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Asymptotic Behavior of Nonexpansive Mappings and Some Geometric Properties in Banach Spaces

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Introduction

Throughout this paper, X denotes a real Banach space with the dual space X^* and the bidual space X^{**} and C is a closed convex subset of X. For $0\leqq\gamma\leqq 1$ we consider a mapping $T: C\to C$ such that $\|Tx-Ty\|\leqq\gamma\|x-y\|$ for all $x, y\in C$. A mapping T is called nonexpansive (resp. contraction) if $\gamma=1$ (resp. $\gamma<1$). Let $A\subset X\times X$ be an accretive operator satisfying the range condition $R(I+\lambda A)\supset\overline{D(A)}$ (the closure of the domain of A) for all $\lambda>0$, where I is the identity, $J_\lambda=(I+\lambda A)^{-1}$ be the resolvent, and let $A_\lambda=(I-J_\lambda)/\lambda$ be the Yosida approximation. A one-parameter family $\{T(t); t\geqq 0\}$ denotes the nonexpansive semigroup on $\overline{D(A)}$ generated by $-A$, i.e., $T(t)x=\lim_{\lambda\to 0+}J_\lambda^{[t/\lambda]}x$ for $x\in\overline{D(A)}$ and $t\geqq 0$ (see [1]). We use lim and w-lim for convergence in the strong and weak topology, respectively. We define $S(X)=\{x\in X; \|x\|=1\}$ and $d(0, R(A))=\inf\{\|x\|; x\in R(A)\}$, where $R(A)$ denotes the range of A.

Our main purpose is to show the following results.

THEOREM 1. *Let the sequence $\{x_n\}_{n\geqq 0}$ be defined by $x_{n+1}=c_nTx_n+(1-c_n)x_n$, where $x_0\in C$ and $\{c_n\}_{n\geqq 0}$ is a real sequence such that $0<c_n\leqq 1$ and $a_n=\sum_{i=0}^n c_i\to\infty$ as $n\to\infty$. Then there exists an $f\in S(X^*)$ such that for any $x, x_0\in C$,*

$$(1)\qquad \lim_{n\to\infty} f(T^n x)/n=\lim_{n\to\infty}\|T^n x\|/n=\inf_{y\in C}\|Ty-y\| =\lim_{n\to\infty} f(x_{n+1})/a_n=\lim_{n\to\infty}\|x_{n+1}\|/a_n\ .$$

COROLLARY 2. (i) *In Theorem 1, if X is reflexive and strictly convex, then* w-$\lim_{n\to\infty} T^n x/n=$ w-$\lim_{n\to\infty} x_{n+1}/a_n=-v$ *for all $x, x_0\in C$, where $\|v\|=\inf_{y\in C}\|Ty-y\|$.*

(ii) *In Theorem 1, if X^* has Fréchet differentiable norm, then*

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$\lim_{n\to\infty} x_{n+1}/a_n = -v$ *for all* $x_0 \in C$, *where* v *is the unique point of least norm in* $\overline{R(I-T)}$.

THEOREM 3. *Let the sequence* $\{x_n\}_{n\geq 0}$ *be defined by* $x_{n+1} = J_{c_n} x_n$, *where* $x_0 \in \overline{D(A)}$ *and* $\{c_n\}_{n\geq 0}$ *is a positive sequence such that* $a_n = \sum_{i=0}^{n} c_i \to \infty$ *as* $n \to \infty$. *Then there exists an* $f \in S(X^*)$ *such that for any* $x, x_0 \in \overline{D(A)}$,

$$(2) \qquad \lim_{n\to\infty} f(J_1^n x)/n = \lim_{n\to\infty} \|J_1^n x\|/n = d(0, R(A)) = \lim_{n\to\infty} f(x_{n+1})/a_n = \lim_{n\to\infty} \|x_{n+1}\|/a_n .$$

COROLLARY 4. (i) *In Theorem* 3, *if* X *is reflexive and strictly convex, then* $\text{w-}\lim_{n\to\infty} J_1^n x/n = \text{w-}\lim_{n\to\infty} x_{n+1}/a_n = -v$ *for all* $x, x_0 \in \overline{D(A)}$, *where* $\|v\| = d(0, R(A))$.

(ii) *In Theorem* 3, *if* X^* *has Fréchet differentiable norm, then* $\lim_{n\to\infty} x_{n+1}/a_n = -v$ *for all* $x_0 \in \overline{D(A)}$, *where* v *is the unique point of least norm in* $\overline{R(A)}$.

