TOKYO J. MATH.
VOL. 7, NO. 2, 1984

The Existence of Periodic Orbits on the Sphere

Kiyoshi HAYASHI

Keio University

Introduction

In the theory of dynamical systems, there remains the open problem, so called Seifert Conjecture: Has any sufficiently smooth flow on S^3 a periodic orbit? This conjecture is based on Seifert's paper [11] which proved the following theorem.

THEOREM 1. *Let $x=(x_1, x_2)$, $y=(y_1, y_2)$ be points of $\boldsymbol{R}^2$ and consider the following equation in $\boldsymbol{R}^4$*

$$(1)\qquad \dot{x}_i=y_i,\ \dot{y}_i=-x_i;\ i=1, 2\ .$$

This system has $S^3=\{(x, y)\in \boldsymbol{R}^4; x_1^2+x_2^2+y_1^2+y_2^2=1\}$ as an invariant set, so we can consider the flow on S^3 induced by (1). Then any flow C^0 near the above flow on S^3 has at least one periodic orbit.

The system (1) is the Hamiltonian system with Hamiltonian

$$(2)\qquad H(x, y)=\frac{1}{2}(y_1^2+y_2^2)+\frac{1}{2}(x_1^2+x_2^2)\ ,$$

which describes the harmonic oscilaters.

More strongly, (2) is derived from the Lagrangian system

$$(3)\qquad \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}(T-U)=\frac{\partial}{\partial x_i}(T-U);\ i=1, 2$$

where

$$(4)\qquad T=\frac{1}{2}(\dot{x}_1^2+\dot{x}_2^2)\quad \text{and}\quad U=\frac{1}{2}(x_1^2+x_2^2)\ ,$$

with $y_i=(\partial T/\partial \dot{x}_i)=\dot{x}_i$ $(i=1, 2)$.

Received July 14, 1983

For a Lagrangian system, Seifert also obtained the following result [10].

THEOREM 2. *Let G be an open subset of $\boldsymbol{R}^n$ and consider a Lagrangian system of n degrees of freedom*

$$(5) \qquad \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}(T-U)=\frac{\partial}{\partial x_i}(T-U);\ i=1, 2, \cdots, n$$

where $x=(x_1, x_2, \cdots, x_n)\in G$ and

$$(6) \qquad T=\sum_{i,j=1}^{n} a_{ij}(x)\dot{x}_i\dot{x}_j\ , \quad a_{ij}\colon G\longrightarrow \boldsymbol{R};\ C^\infty\text{-function}$$

and $(a_{ij}(x))$ is symmetric and positive definite for all $x\in G$, and

$$(7) \qquad U\colon G\longrightarrow \boldsymbol{R};\ C^\infty\text{-function}\ .$$

Assume that, for some $e\in \boldsymbol{R}$, the set $W=\{x\in G;\ U(x)\leqq e\}$ is homeomorphic to the n-disk D^n. Then there exists at least one periodic solution of (5) *with total energy $T+U=e$.*

[10] originally treated C^ω-case, but it is not essential (See [9]). His periodic solution is a so called brake orbit [9], which stops at the boundary $\partial W=\{U=e\}$. [4], [15] prove Theorem 2 under the assumption that W is any smooth compact manifold with boundary.

In a footnote of [10], Seifert stated that:

(8) In the situation of Theorem 2, there may be n periodic orbits.

In this note, we give two theorems, stated in § 1, one of which generalizes Theorem 1 and another one answers the question (8) for special type of Lagrangians including (4). See also [15]. Both of them give periodic orbit(s) on an odd dimensional sphere.

§ 1. Statements of the theorems.

First we generalize Theorem 1. Let μ_i, $i=1, 2, \cdots, n$, be arbitrary positive numbers. We consider the following equation in $\boldsymbol{R}^{2n}$,

$$(9) \qquad \dot{x}_i=\mu_i y_i\ , \quad \dot{y}_i=-\mu_i x_i\ ; \quad i=1, 2, \cdots, n\ .$$

This defines the dynamical system on S^{2n-1}, which is derived from the tangent vector field

$$(10) \qquad \sum_{i=1}^{n}\mu_i\left(y_i\frac{\partial}{\partial x_i}-x_i\frac{\partial}{\partial y_i}\right).$$

In this case, there are at least n periodic orbits on S^{2n-1}.

