamma_1)\leqq 3R/4$ separately.

Case 1. Let $\operatorname{dist}(x_0,\Gamma_1)>3R/4$. Then $\operatorname{dist}(x_0,\partial\Omega)>3R/4$, and hence we can proceed as in the proof of Theorem 4.1. of [4], and get for $l=mn/(m+n)$

$$\begin{aligned}(1.2)\qquad ⨍_{B_{R/2}}(1+|Du|)^m dx&\leqq\gamma_1\left(⨍_{B_{3R/4}}(1+|Du|)^l dx\right)^{m/l}\\&\leqq\gamma_1(4/3)^{mn/l}\left(⨍_{B_R}(1+|Du|)^l dx\right)^{m/l}.\end{aligned}$$

Case 2. Let $\operatorname{dist}(x_0,\Gamma_1)\leqq 3R/4$, and $0<t<r<R$, and η be a C^∞-function with $\operatorname{supp}\eta\subset B_r$, $0\leqq\eta\leqq 1$, $\eta=1$ on B_t, $|D\eta|\leqq 2/(r-t)$. Put $u_R^i=⨍_{B_R}u^i dx$ and

$$\bar{u}_R^i=\begin{cases}u_R^i & \text{for } 1\leqq i\leqq s\,,\\ 0 & \text{for } s+1\leqq i\leqq N\,.\end{cases}$$

If we put $v=u-\eta(u-\bar{u}_R)$, then $u-v\in V^m$, and hence from the minimality of u and (1.1) we get

$$\int_{B_r}|Du|^m dx\leqq\gamma_2\left\{\int_{B_r-B_t}|Du|^m dx+\left(\frac{2}{r-t}\right)^m\int_{B_r}|u-\bar{u}_R|^m dx+|Br|\right\}.$$

By the hole-filling method (cf. [6]) we obtain

$$\int_{B_{R/2}}|Du|^m dx\leqq\gamma_3\left\{R^{-m}\int_{B_R}|u-\bar{u}_R|^m dx+|B_R|\right\}.$$

Since dist $(x_0, \Gamma_1)\leqq 3R/4$, we can use the Sobolev-Poincarè inequality for $u-\bar{u}_R$ to get

$$(1.3)\qquad \fint_{B_{R/2}}(1+|Du|)^m dx\leqq\gamma_4\left(\fint_{B_R}(1+|Du|)^l dx\right)^{m/l},\qquad l=\frac{mn}{(m+n)}.$$

From (1.2) and (1.3) we get for all $x_0\in\Omega\cup\Gamma_1$, $0<R<\text{dist}\,(x_0, \Gamma_0)$

$$\fint_{B_{R/2}(x_0)}(1+|Du|)^m dx\leqq\gamma\left(\fint_{B_R(x_0)}(1+|Du|)^l dx\right)^{m/l},$$

where $\gamma=\max\{\gamma_4, (4/3)^{mn/l}\gamma_1\}$.

Theorem 1.1 now follows from Proposition 5.1 of [6].

§2. Quadratic functionals.

In this section we shall prove some regularity results for minima of quadratic functional

$$(2.1)\qquad F(u):=\int_\Omega\sum_{i=1}^N A^{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i dx,\qquad A^{\alpha\beta}=A^{\beta\alpha}.$$

We assume that the coefficients $A^{\alpha\beta}$ are bounded continuous functions in $\Omega\times\boldsymbol{R}^N$ and satisfy the condition

$$(2.2)\qquad A^{\alpha\beta}(x, u)\xi_\alpha\xi_\beta\geqq\lambda|\xi|^2\qquad\forall\xi\in\boldsymbol{R}^n,\qquad\lambda>0.$$

Moreover we assume that there exists a continuous, increasing, concave function $\omega\colon\boldsymbol{R}^+\to\boldsymbol{R}^+$ satisfying $\omega(0)=0$, $\omega(t)\leqq M$, and

$$(2.3)\qquad |A^{\alpha\beta}(x, u)-A^{\alpha\beta}(y, v)|\leqq\omega(|x-y|^2+|u-v|^2).$$