Theorems 1 and 3 imply that the asymptotic behavior of x_{n+1}/a_n is reduced to the asymptotic behavior of $T^n x/n$ and $J_1^n x/n$, respectively, in both strong and weak topology. Furthermore, Theorem 1 is valid even if X is a real normed linear space and C is a convex subset of X. Thus it generalizes Kohlberg and Neyman's result [8, Theorem 1.1]. Corollaries 2 and 4 were investigated by [8], [9], [10], [11], [12]. The idea of the proof of Theorem 3 is due to [11], and Kobayashi [5] showed that Theorem 1 follows from [6, Theorem 2.1] by using a different method.

It is known that the following conditions are equivalent (see [2], [3]):

(P) X^* has Fréchet differentiable norm.

(Q) X is reflexive and strictly convex. Furthermore, if $\text{w-}\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \|x_n\| = \|x\|$, then $\lim_{n\to\infty} x_n = x$.

(R) Every sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} f(x_n)$ for some $f \in S(X^*)$ converges strongly to an element of X.

(S) If a sequence $\{x_n\}$ in $S(X)$ and a sequence $\{f_n\}$ in X^* are such that $\lim_{n\to\infty} \|f_n\| = \overline{\lim}_{n\to\infty} \underline{\lim}_{m\to\infty} f_n(x_m) = 1$, then $\{x_n\}$ converges strongly to an element of X.

In Section 2, we state several properties which are equivalent to the property (A), i.e., X is reflexive and strictly convex. They are slight modifications of geometric properties listed in [2], [3].

§ 1. Proofs.

PROOF OF THEOREM 1. Let $d = \inf_{y\in C} \|Ty - y\|$. We follow the argument in Kohlberg and Neyman [8]. For another initial point $y_0 \in C$, we write the associated sequence by $\{y_n\}$. Then for any $x_0, y_0 \in C$, we get

$$(3)\qquad \begin{aligned}&\|x_{n+1}-y_{n+1}\| \leqq \|x_0-y_0\| \quad \text{and}\\ &\|x_{n+1}-x_0\| \leqq 2\|x_0-y_0\|+a_n\|Ty_0-y_0\| .\end{aligned}$$

Therefore, if $f \in S(X^*)$, then for any $x, x_0 \in C$,

$$(4)\qquad \overline{\lim}_{n\to\infty} f(x_{n+1})/a_n \leqq \overline{\lim}_{n\to\infty} \|x_{n+1}\|/a_n \leqq d$$

and letting $c_n \equiv 1$ and replacing x_0 with x, we have

$$(5)\qquad \overline{\lim}_{n\to\infty} f(T^n x)/n \leqq \overline{\lim}_{n\to\infty} \|T^n x\|/n \leqq d .$$

We may assume that C contains 0 and by (3), in order to comlete the proof it is sufficient to show that there exists an $f \in S(X^*)$ such that for $x_0 = 0 \in C$,

$$(6)\qquad f(x_{n+1})/a_n \geqq d \quad \text{and} \quad f(T^{n+1}0)/(n+1) \geqq d \quad \text{for} \quad n \geqq 0 .$$

Since for each $r>0$, $T/(1+r): C \to C$ is a contraction mapping, there exists a unique fixed point $x(r)$. We note that $Tx(r)=(1+r)x(r)$ and $r\|x(r)\| = \|Tx(r)-x(r)\| \geqq d$ for all $r>0$. By (4) and (5) we may assume that $d>0$. Since T is a nonexpansive mapping, we have

$$\begin{aligned}\|x_{n+1}-x(r)\| &= (1+r)\|x_{n+1}-x(r)\| - r\|x_{n+1}-x(r)\|\\ &= (1+r)\|c_n(Tx_n-x(r))+(1-c_n)(x_n-x(r))\| - r\|x_{n+1}-x(r)\|\\ &\leqq c_n(1+r)\|Tx_n-x(r)\|+(1+r)(1-c_n)\|x_n-x(r)\|-r\|x_{n+1}-x(r)\|\\ &\leqq c_n\|Tx_n-(1+r)x(r)\|+c_n r\|Tx_n\|+(1+r)(1-c_n)\|x_n-x(r)\|\\ &\quad -r\|x_{n+1}-x(r)\|\\ &\leqq c_n\|x_n-x(r)\|+c_n r\|Tx_n\|+(1+r)(1-c_n)\|x_n-x(r)\|\\ &\quad -r\|x(r)\|+r\|x_{n+1}\|\\ &= \{1+r(1-c_n)\}\|x_n-x(r)\|-r\|x(r)\|+r(c_n\|Tx_n\|+\|x_{n+1}\|)\end{aligned}$$