Then we have

THEOREM 3. *In the above situation, any flow on S^{2n-1} generated a tangent vector field C^1 near* (10) *has at least one periodic orbit.*

F. Fuller [2] treated the case $\mu_1=\mu_2=\cdots=\mu_n=1$, when all solutions are periodic.

The system (9) is considered as a Hamiltonian system with Hamiltonian $H_0=(1/2)\sum_{i=1}^n \mu_i(x_i^2+y_i^2)$. So the ellipsoid $H_0^{-1}(e)$, $e>0$, is of course an invariant set, but in Theorem 3, we take another invariant set S^{2n-1}.

A. Weinstein [13] considered the Hamiltonian system with Hamiltonian $H=H_0+$[higher order] and proved that for sufficiently small $\varepsilon>0$, there exist at least n periodic solutions on $H^{-1}(\varepsilon^2)$. More general perturbation theory of Hamiltonian systems is given in [14].

J. Moser [5] generalized Weinstein's result replacing Hamiltonian systems with systems having an integral. The proof of Theorem 3 is based on the result of Moser.

Now we consider the Lagrangian system of n degrees of freedom (5) with $T=T(x, \dot{x})$ and $U=U(x)$ as (6) and (7).

DEFINITION. This Lagrangian system is called *rotationally symmetric* if for all $R\in O(n)$, x and $\dot{x}$, we have

$$U(Rx)=U(x)\ , \tag{11}$$

$$T(Rx, R\dot{x})=T(x, \dot{x})\ . \tag{12}$$

For example, $T=(1/2)|\dot{x}|^2$ and U depends only on $|x|$, or

$$T=\frac{1}{2}(|\dot{x}|^2+(\operatorname{grad} U(x), \dot{x})^2) \quad \text{and} \quad U=-(1-x_1^2-x_2^2)^{1/2}\ ,$$

which describes a spherical pendulum.

Then we have

THEOREM 4. *We consider a Lagrangian system* (5) *and assume that the system is rotationally symmetric and for some $e\in \boldsymbol{R}$, which is a regular value of U, $W=\{x;\ U(x)\leqq e\}$ is homeomorphic to the n-disk. Then any Lagrangian system C^2 near the above system has at least n periodic solutions with total energy e.*

The meaning of "C^2 *near*" is clarified in the proof, §4.

§ 2. The proof of Theorem 3.

The proof of Theorem 3 is based on the following Moser's result [6].

PROPOSITION 1. *Let $f=f(z)$, $z\in \boldsymbol{R}^{2n}$, be a C^1 function defined on a neighborhood of the origin $z=0$ in $\boldsymbol{R}^{2n}$ satisfying*

$$f(0)=0\ ,\quad f_z(0)=C \tag{13}$$

where C is defined by

$$C=\begin{bmatrix} 0 & \mu_1 & & & & & \\ -\mu_1 & 0 & & & & & \\ & & 0 & \mu_2 & & & \\ & & -\mu_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \mu_n \\ & & & & & -\mu_n & 0 \end{bmatrix} \tag{14}$$

where μ_i, $i=1,\cdots,n$, are arbitrary positive numbers.

Consider the following autonomous equation

$$\dot{z}=f(z)\ . \tag{15}$$

If there exists an integral $G=G(z)$ for the equation (15) *defined on a neighborhood of $z=0$ satisfying*

$$G(0)=0\ ,\quad G_z(0)=0 \quad \text{and} \quad G_{zz}(0)\text{: positive definite}\ , \tag{16}$$

then there exists $\delta>0$ such that for any $\varepsilon\in(0,\delta)$, the integral surface $G^{-1}(\varepsilon^2)$ contains at least one periodic orbit.

We define a domain $\Omega=\Omega_{r,\delta}$ by

$$\Omega_{r,\delta}=\{(z,\varepsilon)\in \boldsymbol{R}^{2n}\times \boldsymbol{R};\ |z|<r,\ |\varepsilon|<\delta\}\ , \tag{17}$$

where $|z|^2=z_1^2+\cdots+z_{2n}^2$, and denote by $\mathfrak{B}_0$ the Banach space of all real valued bounded continuous functions defined on $\Omega_{r,\delta}$ with the norm

$$|u|_{0,r,\delta}=\sup\{|u(z,\varepsilon)|;\ (z,\varepsilon)\in\Omega_{r,\delta}\}\ . \tag{18}$$