THEOREM 2.1. *Under the same hypotheses as above, let* $u\in H^{1,2}(\Omega, \boldsymbol{R}^N)$ *be a solution of* (**). *Then there exists a relatively open subset* $\Omega_0\subset\Omega\cup\Gamma_1$ *such that* $u\in C^{0,\alpha}(\Omega_0, \boldsymbol{R}^N)$ *for some* $\alpha\in(0, 1)$. *Moreover*

$$(\Omega \cup \Gamma_1)-\Omega_0=\left\{x \in \Omega \cup \Gamma_1 \colon \liminf_{R\to 0} R^{2-n}\int_{B_R(x)\cap\Omega} |Du|^2 dx > \varepsilon_0\right\} ,$$

where ε_0 is a positive constant independent of u.

PROOF. On account of Theorem 5.1 of [4], we have only to investigate the behavior of u in a neighborhood of Γ_1. Let $x_0 \in \Gamma_1$, $R<(1/2)\,\mathrm{dist}\,(x_0, \Gamma_0)$ and choose coordinates such that $x_0=0$, $\Gamma_1 \cap B_{2R}(x_0) \subset \{x \in \boldsymbol{R}^n \colon x^n=0\}$.

We use the following notations: for $x \in \boldsymbol{R}^n$, $x_*=(x_*^1, \cdots, x_*^n)=(x^1, \cdots, x^{n-1}, -x^n)$ and

$$\bar{u}(x)=\begin{cases} u(x) & \text{if } x \in \Omega ,\\ u(x_*) & \text{if } x \in B_{2R}(0)-\Omega ,\end{cases}$$

$$\sigma^{\alpha\beta}=\begin{cases} 1 & \text{if } 1\leqq\alpha, \beta\leqq n-1 ,\\ -1 & \text{if } \alpha=n \text{ or } \beta=n \text{ and } \alpha\neq\beta ,\\ 1 & \text{if } \alpha=\beta=n ,\end{cases}$$

$$\bar{A}^{\alpha\beta}(x)=\begin{cases} A^{\alpha\beta}(0, u_R) & \text{if } x \in \Omega ,\\ \sigma^{\alpha\beta}A^{\alpha\beta}(0, u_R) & \text{if } x \in B_{2R}-\Omega \end{cases}$$

where $u_R=\int_{B_R(0)} u dx$. Let $v \in H^{1,2}(B_R(0)\cap\Omega, \boldsymbol{R}^N)$ be a solution of the problem

$$\begin{cases} \int_{B_R(0)\cap\Omega} A^{\alpha\beta}(0, u_R)D_\alpha v^i D_\beta \psi^i dx=0 & \text{for all } \psi \in V' \\ u-v \in V' ,\end{cases}$$

where

$$V'=\{v \in H^{1,2}(B_R(0)\cap\Omega, \boldsymbol{R}^N) \colon v=0 \text{ on } \partial B_R(0)\cap\Omega,\ v(\partial\Omega\cap B_R)\subset\Sigma\} .$$

Then for $1\leqq i\leqq s$, $\bar{v}^i$ are solutions problems

$$\begin{cases} \int_{B_R} \bar{A}^{\alpha\beta}(x)D_\alpha \bar{v}^i D_\beta \psi dx=0 & \text{for all } \psi \in H_0^{1,2}(B_R(0)) ,\\ \bar{v}^i-\bar{u}^i \in H_0^{1,2}(B_R(0)) ,\end{cases}$$

and for $s+1\leqq i\leqq N$, v^i are solutions of the problems

$$\begin{cases} \int_{B_R\cap\Omega} A^{\alpha\beta}(0, u_R)D_\alpha v^i D_\beta \psi dx=0 & \text{for all } \psi \in H_0^{1,2}(B_R\cap\Omega) ,\\ v^i=u^i \text{ on } \partial B_R\cap\Omega,\ v^i=0 & \text{on } \partial\Omega\cap B_R ,\end{cases}$$

where $B_R = B_R(0)$.