for $n \geqq 0$. Then by induction we have

$$\|x_{n+1}-x(r)\| \leqq \|x(r)\| - a_n r\|x(r)\| + O(r) ,$$

where $O(r) = r\sum_{m=0}^{n} \{(c_m\|Tx_m\|+\|x_{m+1}\|)\prod_{k=m+1}^{n}[1+r(1-c_k)]\}$ for $n \geqq 0$. Moreover, letting $c_n \equiv 1$, we have

$$\|T^n 0 - x(r)\| \leqq \|x(r)\| - nr\|x(r)\| + O(r) \quad \text{for} \quad n \geqq 1 .$$

Let f_x be an element of $S(X^*)$ such that $f_x(x) = \|x\|$ for $x \neq 0$. Since $\|x-y\| \leqq \|x\| - \beta$ implies $f_x(y) \geqq \beta$, we see that

$$f_{x(r)}(x_{n+1})/a_n \geqq d+O(r) \quad \text{and} \quad f_{x(r)}(T^n 0)/n \geqq d+O(r) .$$

By the Banach-Alaoglu theorem, there exists an accumulation point $f \in X^*$ of $\{f_{x(r)}\}$ as $r \to 0+$ in the w*-topology such that $\|f\| \leqq 1$. Then f satisfies (6) and hence $f/\|f\|$ also satisfies (6). Q.E.D.

REMARK 1. The weak-star accumulation point f of $\{f_{x(r)}\}$ belongs to $S(X^*)$. In fact, since $\|x_{n+1}\|/a_n \to d$ as $n \to \infty$, $\|f\| \geqq f(x_{n+1})/\|x_{n+1}\| \geqq a_n d/\|x_{n+1}\| \to 1$ as $n \to \infty$.

PROOF OF COROLLARY 2. (i) Let $d = \inf_{y \in C} \|Ty - y\|$. It is known that X is reflexive and strictly convex if and only if X has the property (D.2) (see Section 2). Consequently, it follows from Theorem 1 that there exist $u, v \in X$ such that w-$\lim_{n\to\infty} T^n x/n = u$ and w-$\lim_{n\to\infty} x_{n+1}/a_n = v$ for all $x, x_0 \in C$. Since $f(u) = \|u\| = f(v) = \|v\| = d$, we see that $\|u+v\| = \|u\| + \|v\| = 2d$. By strict convexity of X, we have $u = v$.

(ii) We note that (P), (Q) and (R) are equivalent. Therefore, it follows from (R) and Theorem 1 that there exist $u, v \in X$ such that $\lim_{n\to\infty} T^n x/n = u$ and $\lim_{n\to\infty} x_{n+1}/a_n = v$ for all $x, x_0 \in C$. Using strict convexity of X, we have $u = v$ in the same way as in (i). Moreover, it is known that $\{T^n x/n\}$ is convergent to the unique point of least norm in $\overline{R(T-I)}$ (see [9, Corollary]). Q.E.D.

PROOF OF THEOREM 3. Let $d = d(0, R(A))$. For any $[u, v] \in A$, we put $w = u + c_n v$. Then we have $\|x_{n+1} - u\| = \|J_{c_n} x_n - J_{c_n} w\| \leqq \|x_n - w\| = \|x_n - u - c_n v\| \leqq \|x_n - u\| + c_n\|v\|$. So we get $\|x_{n+1} - u\| \leqq \|x_0 - u\| + a_n\|v\|$. Therefore, if $f \in S(X^*)$, then for any $x, x_0 \in \overline{D(A)}$,

$$(7) \qquad \overline{\lim}_{n\to\infty} f(x_{n+1})/a_n \leqq \overline{\lim}_{n\to\infty} \|x_{n+1}\|/a_n \leqq d$$

and letting $c_n \equiv 1$ and replacing x_0 with x, we have

$$(8) \qquad \overline{\lim}_{n\to\infty} f(J_1^n x)/n \leqq \overline{\lim}_{n\to\infty} \|J_1^n x\|/n \leqq d .$$

Since J_λ is a nonexpansive mapping, to complete the proof it is sufficient to show that there exists an $f \in S(X^*)$ such that for some $x = x_0 \in \overline{D(A)}$,

$$(9) \qquad f(x_{n+1} - x)/a_n \geqq d \quad \text{and} \quad f(J_1^{n+1} x - x)/(n+1) \geqq d \quad \text{for} \quad n \geqq 0 .$$