Also let $\mathfrak{B}_1$ be the Banach space of all C^1 functions in $\mathfrak{B}_0$ which have bounded derivatives with the norm

$$|u|_{1,r,\delta}=\mathrm{Max}\{|u|_{0,r,\delta},\ |u_z|_{0,r,\delta},\ |u_\varepsilon|_{0,r,\delta}\}\ . \tag{19}$$

We put

(20) $$f(z, \varepsilon)=\varepsilon^{-1}f(\varepsilon z) \quad \text{for} \quad \varepsilon\neq 0 , \qquad f(z, 0)=Cz$$

and

(21) $$p(z, \varepsilon)=f(z, \varepsilon)-Cz .$$

Then we have the following.

LEMMA 1. *In Proposition 1, we assume that $G(z)=|z|^2$ and there exists a constant $L\geqq 1$ such that for any small $\varepsilon_1>0$, we have*

(22) $$|p|_{1,2,\varepsilon_1}\leqq L\varepsilon_1 .$$

Then the δ in Proposition 1 depends only on C and L.

The proof of this lemma is obtained by a similar fashion as Moser's proof, modifying slightly to suit our situations. So we omit the proof.

PROOF OF THEOREM 3. Let $\mathfrak{X}$ be the set of all C^1 tangent vector field on $S=S^{2n-1}$ and $q\in\mathfrak{X}$. q is the restriction to S of a C^1 mapping $\bar{q}$ from an open neighborhood of S into $\boldsymbol{R}^{2n}$. For $w\in S$, we denote by $q'(w)$ the restriction of $\bar{q}'(w)\colon \boldsymbol{R}^{2n}\to\boldsymbol{R}^{2n}$ to $T_wS\subset\boldsymbol{R}^{2n}$, which is independent of the extension $\bar{q}$. Also $|q'(w)|$ denotes the operator norm.

We put $|q|_1=\mathrm{Max}\{\mathrm{Max}_{w\in S}|q(w)|, \mathrm{Max}_{w\in S}|q'(w)|\}$, then $(\mathfrak{X}, |\ \ |_1)$ is a Banach space.

Put $p(z)=|z|^3q(z/|z|)$ and $f(z)=Cz+p(z)$. $p(0)$ should be regarded as 0 automatically and hereafter such a remark shall be omitted. $p=p(z)$ is of C^1 class on $|z|<\infty$ and

$$p'(z)\cdot h=|z|^2\{3(w, h)q(w)+\bar{q}'(w)\cdot(h-(w, h)w)\}$$

where $w=z/|z|$ and $h\in\boldsymbol{R}^{2n}$. Since $h-(w, h)w$ is the orthogonal projection of h onto T_wS, we have

$$|p'(z)\cdot h|\leqq|z|^2(3|q(w)|+|q'(w)|)|h| ,$$

hence $|p'(z)|\leqq 4|z|^2|q|_1$.

Now we consider any vector field $q\in\mathfrak{X}$ with $|q|_1=1$. In this case

$$\begin{aligned} p(z, \varepsilon)&=\varepsilon^{-1}p(\varepsilon z)\\ &=(\operatorname{sgn}\varepsilon)\varepsilon^2 p(\operatorname{sgn}\varepsilon\cdot z) ,\end{aligned}$$

so $p_z=\varepsilon^2 p'(\operatorname{sgn}\varepsilon\cdot z)$ and $p_\varepsilon=2|\varepsilon|p(\operatorname{sgn}\varepsilon\cdot z)$.

Thus we have $|p|_{1,2,\varepsilon_1} \leqq 16\varepsilon_1$ for small $\varepsilon_1 > 0$.

$f(z)$ satisfies the condition in Proposition 1 and we can take $G(z)=|z|^2$ as an integral. Thus, by Proposition 1 and Lemma 1, there exists $\delta>0$, depending only on C, such that for any $0<\varepsilon<\delta$, the integral surface $G^{-1}(\varepsilon^2)$ contains a periodic orbit.

The vector field on S corresponding to the vector field on $G^{-1}(\varepsilon^2)$ by the transformation

$$w=\varepsilon^{-1}z \quad (w \in S,\ z \in G^{-1}(\varepsilon^2))$$

is $\varepsilon^{-1}f(\varepsilon w)=\varepsilon^{-1}(C\varepsilon w+\varepsilon^3 q(w))=Cw+\varepsilon^2 q(w)$. This vector field on S has a periodic orbit and δ is independent of $q \in \{q \in \mathfrak{X};\ |q|_1=1\}$, hence any C^1 vector field belonging to the δ^2-neighborhood of C in the Banach space $\mathfrak{X}$ has a periodic orbit.