For $\bar{v}^i$, $1 \leqq i \leqq s$, using Theorem of De Giorgi-Nash, we obtain for some $\beta \in (0, 1)$

$$\int_{B_\rho} |D\bar{v}^i|^2 dx \leqq C_2\left(\frac{\rho}{R}\right)^{n-2+2\beta} \int_{B_R} |D\bar{v}^i|^2 dx\,, \qquad \text{for all } \rho \leqq R\,, \tag{2.4}$$

and therefore

$$\int_{B_\rho \cap \Omega} |Dv^i|^2 dx \leqq C_3\left(\frac{\rho}{R}\right)^{n-2+2\beta} \int_{B_R \cap \Omega} |Dv^i|^2 dx\,. \tag{2.5}$$

For v^i, $s+1 \leqq i \leqq N$, we have

$$\int_{B_\rho \cap \Omega} |Dv^i|^2 dx \leqq C_4\left(\frac{\rho}{R}\right)^{n} \int_{B_R \cap \Omega} |Dv^i|^2 dx \tag{2.6}$$

(cf. [2]).

From (2.5) and (2.6), we obtain

$$\int_{B_\rho \cap \Omega} |Dv|^2 dx \leqq C_5\left(\frac{\rho}{R}\right)^{n-2+2\beta} \int_{B_R \cap \Omega} |Dv|^2 dx\,, \tag{2.7}$$

for some $\beta \in (0, 1)$.

Putting $w = u - v$, we have $w \in V'$, hence

$$\int_{B_R \cap \Omega} \sum_{i=1}^{N} A^{\alpha\beta}(0, u_R) D_\alpha v^i D_\beta w^i dx = 0\,.$$

Thus we have

$$\begin{aligned}
&\int_{B_R \cap \Omega} \sum_{i=1}^{N} A^{\alpha\beta}(0, u_R) D_\alpha w^i D_\beta w^i dx \\
&\quad = \int_{B_R \cap \Omega} \sum_{i=1}^{N} A^{\alpha\beta}(0, u_R) D_\alpha u^i D_\beta w^i dx \\
&\quad = \int_{B_R \cap \Omega} \sum_{i=1}^{N} [A^{\alpha\beta}(0, u_R) - A^{\alpha\beta}(x, u)] D_\alpha (u+v)^i D_\beta w^i dx \\
&\qquad + \int_{B_R \cap \Omega} \sum_{i=1}^{N} [A^{\alpha\beta}(x, v) - A^{\alpha\beta}(x, u)] D_\alpha v^i D_\beta v^i dx \\
&\qquad + \int_{B_R \cap \Omega} \sum_{i=1}^{N} A^{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i dx - \int_{B_R \cap \Omega} \sum_{i=1}^{N} A^{\alpha\beta}(x, v) D_\alpha v^i D_\beta v^i dx\,.
\end{aligned}$$

Since u minimizes F and $u - v \in V'$, the sum of the last two terms is non-negative. Thus we get

$$\int_{B_R \cap \Omega} |Dw|^2 \mathrm{dx} \tag{2.8}$$

$$\leqq C_5 \int_{B_R \cap \Omega} [|Du|^2 + |Dv|^2] \times [\omega(R^2 + |u - u_R|^2) + \omega(R^2 + |u - v|^2)] dx \, .$$