If $d = 0$, the result follows from (7) and (8), and hence we assume that $d > 0$. Let $x = x_0 \in \overline{D(A)}$, and let n be fixed. We set $\alpha = \alpha_n = \max\{1, c_0, \cdots, c_n\}$ and $y_\lambda = y_\lambda^n = (1/(1+\lambda))x + (\lambda/(1+\lambda))J_{(1+\lambda)\alpha_n} x$ for $\lambda > 0$. Then we have $J_{(1+\lambda)\alpha} x = J_\alpha y_\lambda$ by the resolvent identity, and $\lambda(y_\lambda - J_\alpha y_\lambda) = x - y_\lambda$. We note that $\|x - y_\lambda\| = \lambda\alpha\|A_\alpha y_\lambda\| \geqq \lambda\alpha d$ for all $\lambda > 0$, because $A_\alpha y_\lambda \in AJ_\alpha y_\lambda$. Using the resolvent identity and $(1+\lambda)y_\lambda=x+\lambda J_\alpha y_\lambda$, we have

$$\begin{aligned}\|x_{i+1}-y_\lambda\|&=(1+1/\lambda)\|x_{i+1}-y_\lambda\|-(1/\lambda)\|x_{i+1}-y_\lambda\|\\&=(1/\lambda)\|(1+\lambda)x_{i+1}-x-\lambda J_\alpha y_\lambda\|-(1/\lambda)\|x_{i+1}-y_\lambda\|\\&\leqq\|x_{i+1}-J_\alpha y_\lambda\|+(2/\lambda)\|x_{i+1}-x\|-(1/\lambda)\|x-y_\lambda\|\\&\leqq\|x_i-y_\lambda\|+(1-c_i/\alpha)(1/\lambda)\|x-y_\lambda\|-(1/\lambda)\|x-y_\lambda\|+(2/\lambda)\|x_{i+1}-x\|\\&=\|x_i-y_\lambda\|-c_i(1/\lambda\alpha)\|x-y_\lambda\|+(2/\lambda)\|x_{i+1}-x\| .\end{aligned}$$

Therefore, we obtain

$$\|x_{i+1}-y_\lambda\|\leqq\|x-y_\lambda\|-a_i(1/\lambda\alpha)\|x-y_\lambda\|+(2/\lambda)\sum_{k=0}^{i}\|x_{k+1}-x\| \tag{10}$$

and letting $c_i\equiv 1$, we have

$$\|J_1^{i+1}x-y_\lambda\|\leqq\|x-y_\lambda\|-(i+1)(1/\lambda\alpha)\|x-y_\lambda\|+(2/\lambda)\sum_{k=0}^{i}\|J_1^{k+1}x-x\| , \tag{11}$$

for $i=0, 1, \cdots, n$. Let f_λ be an element of $S(X^*)$ such that $f_\lambda(y_\lambda-x)=\|y_\lambda-x\|$. Then by (10) and (11) we have

$$f_\lambda(x_{i+1}-x)/a_i\geqq d+0(1/\lambda) \quad\text{and}\quad f_\lambda(J_1^{i+1}x-x)/(i+1)\geqq d+0(1/\lambda)$$

for $i=0, 1, \cdots, n$. By the Banach-Alaoglu theorem, there exists an accumulation point $f\in X^*$ of $\{f_\lambda\}$ as $\lambda\to\infty$ in the w*-topology such that $\|f\|\leqq 1$. Then f satisfies

$$f(x_{i+1}-x)/a_i\geqq d \quad\text{and}\quad f(J_1^{i+1}x-x)/(i+1)\geqq d \quad\text{for}\quad i=0, 1, \cdots, n \tag{12}$$

and hence $f_n=f/\|f\|$ also satisfies (12). Furthermore, an accumulation point $g\in X^*$ of $\{f_n\}$ in the w*-topology satisfies (9) and so $g/\|g\|$ is the desired element of $S(X^*)$. Q.E.D.

REMARK 2. The weak-star accumulation point g of $\{f_n\}$ belongs to $S(X^*)$. In fact, since $\|x_{n+1}-x\|/a_n\to d$ as $n\to\infty$, $\|f\|\geqq f(x_{n+1}-x)/\|x_{n+1}-x\|\geqq a_nd/\|x_{n+1}-x\|\to 1$ as $n\to\infty$.