This completes the proof of Theorem 3.

§ 3. Mini-max principle with involution.

The n solutions of Theorem 4 are obtained as critical points of a function which is invariant under an involution reflecting the reversibility of the system.

Let X be a Hausdorff space and $\xi: X \to X$ be a continuous involution, that is, $\xi\circ\xi=\mathrm{id}$. We denote by $(S^\infty \times X)_\Pi$ the orbit space of $S^\infty \times X$ under the involution $(\zeta, x)\mapsto(-\zeta, \xi x)$. For an invariant subset $A\subset X$, we define the equivariant (co)homology groups by $H_*^\Pi(X, A)=H_*((S^\infty\times X)_\Pi, (S^\infty\times A)_\Pi)$ and $H_\Pi^*(X, A)=H^*((S^\infty\times X)_\Pi, (S^\infty\times A)_\Pi)$. The coefficient field $\boldsymbol{Z}_2$ is always understood.

Then we have the following equivariant version of Mini-Max Principle [5].

LEMMA 2. *Let Λ be a complete Hilbert manifold and $f: \Lambda\to[0, \infty)$ a smooth function satisfying the condition (C) of Palais-Smale. Assume that there is a smooth involution $\xi: \Lambda\to\Lambda$ satisfying*

(i) *$f\circ\xi=f$,*

(ii) *ξ is isometric,*

(iii) *for small $\varepsilon>0$, Λ^0 is a deformation retract of Λ^ε and the homotopy using there is equivariant ($\Lambda^a=f^{-1}[0, a]$),*

(iv) *if $df(\lambda)=0$ and $f(\lambda)\geqq\varepsilon$, then $\xi\lambda\neq\lambda$.*

Then the equivariant version of pairwise subordinated homology classes [5] give critical points of f. That is, if there exist $b\in H_*^\Pi(\Lambda, \Lambda^\varepsilon)$ and $\theta_1, \cdots, \theta_r \in H_\Pi^*(\Lambda-\Lambda^0)$ with $\deg\theta_i>0$ and $(\theta_1\cup\cdots\cup\theta_r)\cap b\neq0$, then there exist at least $r+1$ critical points with $f\geqq\varepsilon$. (In counting critical *points, we identify* λ *and* $\xi\lambda$).

PROOF. We put $\tilde{\Lambda}=(S^\infty\times\Lambda)_\Pi$ and define $\tilde{f}\colon\tilde{\Lambda}\to[0,\infty)$ by $\tilde{f}[\zeta,\lambda]=f(\zeta,\lambda)$, where $[\zeta,\lambda]$ is the element of $\tilde{\Lambda}$ represented by (ζ,λ). Define

$$c=\inf_{z\in b}\operatorname{Max}\tilde{f}(|z|)\,, \tag{23}$$

where $|z|=\cup_i\operatorname{Im}\sigma_i$ if $z=\sum_i\sigma_i$. As in [5], since $b\in H_*(\tilde{\Lambda},\tilde{\Lambda}^\varepsilon)$, where $\tilde{\Lambda}^\varepsilon=(S^\infty\times\Lambda^\varepsilon)_\Pi$, we have $c\geqq\varepsilon$.

First we claim that

$$c \text{ is a critical value of } f\,. \tag{24}$$

Let $\phi_s\colon\Lambda\to\Lambda$, $0\leqq s<\infty$, be the deformation generated by $-\operatorname{grad}f$ and put

$$K_c=\{\lambda\in\Lambda;\ f(\lambda)=c \quad\text{and}\quad df(\lambda)=0\}\,.$$

In the proof of 2.1.2. in [5], the following fact is given.

(25) Let U be an open neighborhood of $K_c(c\geqq0)$ and $\rho>0$ be sufficiently small. Then for every $\lambda\in\Lambda^{c+\rho}$, there exists a neighborhood U_λ of λ and $s_\lambda\geqq0$ such that $\phi_sU_\lambda\subset U\cup\Lambda^{c-}$ for $s\geqq s_\lambda(\Lambda^{c-}=f^{-1}[0,c))$.