Using the inequality (2.7) and Theorem 1.1, we can proceed as the proof of Theorem 5.1 of [4], and from (2.8) we get

$$\begin{aligned} &\int_{B_\rho \cap \Omega} (1 + |Du|^2) dx \\ (2.9) \qquad &\leqq C_6 \left[\left(\frac{\rho}{R} \right)^{n-2+2\beta} + \omega\left(R^2 + C_7 R^{2-n} \int_{B_R \cap \Omega} |Du|^2 dx \right)^{1-2/p} \right] \\ &\times \int_{B_{2R} \cap \Omega} (1 + |Du|^2) dx \, , \end{aligned}$$

for every $0 < \rho < R < 2R < \text{dist}\,(0, \Gamma_0)$. By a well known lemma (cf. [3] p. 18) it now follows that for any $\alpha \in (0, \beta)$ there exist positive numbers ε_1 and R_0 with the following property: Put $\Phi(x_0, r) = r^{2-n} \int_{B_r(x_0) \cap \Omega} |Du|^2 dx$. For $x_0 \in \Gamma_1$ if $\Phi(x_0, r) \leqq \varepsilon_1$ for some $r < \min\{\text{dist}\,(x_0, \Gamma_0), R_0\}$, then

$$(2.10) \qquad \Phi(x_0, \rho) \leqq C_8 \varepsilon_1 \left(\frac{\rho}{r} \right)^{2\alpha} \qquad \text{for every} \quad \rho < r \, .$$

For an interior point x_1 we get by [4], the following: If $\Phi(x_1, r) \leqq \varepsilon_1$ for some $r < \min\{R_0, \text{dist}\,(x_1, \partial\Omega)\}$, then

$$(2.11) \qquad \Phi(x_1, \rho) \leqq C_9 \Phi(x_1, r) \left(\frac{\rho}{r} \right)^{2\alpha} \qquad \text{for every} \quad \rho < r \, .$$

Now we want to prove the following result: There exists a positive constant ε_0 such that if $x_1 \in \Omega \cup \Gamma_1$, and $\Phi(x_1, r_0) \leqq \varepsilon_0$ for some $r_0 < \min\{R_0, \text{dist}\,(x_1, \Gamma_0)\}$, then

$$(2.12) \qquad \Phi(x_1, \rho) \leqq C_{10} \left(\frac{p}{r_0} \right)^{2\alpha} \qquad \text{for every} \quad \rho < r_0 \, .$$

The assertion of Theorem 2.1 follows from (2.12) together with the integral characterization of Hölder continuous functions due to Campanato and Morrey (cf. [2]).

To prove (2.12), we follow the argument due to [8]: Let

$$(2.13) \qquad \varepsilon_0 = \sigma^{n-2} \varepsilon_1 \, ,$$

where $\sigma < 1/8$ is determined in such a way that

$$(2.14) \qquad \sigma^{2\alpha} 4^{n-2+2\alpha} C_8 < 1 \, .$$

It is sufficient to prove (2.12) for $\rho<\sigma r_0$. Therefore we restrict ourselves to the case that that $\rho<\sigma r_0$. Suppose that $\Phi(x_1, r_0)<\varepsilon_0$ and choose $x_0 \in \Gamma_1$ with $d:=\operatorname{dist}(x_1, \Gamma_1)=|x_1-x_0|$.

Case 1. If $d>\sigma r_0>\rho$, then (2.11) can be applied with $r=\sigma r_0$. Therefore

$$\Phi(x_0, \rho)\leqq C_9'\varepsilon_1\left(\frac{\rho}{r_0}\right)^{2\alpha} .$$

Case 2. If $\sigma r_0\geqq d$ then $B_d(x_1)\subset B_{2d}(x_0)\subset B_{r_0/2}(x_0)\subset B_{r_0}(x_1)$ and therefore $\Phi(x_0, r_0/2)\leqq 2^{n-2}\Phi(x_1, r_0)<\varepsilon_1$.