PROOF OF COROLLARY 4. The proof is similar to that of Corollary 2. That v is the unique point of least norm in $\overline{R(A)}$ follows from [9, Theorem 2] and the estimate $\|T(n)x-J_1^nx\|\leqq 2\|x-u\|+\sqrt{n}|||Au|||$, where $u\in D(A)$ and $|||Au|||=\inf\{\|v\|; v\in Au\}$. Here $\{T(t); t\geqq 0\}$ is the nonexpansive semigroup generated by $-A$ and this estimate is obtained from [7, Lemma 2.1]. Q.E.D.

§ 2. Geometric properties.

In this section, we list some conditions which are equivalent in a real Banach space X. Among others (B.3) and (D.2) are useful to study the asymptotic behavior of an integral solution of

$$(13) \qquad (d/dt)u(t)+Au(t)+g(t)u(t) \ni g(t)x\ , \quad u(0)=x_0 \in \overline{D(A)}\ ,$$

at the origin and at infinity, respectively (see [6]). Here $g\colon [0, \infty)\to [0, \infty)$ is a nonincreasing function such that $\lim_{t\to\infty} g(t)=0$ and $x \in X$.

We denote the closed convex hull of a subset M of X by $\operatorname{clco} M$. A mapping $x \to f_x$ of $X\backslash\{0\}$ to $X^*\backslash\{0\}$ is called a support mapping if (i) $\|x\|=1$ implies $\|f_x\|=1=f_x(x)$ and (ii) $\lambda \geqq 0$ implies $f_{\lambda x}=\lambda f_x$.

We consider the following properties:

(A) X is reflexive and strictly convex.

(B.1) Every sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} \|x_n\|=\lim_{n\to\infty} \inf\{\|x\|; x \in \operatorname{clco}\{x_m; m \geqq n\}\}$ converges weakly to an element of X.

(B.2) For any decreasing sequence of convex sets $\{K_n\}$ in X, every sequence $\{y_n\}$ of elements satisfying $y_n \in K_n$ $(n \geqq 1)$ and $\lim_{n\to\infty} \|y_n\|= \lim_{n\to\infty} \inf\{\|x\|; x \in K_n\}$ converges weakly to an element of X.

(B.3) If a sequence $\{x_n\}$ in $S(X)$ and a sequence $\{f_n\}$ in X^* are such that $\lim_{n\to\infty} \|f_n\|=\overline{\lim}_{n\to\infty} \underline{\lim}_{m\to\infty} f_n(x_m)=1$, then $\{x_n\}$ converges weakly to an element of X.

(C) If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \|x_n\|=1$ satisfies $\|1/n \sum_{i=1}^{n} x_{k_i}\| \geqq 1$ for any finite set of distinct indices $k_1<k_2<\cdots<k_n$, then $\{x_n\}$ converges weakly to an element of X.

(D.1) If a sequence $\{x_n\}$ in X and a sequence $\{f_n\}$ in $S(X^*)$ are such that $\lim_{n\to\infty} \|x_n\|=\underline{\lim}_{m\to\infty} \underline{\lim}_{n\to\infty} f_n(x_m)$, then $\{x_n\}$ converges weakly to an element of X.

(D.2) Every sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} \|x_n\|=\lim_{n\to\infty} f(x_n)$ for some $f \in S(X^*)$ converges weakly to an element of X.

(D.3) For any convex set K in X, every sequence $\{x_n\}$ in K satisfying $\lim_{n\to\infty} \|x_n\|=\inf\{\|x\|; x \in K\}$ converges weakly to an element of X.

(D.4) For any closed hyperplane (or closed half-space) H in X, every sequence $\{x_n\}$ in H satisfying $\lim_{n\to\infty} \|x_n\|=\inf\{\|x\|; x \in H\}$ converges weakly to an element of X.

(E) X is reflexive and every support mapping $x' \to f_{x'}$ of $X^*\backslash\{0\}$ to $X^{**}\backslash\{0\}$ is norm to weak-star continuous from $S(X^*)$ to $S(X^{**})$.

(F) X is reflexive and X^* is smooth.

We study the relationship between the properties (A)–(F). Our results and proofs are parallel to [3].

