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# A Characterization of Cyclical Monotonicity by the Gâteaux Derivative

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## Introduction

Let $X$ be a real Banach space and $X'$ be its dual space. In this paper, we characterize the (maximal) cyclical monotonicity of a $w^*$-Gâteaux differentiable (nonlinear) operator: $X \to X'$, by means of the Gâteaux derivative. Our result is a nonlinear version of the well-known proposition; A linear and densely defined maximal monotone operator in a Hilbert space is cyclically monotone if and only if it is self-adjoint.

We give an equivalent condition for a $w^*$-Gâteaux differentiable operator from $X$ to $X'$ to be cyclically monotone, under some assumptions. Furthermore we give sufficient conditions for a ($w$-)Gâteaux differentiable operator in a Hilbert space to be maximal cyclically monotone. For instance, our Corollary 1 says that an operator $A$ in a Hilbert space is maximal cyclically monotone, if $\overline{\delta A(x)}$, the minimal closed extension of the Gâteaux derivative of $A$ at $x$, is positive self-adjoint for each $x$ in the domain of $A$, under a suitable assumption.

## §1. Preliminaries.

Throughout this paper we use the following notations and definitions.

$X$ denotes a real Banach space with norm $\| \ \|$, and $X'$ denotes its dual space. We denote by $(x, f)$ the pairing between $x \in X$ and $f \in X'$. Especially if $X$ is a real Hilbert space, $( \ , \ )$ is the inner product and we use the notation $H$ instead of $X$.

For a subset $S$ of $X$, $\bar{S}$ denotes the closure of $S$ in $X$.

Let $A$ be an operator from $X$ to $X'$. $D(A)$ denotes the domain of $A$ and $R(A)$ denotes the range of $A$. We denote the minimal closed extension of $A$ by $\bar{A}$.

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Let $A$ be a linear operator from $X$ to $X'$. $A$ is said to be *symmetric* if $(x, Ay)=(y, Ax)$ for every $x$ and $y$ in $D(A)$. $A$ is said to be *positive* if $(x, Ax)\geqq 0$ for every $x$ in $D(A)$.

A (multi-valued) operator $A$ in $H$ is said to be *monotone* if $(x_1-x_2, x_1'-x_2')\geqq 0$ whenever $x_i'\in Ax_i$, $i=1, 2$. $A$ monotone operator $A$ is said to be *maximal monotone* if it has no monotone extensions in $H$. It is well-known that a monotone operator $A$ in $H$ is maximal monotone if and only if $R(I+\lambda A)=H$ for some $\lambda>0$.

A (multi-valued) operator $A: X\to X'$ is said to be *cyclically monotone* if $\sum_{i=1}^{n}(x_i-x_{i-1}, x_i')\geqq 0$ whenever $x_i'\in Ax_i$, $x_n=x_0$, $x_n'=x_0'$. $A$ cyclically monotone operator $A$ is said to be *maximal cyclically monotone* if it has no cyclically monotone extensions from $X$ to $X'$.

Let $\phi: X\to(-\infty, \infty]$ be a convex functional. Also assume that $\phi$ is proper, i.e. that its effective domain $D(\phi)=\{x\in X; \phi(x)<\infty\}$ is nonempty. Then the *subdifferential* of $\phi$ is defined by

$$\partial\phi(x)=\{z\in X'; \phi(w)-\phi(x)\geqq(w-x, z) \text{ for all } w\in X\}\ .$$

$\partial\phi: X\to X'$ is cyclically monotone. Furthermore, it holds that an operator $A: X\to X'$ is maximal cyclically monotone if and only if $A=\partial\phi$ for some lower-semicontinuous proper convex functional $\phi$.