If $\sigma r_0>\rho\geqq d/2$, we apply (2.10) and arrive at

$$\Phi(x_1, \rho)\leqq 4^{n-2}\Phi(x_0, 4\rho)\leqq 4^{n-2}C_8\varepsilon_1\left(\frac{8\rho}{r_0}\right)^{2\alpha} .$$

Now let $\sigma r_0\geqq d\geqq 2\rho$. Using (2.10) with $r=r_0/2$, we get

$$\Phi(x_1, d/2)\leqq 4^{n-2}\Phi(x_0, 2d)\leqq 4^{n-2}C_8\varepsilon_1\left(\frac{4d}{r_0}\right)^{2\alpha}\leqq 4^{n-2}C_8\varepsilon_1(4\sigma)^{2\alpha}\leqq\varepsilon_1 .$$

Hence we can apply (2.11) with $r=d/2$ and obtain

$$\Phi(x_1, \rho)\leqq 4^{n-2}C_8C_9\varepsilon_1\left(\frac{4d}{r_0}\right)^{2\alpha}\left(\frac{2\rho}{d}\right)^{2\alpha}\leqq C_{10}\varepsilon_1\left(\frac{\rho}{r_0}\right)^{2\alpha} .$$

Thus we get (2.12) for all case.

REMARK. To apply the integral characterization of Hölder continuous functions the following consideration is necessary: Let $x_1 \in \Omega\cup\Gamma_1$ and assume that $\Phi(x_1, r_0)\leqq\varepsilon_0$ for some $r_0<\min\{R_0, \operatorname{dist}(x_1, \Gamma_0)\}$. Because of the continuity of $\Phi(x, r_0)$ with respect to x, there exists a number $\delta>0$, with $\delta+r_0<\min\{R_0, \operatorname{dist}(x_1, \Gamma_0)\}$ such that $\Phi(x, r_0)<\varepsilon_0$ for every $x\in B_\delta(x_1)\cap(\Omega\cup\Gamma_1)$. From (2.12) we get

$$\int_{B_\rho(x)\cap\Omega}|Du|^2dx\leqq\frac{C_{11}}{r_0^2}\rho^{n-2+2\alpha} \tag{2.15}$$

for all $x\in B_\delta(x_1)\cap(\Omega\cup\Gamma_1)$ and all $\rho<r_0$.

§3. Differentiable coefficient case.

In this section we treat the case that the coefficients $A^{\alpha\beta}(x, u)$ are differentiable, so that every bounded minimum u is a solution of Euler equation,

$$(3.1) \qquad \int_\Omega \sum_{i=1}^N A^{\alpha\beta}(x, u) D_\alpha u^i D_\beta \psi^i dx = -\frac{1}{2}\int_\Omega \sum_{1\leq i,h\leq N} A^{\alpha\beta}{}_h(x, u) D_\alpha u^i D_\beta u^i \psi^h dx$$

for every $\psi \in V^2 \cap L^\infty(\Omega)$ (V^m is defined in section 1.), where $A^{\alpha\beta}{}_h(x, u) = \partial A^{\alpha\beta}(x, u)/\partial u^h$.

As usual we suppose that

$$(3.2) \qquad |A^{\alpha\beta}(x, u)| \leqq M , \qquad A^{\alpha\beta}(x, u)\xi_\alpha\xi_\beta \geqq \lambda|\xi|^2 , \qquad \lambda > 0 \ \text{ for all } \ \xi \in \boldsymbol{R}^n .$$

Then we get the following theorem corresponding to Theorem 5.1 of [4].

THEOREM 3.1. *Assume that*

$$(3.3) \qquad -\frac{1}{2}\sum_{1\leq i,h\leq N} u^h A^{\alpha\beta}{}_h(x, u) D_\alpha u^i D_\beta u^i \leqq \lambda^* |Du|^2$$

with $\lambda^* < \lambda$. *Then every bounded solution of the mixed boundary value problem* (**) *is Hölder continuous in* $\Omega \cup \Gamma_1$.