THEOREM 5. *The following equivalence relations hold:*

$$(\mathrm{B.1}) \longleftrightarrow (\mathrm{B.2}) \longleftrightarrow (\mathrm{B.3}) \ ,$$
$$(\mathrm{D.1}) \longleftrightarrow (\mathrm{D.2}) \longleftrightarrow (\mathrm{D.3}) \longleftrightarrow (\mathrm{D.4}) \ .$$

THEOREM 6. *The following implications hold:*

$$(\mathrm{A}) \longrightarrow (\mathrm{B.1}) \longrightarrow (\mathrm{C}) \longrightarrow (\mathrm{D.2}) \longrightarrow (\mathrm{E}) \ .$$

Since (E)$\leftrightarrow$(F)$\leftrightarrow$(A) (see [2]), (A)–(F) *are equivalent.*

REMARK 3. The value of the property (S) was pointed out by Kobayasi [5]. On the other hand, the property (B.3) is useful to investigate the asymptotic behavior of the nonexpansive semigroup generated by $-A$, or more generally, an integral solution of (13) at the origin in the weak topology.

PROOF OF THEOREM 5. (B.1)$\to$(B.2). If $y_n \in K_n$ $(n \geqq 1)$ satisfy the hypothesis of (B.2), then they satisfy the hypothesis of (B.1) (see [3, (C.1)$\leftrightarrow$(C.2)]).

(B.2)$\to$(B.1). It is sufficient to set $K_n = \mathrm{clco}\,\{x_m;\, m \geqq n\}$.

(B.1)$\to$(B.3). If $\{x_n\}$ and $\{f_n\}$ satisfy the hypothesis of (B.3), then $\{x_n\}$ satisfies the hypothesis of (B.1) (see [3, (C.1)$\to$(C.3)]).

(B.3)$\to$(B.1). Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\alpha_n = 1$, where $\alpha_n = \inf\{\|x\|;\, x \in \mathrm{clco}\,\{x_m;\, m \geqq n\}\}$. For each n, the convex set $\mathrm{clco}\,\{x_m;\, m \geqq n\}$ and the open convex set $\{x \in X;\, \|x\| < \alpha_n\}$ are disjoint. So there exists an $f_n \in X^*$ such that $f_n(x) \leqq 1$ for all x with $\|x\| < \alpha_n$ and $f_n(x_m) \geqq 1$ for all $m \geqq n$. Then we note that $1/\|x_n\| \leqq f_n(x_n)/\|x_n\| \leqq \|f_n\| \leqq 1/\alpha_n$, and hence $\lim_{n\to\infty}\|f_n\| = 1$. Since $\underline{\lim}_{m\to\infty} f_n(x_m) \geqq 1$ for every n, $\underline{\lim}_{m\to\infty}\underline{\lim}_{n\to\infty} f_n(x_m) \geqq 1$. On the other hand, we have $\overline{\lim}_{n\to\infty}\underline{\lim}_{m\to\infty} f_n(x_m) \leqq \lim_{n\to\infty}\|f_n\| = 1$. Hence $\lim_{n\to\infty}\underline{\lim}_{m\to\infty} f_n(x_m) = 1$. (The property (B.3) is not changed if we replace "$\{x_n\}$ in $S(X)$" in that condition by "$\{x_n\}$ in X with $\lim_{n\to\infty}\|x_n\| = 1$".)

(D.1)$\to$(D.2) is trivial.

(D.2)$\to$(D.1). Let $\{x_n\}$ and $\{f_n\}$ be such that $f_n \in S(X^*)$ and $\lim_{n\to\infty}\|x_n\| = \underline{\lim}_{m\to\infty}\underline{\lim}_{n\to\infty} f_n(x_m) = 1$. We first note that for any subsequence $\{y_n\}$ of $\{x_n\}$, $\underline{\lim}_{m\to\infty}\underline{\lim}_{n\to\infty} f_n(y_m) \geqq 1$. We now consider subsequences $\{y_n\}$ and $\{z_n\}$ of $\{x_n\}$ such that $\underline{\lim}_{n\to\infty} f_n(y_m) \geqq 1 - 2^{-m}$ and $\underline{\lim}_{n\to\infty} f_n(z_m) \geqq 1 - 2^{-m}$ $(m \geqq 1)$. We define the sequence $\{w_n\}$ by $w_{2n-1} = y_n/(1-2^{-n})$ and $w_{2n} = z_n/(1-2^{-n})$. Then we have $\lim_{n\to\infty}\|w_n\| = 1$ and $\underline{\lim}_{n\to\infty} f_n(w_m) \geqq 1$ for every m. For each $u \in \mathrm{clco}\,\{w_n\}$, we have $\|u\| \geqq f_n(u)$ for every n, $\underline{\lim}_{n\to\infty} f_n(u) \geqq 1$ and hence $\|u\| \geqq 1$. Therefore, there exists a $g \in X^*$ such that $g(x) \leqq 1$ for all x with $\|x\| < 1$ and $g(u) \geqq 1$ for all $u \in \mathrm{clco}\,\{w_n\}$. Then from $1 \geqq \|g\| \geqq g(w_n)/\|w_n\| \geqq 1/\|w_n\|$, we obtain $\|g\|=1$ and $\lim_{n\to\infty} g(w_n)=1=\lim_{n\to\infty}\|w_n\|$. By (D.2) there exists a $v\in X$ such that $\text{w-lim}_{n\to\infty} w_n=v$. Consequently, we easily see that $\text{w-lim}_{n\to\infty} y_n=\text{w-lim}_{n\to\infty} z_n=v$.