DEFINITION. Let $A: X\to X'$ be a single-valued operator with convex domain. We shall say that $A$ is *Gâteaux differentiable* on $D(A)$ if there is a linear operator $\delta A(x): X\to X'$ such that

$$\lim_{\substack{\lambda\to 0\\ x+\lambda y\in D(A)}}\frac{1}{\lambda}\{A(x+\lambda y)-Ax\}=\delta A(x)y \quad \text{for } {}^{\forall}y\in X' \text{ with } x+y\in D(A)\ , \tag{1.1}$$

for every $x\in D(A)$. Furthermore, $\delta A(x)$ is called the *Gâteaux derivative* of $A$ at $x$. If the convergence in (1.1) is in the weak (resp. $w^*$)-topology, we say that $A$ is $w$ (resp. $w^*$)*-Gâteaux differentiable.*

## §2. Theorem and proof.

THEOREM. *Let $A: X\to X'$ be a $w^*$-Gâteaux differentiable operator on convex domain $D(A)$ and $w^*$-continuous on every 2-dimensional subset in $D(A)$. Then the following three conditions are equivalent.*

1°) *$A: X\to X'$ is cyclically monotone.*

2°) *$\delta A(x): X\to X'$ is cyclically monotone for each $x\in D(A)$.*

3°) *$\delta A(x): X\to X'$ is positive symmetric for each $x\in D(A)$.*

REMARK 1. Let $A$ be an operator in a Hilbert space $H$. Suppose that there is a dense Banach space $Y$ such that $Y \subset H = H' \subset Y'$, and $\tilde{A}: Y \to Y'$ such that $A = \tilde{A}_H$ (the restriction of $\tilde{A}$ to $D(\tilde{A}_H) = \{x; \tilde{A}x \in H\}$). If $\tilde{A}: Y \to Y'$ is cyclically monotone, then $A$ is cyclically monotone in $H$. Hence, if $\tilde{A}$ satisfies the hypothesis of Theorem and the condition 2°) or 3°), then $A$ is cyclically monotone.

To prove Theorem, we shall show the following lemmas.

LEMMA 1. *Let $A: X \to X'$ be an operator with convex domain, and be $w^*$-continuous on every 1-dimensional subset in $D(A)$. Suppose that there is $x_0 \in D(A)$ such that*

$$\int_0^1 (y, A(x_0 + sy))ds + \int_0^1 (z, A(x_0 + y + sz))ds = \int_0^1 (y+z, A(x_0 + s(y+z)))ds \tag{2.1}$$

*for every $y, z \in X$ with $x_0 + y$, $x_0 + y + z \in D(A)$. If $\phi$ is defined by*

$$\phi(x) = \int_0^1 (x - x_0, A(x_0 + s(x - x_0)))ds \qquad \text{for} \quad x \in D(A) , \tag{2.2}$$

*then for each $x, y \in D(A)$, the function $t \mapsto \phi(x + t(y - x))$ is differentiable on $[0, 1]$ and*

$$\frac{d}{dt}\phi(x + t(y-x)) = (y - x, A(x + t(y - x))) \qquad \text{for} \quad 0 \leqq t \leqq 1 .$$

PROOF. Let $u$ and $v$ be any elements of $D(A)$. We put $v_1 = v - u$. Taking $y = u - x_0 + tv_1$, $z = hv_1$ $(0 \leqq t,\ t + h \leqq 1)$ in (2.1), we have

$$\phi(u + tv_1) + \int_0^1 (hv_1, A(u + tv_1 + shv_1))ds = \phi(u + tv_1 + hv_1) .$$

Hence, we have that

$$\frac{1}{h}\{\phi(u + (t+h)v_1) - \phi(u + tv_1)\} = \int_0^1 (v_1, A(u + tv_1 + shv_1))ds . \tag{2.3}$$

Since $(v_1, A(u + tv_1 + shv_1))$ is continuous in $h$, by letting $h \to 0$, the right-hand side of (2.3) converges to $(v_1, A(u + tv_1))$. Thus the assertion holds.

LEMMA 2. *Let $A: X \to X'$ be a cyclically monotone operator with convex domain, and be $w^*$-continuous on every 1-dimensional subset in $D(A)$. Then $A$ satisfies the hypothesis of Lemma 1.*