PROOF. On account of Theorem 2.1, it is sufficient to show that for every $x_0 \in \Omega \cup \Gamma_1$ we have

$$(3.4) \qquad \rho^{2-n}\int_{B_\rho(x_0)\cap\Omega} |Du|^2 dx \leqq \varepsilon_0 ,$$

for some $\rho > 0$. Since it is known in [4] that this is the case for $x_0 \in \Omega$, we only have to treat the case that $x_0 \in \Gamma_1$.

Let $R < (1/2)\operatorname{dist}(x_0, \Gamma_0)$, $\eta \in H_0^{1,2}(B_{2R}(x_0))$, $\eta \geqq 0$. Taking $\psi = \eta u$ in (3.1) we get

$$(3.5) \qquad \lambda\int_{B_{2R}(x_0)\cap\Omega} \eta|Du|^2 dx \leqq -\frac{1}{2}\int_{B_{2R}(x_0)\cap\Omega} A^{\alpha\beta} D_\alpha |u|^2 D_\beta \eta dx + \lambda^*\int_{B_{2R}(x_0)\cap\Omega} \eta|Du|^2 dx .$$

Choose coordinate such that $x_0 = 0$ and $\Gamma_1 \cap B_{2R}(x_0) = \Gamma_1 \cap B_{2R}(0) \subset \{x \in \boldsymbol{R}^n : x^n = 0\}$, and define x_*, $\bar{u}$ and $\sigma^{\alpha\beta}$ as in the section 2. Let

$$\bar{A}^{\alpha\beta}(x, v) = \begin{cases} A^{\alpha\beta}(x, v) & \text{if } x \in \Omega , \\ \sigma^{\alpha\beta} A^{\alpha\beta}(x_*, v) & \text{if } x \in B_{2R}(x_0) - \Omega . \end{cases}$$

Then from (3.5), the function $z := M^2(2R) - |\bar{u}|^2$, where $M(t) := \sup_{B_t} |\bar{u}|$, is a non-negative supersolution of an elliptic operator, i.e.

$$\int_{B_{2R}(0)} \bar{A}^{\alpha\beta}(x, \bar{u}) D_\alpha z D_\beta \eta dx \geqq 0$$

for all $\eta \in C_0^\infty(B_{2R}(0))$, $\eta \geqq 0$ and $z \geqq 0$. Therefore from the weak Harnack inequality, we get

$$R^{-n}\int_{B_{2R}(0)} zdx \leqq C_{12} \inf_{B_R(0)} z \ . \tag{3.6}$$

Now let $w \in H_0^{1,2}(B_{2R}(0))$ be a solution of the equation

$$\int_{B_{2R}(0)} \bar{A}^{\alpha\beta}(x, \bar{u}) D_\alpha w D_\beta \psi dx = R^{-2}\int_{B_{2R}(0)} \psi dx \ , \tag{3.7}$$

for all $\psi \in H_0^{1,2}(B_{2R}(0))$. Taking $\psi = wz$ we obtain

$$\frac{1}{2}\int_{B_{2R}} \bar{A}^{\alpha\beta} D_\alpha w^2 D_\beta z dx + \int_{B_{2R}} z \bar{A}^{\alpha\beta} D_\alpha w D_\beta w dx = R^{-2}\int_{B_{2R}} wzdx \tag{3.8}$$

where $B_{2R} = B_{2R}(0)$, $\bar{A}^{\alpha\beta} = \bar{A}^{\alpha\beta}(x, \bar{u})$. It follows from (3.7) and the boundary condition $w|_{\partial B_{2R}} = 0$, that w is a non-negative weak solution of $D_\alpha(\bar{A}^{\alpha\beta} D_\beta w) = -1/R^2 < 0$. By the maximum principle we have $w > 0$ in the interior of B_{2R} and hence, by the weak Harnack inequality, we have $w \geqq \alpha_1 > 0$ in B_R. Moreover we have $w \leqq \alpha_2$ in B_{2R}. α_1 and α_2 are constants independent of R.