(D.2)$\to$(D.3). If $x_n\in K$ $(n\geqq 1)$ satisfy the hypothesis of (D.3), then $\{x_n\}$ satisfies the hypothesis of (D.2) (see [3, (E.2)$\to$(E.3)]).

(D.3)$\to$(D.4) is trivial.

(D.4)$\to$(D.2). Let $\|f\|=\lim_{n\to\infty}\|x_n\|=\lim_{n\to\infty} f(x_n)=1$. Then $y_n=x_n/f(x_n)$ and the closed hyperplane $H=\{z\in H;\ f(z)=1\}$ (or the closed half-space $H=\{z\in X;\ f(z)\geqq 1\}$) satisfy the hypothesis of (D.4) (see [3, (E.4)$\to$(E.2)]). Therefore, it is clear that (D.2) holds. Q.E.D.

PROOF OF THEOREM 6. (A)$\to$(B.1). We first observe that we can use the facts obtained in the proof of (B.3)$\to$(B.1). Now, since X is reflexive, there exist $y, z\in X$ and subsequences $\{n_k\}$, $\{m_k\}$ of $\{n\}$ such that $\text{w-lim}_{k\to\infty} x_{n_k}=y$ and $\text{w-lim}_{k\to\infty} x_{m_k}=z$. Then we can conclude that $\|y\|\geqq 1$, $\|z\|\geqq 1$, because $\alpha_n\to 1$ as $n\to\infty$, where α_n is defined as in the proof of (B.3)$\to$(B.1). Therefore, we have $\|y\|=\|z\|=1$. Noting that $2\leqq f_n(x_{n_k}+x_{m_k})$ for $n_k, m_k\geqq n$, we have $2\leqq f_n(y+z)\leqq\|f_n\|\,\|y+z\|\leqq 2\|f_n\|$. Letting $n\to\infty$, we get $\|y+z\|=2$ and hence, by strict convexity of X, $y=z$.

(B.1)$\to$(C). Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}\|x_n\|=1$ and $\|1/n\sum_{i=1}^n x_{k_i}\|\geqq 1$ for any finite set of distinct indices $k_1<k_2<\cdots<k_n$. Then $\|\sum_{i=1}^n \lambda_i x_{k_i}\|\geqq n+1-\sum_{i=1}^n\|x_{k_i}\|$ holds for any finite set of distinct indices $k_1<k_2<\cdots<k_n$ and for any $\lambda_i\geqq 0$ with $\sum_{i=1}^n\lambda_i=1$. In fact, since $\|x_n\|\geqq 1$ $(n\geqq 1)$, we have $n\leqq\|\sum_{i=1}^n x_{k_i}\|\leqq\|\sum_{i=1}^n\lambda_i x_{k_i}\|+\sum_{i=1}^n(1-\lambda_i)\|x_{k_i}\|\leqq\|\sum_{i=1}^n\lambda_i x_{k_i}\|+\sum_{i=1}^n\|x_{k_i}\|-1$. We next show that every subsequence $\{y_n\}$ of $\{x_n\}$ satisfying $\|y_n\|\leqq 1+2^{-n}$ $(n\geqq 1)$ converges weakly to an element of X. For such a subsequence $\{y_n\}$ of $\{x_n\}$ and for $k_n>k_{n-1}>\cdots>k_1>m$, we have $\|\sum_{i=1}^n\lambda_i y_{k_i}\|\geqq n+1-\sum_{i=1}^n\|y_{k_i}\|\geqq n+1-\sum_{i=1}^n(1+2^{-k_i})>1-2^{-m}$ for any $\lambda_i\geqq 0$ with $\sum_{i=1}^n\lambda_i=1$. Therefore, $\lim_{m\to\infty}\inf\{\|y\|;\ y\in \text{clco}\{y_n;\ n\geqq m\}\}=1=\lim_{n\to\infty}\|y_n\|$ and hence, by (B.1), $\{y_n\}$ converges weakly to an element of X. Now, since $\lim_{n\to\infty}\|x_n\|=1$, we can choose a subsequence $\{u_n\}$ of $\{x_n\}$ satisfying $\|u_n\|\leqq 1+2^{-(2n-1)}$ $(n\geqq 1)$. Therefore, there exists a $u\in X$ such that $\text{w-lim}_{n\to\infty} u_n=u$. Let $\{v_n\}$ be a subsequence of $\{x_n\}$ satisfying $\|v_n\|\leqq 1+2^{-2n}$ $(n\geqq 1)$. We consider the sequence $\{w_n\}$ defined by $w_{2n-1}=u_n$ and $w_{2n}=v_n$. Then clearly $\|w_n\|\leqq 1+2^{-n}$ $(n\geqq 1)$ and so $\{w_n\}$ converges weakly to some $v\in X$. Since $\text{w-lim}_{n\to\infty} u_n=u$, we must have $u=v$. Consequently, we have $\text{w-lim}_{n\to\infty} x_n=u$.