PROOF. Let $x$, $x+y$ and $x+y+z$ be any elements of $D(A)$. We set

$$x_i = x + \frac{i}{n} y \,, \qquad y_j = x + y + \frac{j}{n} z \,, \qquad z_k = x + \frac{k}{n}(y+z)$$

for $i, j, k = 0, 1, \cdots, n$. From the covexity of $D(A)$ we have

$$x_i,\ y_j,\ z_k \in D(A)$$

for $i, j, k = 0, 1, \cdots, n$. From the definition of $x_i$, $y_j$ and $z_k$, we have $x_{i+1} - x_i = (1/n)y$, $y_{j+1} - y_j = (1/n)z$, $z_k - z_{k+1} = -(1/n)(y+z)$ for $i, j, k = 0, 1, \cdots, n-1$. Thus, for the cyclical sequence $\{x = x_0, x_1, \cdots, x_n = x+y = y_0, y_1, \cdots, y_n = x+y+z = z_n, z_{n-1}, \cdots, z_0 = x = x_0\}$, we use the cyclical monotonicity of $A$ to have

$$\sum_{k=0}^{n-1} \left(\frac{1}{n}(y+z),\ Az_k\right) \leqq \sum_{i=1}^{n} \left(\frac{1}{n} y,\ Ax_i\right) + \sum_{j=1}^{n} \left(\frac{1}{n} z,\ Ay_j\right) . \tag{2.4}$$

Similarly, for $\{z_0, z_1, \cdots, z_n = y_n, y_{n-1}, \cdots, y_0 = x_n, x_{n-1}, \cdots, x_0 = z_0\}$, we use the cyclical monotonicity of $A$ to have

$$\sum_{k=1}^{n} \left(\frac{1}{n}(y+z),\ Az_k\right) \geqq \sum_{i=0}^{n-1} \left(\frac{1}{n} y,\ Ax_i\right) + \sum_{j=0}^{n-1} \left(\frac{1}{n} z,\ Ay_i\right) . \tag{2.5}$$

Letting $n \to \infty$ in (2.4), we get

$$\int_0^1 (y+z,\ A(x+t(y+z)))dt \leqq \int_0^1 (y,\ A(x+ty))dt + \int_0^1 (z,\ A(x+y+tz))dt \,.$$

Letting $n \to \infty$ in (2.5), the reverse inequality holds in the above. Hence we obtain (2.1) for any $x \in D(A)$.

LEMMA 3. *Let $u(t, s)$ and $v(t, s)$ be partially differentiable and continuous real-valued functions on a simply connected domain $D \subset \boldsymbol{R}^2$, and suppose that $(\partial u/\partial t) = (\partial v/\partial s)$ on $D$. Then $\int_Q (uds + vdt) = 0$ for every polygon $Q$ in $D$.*

PROOF. If $u$ and $v$ are $C^1$-class functions on $D$, we have the conclusion by Green's theorem. Thus the assertion of Lemma 3 follows by using the mollifier.

LEMMA 4. *Let $A: X \to X'$ be a $w^*$-Gâteaux differentiable operator on convex domain $D(A)$ and $w^*$-continuous on every 2-dimensional subset in $D(A)$. If $\delta A(x)$ is symmetric for each $x \in D(A)$, then $A$ satisfies the assumption of Lemma 1.*

PROOF. Let $x$, $y$ and $z$ be elements of $X$ with $x$, $x+y$ and $x+y+z \in D(A)$. We set

$$P=\int_0^1 (y, A(x+sy))ds+\int_0^1 (z, A(x+y+sz))ds -\int_0^1 (y+z, A(x+s(y+z)))ds .$$

We have only to prove that $P=0$. If $y$ and $z$ are linearly dependent, this is trivial from the definition of the integral. Hence, we may assume that $y$ and $z$ are linearly independent. We set

$$g(t, s)=(y, A(x+ty+sz))$$
$$h(t, s)=(z, A(x+ty+sz)) .$$

Since $D(\delta A(x))\supset D(A)-x$ for every $x \in D(A)$, $D(A)$ is convex and $A$ is $w^*$-continuous on every 2-dimensional subset in $D(A)$, we easily see that $g$ and $h$ are partially differentiable and continuous on domain $D\supset\{(t, s); 0\leqq s\leqq t\leqq 1\}$. Moreover we have

$$\frac{\partial}{\partial s}g(t, s)=(y, \delta A(x+ty+sz)z)$$
$$\frac{\partial}{\partial t}h(t, s)=(z, \delta A(x+ty+sz)y) .$$

Noting that $\delta A(x+ty+sz)$ is symmetric, these imply that

$$\frac{\partial}{\partial s}g(t, s)=\frac{\partial}{\partial t}h(t, s) \quad \text{on} \quad D .$$

Hence, applying Lemma 3 to $u=h$, $v=g$ and $Q=\{(t, 0); 0\leqq t\leqq 1\}\cup\{(1, s); 0\leqq s\leqq 1\}\cup\{(t, t); 0\leqq t\leqq 1\}$, we obtain that

$$P=\int_Q (g(t, s)dt+h(t, s)ds)=0 .$$

Now we shall prove Theorem.