Now by (i) and (ii), we can define $\tilde{\phi}_s\colon\tilde{\Lambda}\to\tilde{\Lambda}$ by $\tilde{\phi}_s[\zeta,\lambda]=[\zeta,\phi_s(\lambda)]$. To prove (24), we assume c is not a critical value, that is $K_c=\varnothing$.

By the definition of c, there is a chain $z\in b$ such that $|z|\subset\tilde{\Lambda}^{c+\rho}$, where $\rho>0$ is in (25) when we take $U=\varnothing$.

For any $[\zeta,\lambda]\in|z|$, $\pi(S^\infty\times U_\lambda)$ is an open neighborhood of $[\zeta,\lambda]$, where $\pi\colon S^\infty\times\Lambda\to\Lambda$ is the projection, and $\tilde{\phi}_s(\pi(S^\infty\times U_\lambda))\cup\tilde{\Lambda}^{c-}$ for $s\geqq s_\lambda$ by (25).

Since $|z|$ is compact, $\phi_{s'}(|z|)\subset\Lambda^{c-}$ for some $s'>0$, but $\phi_{s'*}(z)\in b$.

This contradiction gives (24).

Now $\theta\in H^*(\tilde{\Lambda}-\tilde{\Lambda}^0)$, $\deg\theta>0$, and let $a=\theta\cap b$ be the nonzero element of $H_*(\tilde{\Lambda},\tilde{\Lambda}^\varepsilon)$. This cap product can be taken by (iii) as in [5].

Let c' be the critical value defined by (23) replacing b with a. Then we have $\varepsilon\leqq c'\leqq c$ as in [5].

Finally we give

(26) if $c'=c$, then there exist infinitely many critical points in $f^{-1}(c)$.

To prove (26), assume that there are only finite critical points

$$\lambda_1,\lambda_2,\cdots,\lambda_k;\ \xi\lambda_1,\cdots,\xi\lambda_j$$

in the level $f=c$ ($df(\lambda)=0$ implies $df(\xi\lambda)=0$ and $\lambda_j\neq\xi\lambda_j$ by (iv)).

We can choose contractible neighborhoods U_j of λ_j in $\Lambda-\Lambda^0$ so that

$U_1, \cdots, U_k; \xi U_1, \cdots, \xi U_k$ are all disjoint.

Put $U = \cup_{j=1}^{k} (U_j \cup \xi U_j)$ and $W = \pi(S^\infty \times U)$. Then $W \approx S^\infty \times (U_1 \cup \cdots \cup U_k)$, hence $H^{\deg \theta}(W) = 0$.

For $\rho > 0$ in (25), there is $z \in b$ such that $|z| \subset \Lambda^{c+\rho}$. As in the proof of (24), we have $\tilde{\phi}_{s'}(|z|) \subset W \cup \tilde{\Lambda}^{c-}$ for some $s' \geqq 0$. This derives a contradiction, as in the proof of 2.1.10 in [5], proving (26).

These arguments yield the lemma as in [5]. Q.E.D.

Using Lemma 2, we have

LEMMA 3. *Let V be an open subset of a Riemannian manifold such that for any x and y in V, there exists the unique shortest geodesic whose length equals to $d(x, y)$, the Riemannian distance, and $f(x, y) = d(x, y)^2$ is smooth in x and y. Then for any compact submanifold N in V, there exist at least* $\dim N + 1$ *nonconstant geodesics starting from and ending at N orthogonally.*

In [3], the same result, replacing V by the Euclidian space with complete Riemannian metric, is given. Theorem 1 in [3] is based on [8], but Lemma 2 in this note also give the theorem.

PROOF OF LEMMA 3. We apply Lemma 2 for $\Lambda = N \times N$, $f = f(x, y)$ and $\xi(x, y) = (y, x)$. Critical points of f with $f > 0$ gives the desired geodesics.

The assumptions (i), (ii) and (iv) are easily seen. As in the proof of Theorem 8.48 in [12], the following estimate gives (iii).

(27) For some $\varepsilon > 0$, $f(x, y) \leqq 2|\operatorname{grad} f(x, y)|^2$, if $x, y \in N$ and $f(x, y) \leqq \varepsilon$.

This is given since $\operatorname{grad} f(x, y)$ has order $d(x, y)$ and N is compact. Therefore Theorem 2 in [3] and the naturality of the (co)homology theory give the lemma. Q.E.D.

§ 4. Proof of Theorem 4.