Now let $\eta = w^2$, from (3.8) and boundedness of w, we get

$$\int_{B_{2R}} \bar{A}^{\alpha\beta} D_\alpha \eta D_\beta z dx \leqq C_{13} R^{-2} \int_{B_{2R}} zdx \ ,$$

which, together with (3.5) and (3.6), implies

$$\begin{aligned} (\lambda - \lambda^*) \int_{B_R(0)\cap\Omega} |Du|^2 dx &\leqq -C_{14} \int_{B_{2R}(0)\cap\Omega} A^{\alpha\beta} D_\alpha |u|^2 D_\beta \eta dx \\ &\leqq C_{15} \int_{B_{2R}(0)} \bar{A}^{\alpha\beta} D_\alpha z D_\beta \eta dx \\ &\leqq C_{16} R^{-2} \int_{B_{2R}(0)} zdx \\ &\leqq C_{17} R^{n-2} \inf_{B_R(0)} z \\ &\leqq C_{17} R^{n-2} [M^2(2R) - M^2(R)] \ . \end{aligned} \tag{3.9}$$

On the other hand we have

$$\sum_{k=0}^{\infty} [M^2(2^{1-k}R) - M^2(2^{-k}R)] \leqq M^2(2R) \leqq \sup_{\Omega} |u|^2 \ , \tag{3.10}$$

and inequalities (3.9) and (3.10) imply (3.4) with $\rho=2^{-k}R$ for some $k>0$. This completes the proof.

Combining Theorem 3.1. with the results of [4] and [8], we get regularity for minima of the solution of (**) for any point of $\bar{\Omega}-(\Gamma_0\cap\bar{\Gamma}_1)$:

THEOREM 3.2. *Let Ω be a bounded domain in $\boldsymbol{R}^n$, $n\geqq 2$ with boundary $\partial\Omega$ of class C^2, Γ_1 a relatively open subset of $\partial\Omega$, and $\Gamma_0=\partial\Omega-\Gamma_1$. Let*

$$F(u):=\int_\Omega \sum_{i=1}^N A^{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i dx\ , \qquad A^{\alpha\beta}=A^{\beta\alpha}\ ,$$

be a quadratic functional, where $A^{\alpha\beta}(x, u)$ are differentiable and satisfy (3.2) *and* (3.3). *Assume that Dirichlet boundary condition ϕ of* (**) *is in class $H^{1,p}(\Omega, \boldsymbol{R}^N)$, $p>n$. Then any bounded solution of* (**) *is in class $C^{0,\alpha}(\bar{\Omega}-(\Gamma_0\cap\bar{\Gamma}_1)$ for some $\alpha\in(0, 1)$.*

References

[1] A. BALDES, Harmonic mappings with partially free boundary, Manuscripta Math., **40** (1982), 255–275.

[2] S. CAMPANATO, Equazioni ellisse del secondo ordine e spazi $L^{2,\lambda}$, Ann. Mat. Pura Appl., **69** (1965), 321–382.

[3] M. GIAQUINTA, Remarks on the regularity of weak solutions to some variational inequalities, Math. Z., **177** (1981), 15–31.

[4] M. GIAQUINTA and E. GIUSTI, On the regularity of minima of variational integrals, Acta Math., **148** (1982), 31–46.

[5] M. GIAQUINTA and E. GIUSTI, The singular set of the minima of certain quadratic functionals, Preprint 435 SBF 72, Bonn, 1981.

[6] M. GIAQUINTA and G. MODICA, Regularity results for some class of higher order nonlinear elliptic systems, J. Reine Angew. Math., **311/312** (1979), 145–169.

[7] S. HILDEBRANDT and K.-O. WIDMAN, Some regularity results for quasilinear elliptic systems of second order, Math. Z., **142** (1975), 67–86.

[8] J. JOST and M. MEIER, Boundary regularity for minima of certain quadratic functionals, Math. Ann., **262** (1983), 549–561.

Present Address:
DEPARTMENT OF MATHEMATICS
KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA
223