PROOF OF THEOREM. "3°) implies 1°)." Suppose that 3°) holds. Then it holds by Lemma 4 that $A$ satisfies the hypothesis of Lemma 1. Let $\phi$ be the functional on $D(A)$, defined by (2.2), which satisfies the conclusion of Lemma 1. We extend $\phi$ on $X$, (which denotes the same $\phi$) as follows: $\phi(x)=\infty$ for $x\notin D(A)$. We devide the proof of 3°)$\Rightarrow$1°) into the following two steps.

1) We shall show that $\phi: X\to(-\infty, \infty]$ is convex and proper. Since $D(A)\neq\emptyset$, $\phi$ is proper. Thus we only need to prove the convexity of $\phi$, i.e.,

$$t\phi(x)+(1-t)\phi(y)\geqq\phi\{tx+(1-t)y\}$$

for $x, y\in X$, $0\leqq t\leqq 1$. Since $D(A)$ is convex and $\phi(x)=\infty$ for $x\notin D(A)$, the last inequality is trivial when $x$ or $y\notin D(A)$. Thus we have only to show that $\phi$ is convex on $D(A)$. Let $x$ and $y$ be any elements of $D(A)$. Then, for $0\leqq t\leqq 1$, we have

$$\begin{aligned}\frac{d^2}{dt^2}\phi(x+t(y-x))&=\frac{d}{dt}(y-x,\ A(x+t(y-x)))\\&=(y-x,\ \delta A(x+t(y-x))(y-x))\\&\geqq 0\ .\end{aligned}\tag{2.6}$$

At the last inequality of (2.6), we used the positivity of $\delta A(x+t(y-x))$. (2.6) implies that $\phi$ is convex on $D(A)$.

2) We shall show that $A\subset\partial\phi$. Let $x, y\in D(A)$, and $t\in(0, 1)$. From 1), we have

$$\phi(x+t(y-x))=\phi((1-t)x+ty)\leqq(1-t)\phi(x)+t\phi(y)\ .$$

Therefore,

$$\frac{1}{t}\{\phi(x+t(y-x))-\phi(x)\}\leqq\phi(y)-\phi(x)\ .$$

Letting $t\downarrow 0$, it follows from the property of $\phi$ that

$$(y-x,\ Ax)\leqq\phi(y)-\phi(x)\ .$$

This inequality is obviously true for $y$ which is not in $D(A)$. Therefore, $x\in D(\partial\phi)$ and $Ax\subset\partial\phi(x)$ if $x\in D(A)$. This implies that $A\subset\partial\phi$. Hence, $A$ is cyclically monotone.

"1°) imples 2°)." Suppose that 1°) is satisfied. Let $x$ be any fixed element of $D(A)$. We must show that $\delta A(x): X\to X'$ is cyclically monotone. Let $x_0, x_1, \cdots, x_n=x_0\in D(\delta A(x))$. Then there is an $\eta>0$ such that

$$x+tx_i\in D(A) \quad \text{for} \quad |t|<\eta \quad (i=1, \cdots, n)\,.$$

Since $A$ is cyclically monotone, we have

$$\sum_{i=1}^{n}(t(x_i-x_{i-1}),\ A(x+tx_i))\geqq 0\,.$$

Therefore,

$$\sum_{i=1}^{n}(t(x_i-x_{i-1}),\ A(x+tx_i)-A(x))\geqq 0\,.$$

Dividing this inequality by $t^2(>0)$, and letting $t\downarrow 0$, we obtain

$$\sum_{i=1}^{n}(x_i-x_{i-1},\ \delta A(x)x_i)\geqq 0\,.$$

This implies that $\delta A(x)$ is cyclically monotone.