Consider the system written in Theorem 4. By (11), $U = U(x)$ can be written as $U(x) = U_1(|x|)$ for some smooth function $U_1 = U_1(r)$, and since $W \approx D^n$, U_1 satisfies

$$(28) \qquad \begin{aligned} &U_1(r) < e \quad \text{for} \quad 0 \leqq r < r_0,\ U(r_0) = e \quad \text{and} \\ &U_1'(r_0) > 0 \quad \text{for some} \quad r_0 > 0\,. \end{aligned}$$

Therefore there are $\rho > 0$ and $\delta > 0$ with

(29) $U_1 \leqq e-2\delta$ on $[0, r_0-\rho]$, $U_1 \geqq e+2\delta$ on $[r_0+\rho, r_0+2\rho]$ and $U_1' \geqq 2\delta/r_0$ on $[r_0-2\rho, r_0+2\rho]$.

Put $W'=\{x \in \boldsymbol{R}^n;\ |x|<r_0+2\rho\}$ and denote by $\mathfrak{B}^k(W', \boldsymbol{R}^j)$ the set of C^k-functions $u: W' \to \boldsymbol{R}^j$ with

(30) $$\|u\|_k = \operatorname*{Max}_{0 \leq k' \leq k} \left\{ \sup_{x \in W'} |D^{k'}u(x)| \right\} < \infty \ .$$

For notations used in this section, see [1].

Then there is a neighborhood $\mathscr{U}$ of U in $\mathfrak{B}^1(W', \boldsymbol{R})$ such that for any $\tilde{U}$ in $\mathscr{U}$, e is a regular value of $\tilde{U}$ and there is a smooth function

$$\tilde{B}: \partial W \to (1-\rho/r_0,\ 1+\rho/r_0)$$

such that $\tilde{W}=\{x \in W';\ \tilde{U}(x) \leqq e\}$ is written as $\{\alpha b \in W';\ b \in \partial W \text{ and } 0 \leqq \alpha \leqq \tilde{\beta}(b)\}$. Furthermore the mapping $\tilde{U} \mapsto \tilde{\beta}$ from $\mathscr{U}$ into $C^1(\partial W, R)$ is continuous. This is given by the implicit function theorem applied to the function $F=F(\tilde{U}, b, \alpha)=e-\tilde{U}(\alpha b)$, which is C^1 by Theorem 10.3 in [1]. $\tilde{W}$ is also diffeomorphic to the n-disk D^n.

The system (5) is characterized by the functions $U(x)$ and $a_{ij}(x)$, $1 \leqq i \leqq j \leqq n$. We put $\boldsymbol{Z}=(U, a_{ij})$, then $Z \in \mathfrak{B}^2(W', \boldsymbol{R}^{1+n(n+1)/2}) \equiv \mathfrak{B}^2$.

"C^2 near" in Theorem 4 means "near with respect to the norm of $\mathfrak{B}^2$".

For sufficiently small neighborhood $\mathscr{W}$ of Z in $\mathfrak{B}^2$, $\tilde{Z}=(\tilde{U}, \tilde{a}_{ij}) \in \mathscr{W}$ implies $\tilde{U} \in \mathscr{U}$ and $(\tilde{a}_{ij})$ is positive definite.

For $b \in \partial W=\{x;\ |x|=r_0\}$ and $\tilde{Z} \in \mathscr{W}$, let $\Phi(\tilde{Z}, b, t)$ be the solution of $(\tilde{5})$, the system which is given by replacing Z with $\tilde{Z}$ in (5), with

$$\Phi(\tilde{Z}, b, 0)=\tilde{\beta}(b)b \quad \text{and} \quad \frac{\partial}{\partial t}\Phi(\tilde{Z}, b, 0)=0 \ .$$

Since the system corresponding to Z is rotationally symmetric, $\Phi(Z, b, t)$ can be written as

(31) $$\Phi(Z, b, t)=h(t)b \ , \quad 0 \leqq t \leqq K_0 \ , \quad \text{for some} \quad K_0>0 \ ,$$

where $h=h(t)$ is a smooth function with

(32) $$h(0)=1 \ , \quad \dot{h}(0)=0 \ , \quad \ddot{h}(0)<0 \ ,$$
$$\dot{h}(t)<0 \quad \text{for} \quad 0<t \leqq K_0 \quad \text{and} \quad h(K_0)=0 \ .$$

The solution $\Phi(\tilde{Z}, b, t)$ is a geodesic w.r.t. the metric

(33) $$ds^2=(e-\tilde{U}(x))\tilde{a}_{ij}(x)dx_i dx_j \ ,$$

after a time change.