(C)$\to$(D.2). Let a sequence $\{x_n\}$ in X and $f\in X^*$ be such that $\|f\|=1$ and $\lim_{n\to\infty}\|x_n\|=\lim_{n\to\infty} f(x_n)$. We may assume that this common limit is not 0. Let $y_n=x_n/f(x_n)$. Then $\lim_{n\to\infty}\|y_n\|=1$ and $1=$

$f(1/n \sum_{i=1}^{n} y_{k_i}) \leqq \| 1/n \sum_{i=1}^{n} y_{k_i} \|$ for any indices $k_1 \leqq k_2 \leqq \cdots \leqq k_n$. By (C), $\{y_n\}$ converges weakly to an element of X, and hence so is $\{x_n\}$.

(D.2)$\to$(E). That X is reflexive follows from a slight modification of [3, (E)$\to$(R)]. In fact, let L be a closed linear subspace of X and $g \in X^*$ such that $\sup\{g(x); x \in L \cap S(X)\}=1$. Then we can take a sequence $\{x_n\}$ in $L \cap S(X)$ such that $\lim_{n\to\infty} g(x_n)=1$. By Hahn-Banach's extension theorem for linear functionals, there exists an $f \in X^*$ such that $\|f\|=1$ and $f(x)=g(x)$ for $x \in L$. Then we have $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} f(x_n)=1$ and $\|f\|=1$. By (D.2) there exists an $x_0 \in X$ such that w-$\lim_{n\to\infty} x_n = x_0$. Since $x_0 \in L$, we obtain $1=\lim_{n\to\infty} f(x_n)=f(x_0)=g(x_0)$. Moreover, we have $\|x_0\|=1$, because $1=f(x_0) \leqq \|x_0\| \leqq \lim_{n\to\infty} \|x_n\|=1$. Consequently, we see that g attains its supremum on the unit sphere of L. It follows from James [4, Theorem 2] that X is reflexive.

Now, let $x' \to f_{x'}$ be a support mapping of $X^*\backslash\{0\}$ to $X^{**}\backslash\{0\}$, and let $\{x'_n\}$ and z' be such that $x'_n, z' \in S(X^*)$ $(n \geqq 1)$ and $\lim_{n\to\infty} x'_n = z'$. We consider the sequence $\{y'_n\}$ defined by $y'_{2n-1}=x'_n$ and $y'_{2n}=z'$ $(n \geqq 1)$. Since we have $|f_{y'_n}(z')-1|=|f_{y'_n}(z')-f_{y'_n}(y'_n)| \leqq \|z'-y'_n\|$, we obtain $\lim_{n\to\infty} f_{y'_n}(z')=1$. Noting that $\|f_{y'_n}\|=1$ and viewing $f_{y'_n}$ $(n \geqq 1)$ as members of X, it follows from (D.2) that $\{f_{y'_n}\}$ converges weakly to some $f_0 \in X = X^{**}$. By the definition of $\{y'_n\}$, we must have $f_0 = f_{z'}$. Consequently, $\{f_{x'_n}\}$ converges weak-star to $f_{z'}$ in X^{**}. Q.E.D.

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