The following fact is obtained by the standard way.

(34) There are $r_1>0$ and a neighborhood $\mathscr{W}_1\subset\mathscr{W}$ of Z in $\mathfrak{B}^2$ such that for any $\tilde{Z}\in\mathscr{W}_1$, the set $V=\{x\in\boldsymbol{R}^n;\ |x|<2r_1\}$ has the property of V in Lemma 3 w.r.t. the metric (33).

We take K_1, $0<K_1<K_0$, so that $h(K_1)=r_1$. Let S_1 be the length of the curve $\Phi(Z, b, t)$, $0\leqq t\leqq K_1$, w.r.t. (33) for Z and, for $\tilde{Z}\in\mathscr{W}_1$ and $b\in\partial W$, let $t_1=t_1(\tilde{Z}, b)$ be the time satisfying

$$[\text{the length of the curve } \Phi(\tilde{Z}, b, t),\ 0\leqq t\leqq t_1,\ \text{w.r.t. (33)}]=S_1\ .$$

Then $t_1(Z, b)=K_1$ for all $b\in\partial W$. And put $Q(\tilde{Z}, b)=\Phi(\tilde{Z}, b, t_1(\tilde{Z}, b))$. We claim

(35) There is a neighborhood $\mathscr{W}_2\subset\mathscr{W}_1$ of Z in $\mathfrak{B}^2$ such that for any $\tilde{Z}\in\mathscr{W}_2$, $Q(\tilde{Z}, \cdot)$ is an embedding from ∂W into V.

This is also given by the implicit function theorem.

The image of the embedding $\tilde{N}=\{Q(\tilde{Z}, b);\ b\in\partial W\}$ is a compact submanifold of V with dimension $n-1$. So, by Lemma 3, there exist n geodesics w.r.t. the metric (33), starting from and ending at $\tilde{N}$ orthogonally.

This proves the Theorem as [10] or [4]. Q.E.D.

References

[1] R. Abraham and J. Robbin, Transversal Mapping and Flows, Benjamin, New York, 1967.

[2] F. Fuller, An index of fixed point type for periodic orbits, Amer. J. Math, **89** (1967), 133–148.

[3] K. Hayashi, Double normals of a compact submanifold, Tokyo J. Math., **5** (1982), 419–425.

[4] K. Hayashi, Periodic solution of Classical Hamiltonian systems, Tokyo J. Math., **6** (1983), 473–486.

[5] W. Klingenberg, Lectures on Closed Geodesics, Springer, Berlin-Heidelberg-New York, 1978.

[6] J. Moser, Periodic orbits near an equilibrium and a Theorem by Alan Weinstein, Comm. Pure Appl. Math., **29** (1976), 727–747.

[7] P. Rabinowitz, Periodic solutions of Hamiltonian systems, Comm. Pure Appl. Math., **31** (1978), 157–184.

[8] A. Ried, Lot Geodetische, Arch. Math., **19** (1968), 103–112.

[9] O. Ruiz, Existence of brake orbits in Finsler mechanical systems, Lecture Notes in Math. 597, Geometry and Topology, Springer, Berlin-Heidelberg-New York, 1976, 542–567.

[10] H. Seifert, Periodische Bewegungen mechanischer Systeme, Math. Z., **51** (1948), 197–216.

[11] H. SEIFERT, Closed integral curves in 3-space and isotropic two-dimensional deformations, Proc. Amer. Math. Soc., **1** (1950), 287–302.
[12] J. SCHWARTZ, Nonlinear Functional Analysis, Gordon and Breach, New York-London-Paris, 1969.
[13] A. WEINSTEIN, Normal modes for nonlinear Hamiltonian systems, Invent. Math., **20** (1973), 47–57.
[14] A. WEINSTEIN, Bifurcations and Hamilton's Principle, Math. Z., **159** (1978), 235–248.
[15] H. GLUCK and W. ZILLER, Existence of periodic motions of conservative system, Seminar on Minimal Submanifolds, Princeton University Press, 1983, 65–98.

Present Address:
DEPARTMENT OF MATHEMATICS
KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU
YOKOHAMA 223