"2°) implies 3°)". Suppose that 2°) holds. Let $x$ be any fixed element of $D(A)$. We must show that $\delta A(x)$ is positive symmetric. We set $B=\delta A(x)$. The monotonicity of $B$ means that $B$ is positive. Thus, we have only to show that $(y, Bz)=(z, By)$ for $^\forall y$, $^\forall z\in D(B)$. Applying Lemmas 1, 2 with $A=B$ and $x_0=0$, we have

$$\frac{d}{dt}\phi(y+tz)\,|_{t=0}=(z,\ By)\,,$$

where $\phi(w)=\int_0^1(w,\ B(tw))dt=(1/2)(w,\ Bw)$ for $w\in D(B)$. Therefore we obtain that

$$\begin{aligned}(z,\ By)&=\lim_{t\to 0}\frac{1}{t}\{\phi(y+tz)-\phi(y)\}\\&=\lim_{t\to 0}\frac{1}{2t}\{(y+tz,\ B(y+tz))-(y,\ By)\}\\&=\frac{1}{2}(y,\ Bz)+\frac{1}{2}(z,\ By)\,.\end{aligned}$$

This yields that $(z, By)=(y, Bz)$, and the proof is complete.

From the next two theorems and our Theorem, we get a sufficient condition for the maximal cyclical monotonicity.

THEOREM A (see [4]). *Let* $B\colon H\to H$ *be a positive definite* (i.e., $\inf_{x\in D(B),\|x\|=1}(x, Bx)>0$), *self-adjoint operator. Then* $R(B)=H$.

THEOREM B (see F. E. Browder [2] Corollary 2 to Theorem 2). *Let*

*A be a Gâteaux differentiable operater in $H$ with convex domain and closed range. If $R(\delta A(x))$ is dense in $H$ for every $x \in D(A)$, then $R(A)=H$.*

COROLLARY 1. *Let $A$ be a Gâteaux differentiable closed operator in $H$ with convex domain, and suppose that $A$ is w-continuous on every two dimensional subset in $D(A)$. If $\overline{\delta A(x)}$ is positive self-adjoint for each $x \in D(A)$, then $A$ is maximal cyclically monotone, i.e., there is a proper lower-semicontinuous convex functional $\phi: H \to (-\infty, \infty]$ such that $A=\partial\phi$.*

PROOF. By Theorem, we have that $A$ is cyclically monotone. Thus it suffices to show that $R(I+A)=H$. Since $I+\overline{\delta A(x)}$ is a positive definite, self-adjoint operator in $H$, it follows from Theorem A that $R(I+\overline{\delta A(x)})=H$, which implies that $R(I+\delta A(x))$ is dense in $H$. From the monotonicity and the closedness of $A$, it is easily seen that $R(I+A)$ is closed in $H$. Therefore, we apply Theorem B to an operator $I+A$ to get $R(I+A)=H$.

REMARK 2. Let $x_0$ be an element of $D(A)$. If we define $\phi$ as

$$\phi(x)=\begin{cases} \int_0^1 (x-x_0, A(x_0-t(x-x_0)))dt & \text{for } x \in D(A)\,, \\ \liminf\limits_{y\to x, y \in D(A)} \phi(y) & \text{for } x \in \overline{D(A)}\backslash D(A)\,, \\ \infty & \text{for } x \notin D(A)\,, \end{cases}$$

then $\phi$ satisfies the conclusion of Corollary 1.

In fact, from the proof of Theorem, $\phi: H\to(-\infty, \infty]$ is proper, convex and $A\subset\partial\phi$. Hence $\phi(y)\geqq\phi(x)+(y-x, Ax)$ for $x, y \in D(A)$, which implies that $\liminf_{y\to x, y\in D(A)} \phi(y)\geqq\phi(x)$ for $x \in D(A)$. Thus $\phi$ is lower-semicontinuous, and the maximal monotonicity of $A$ implies that $A=\partial\phi$.

The next corollary also follows from Theorem.

COROLLARY 2. *Let $Y$ be a reflexive Banach space such that $Y\subset H\subset Y'$ with the continuous and dense inclusion. Let $\tilde{A}: Y\to Y'$ be an operator which is everywhere defined on $Y$, coercive, w-Gâteaux differentiable and w-continuous on every 2-dimensional subset of $Y$. If $\delta\tilde{A}(x): Y\to Y'$ is a positive symmetric operator for each $x \in Y$, then $A=\tilde{A}_H$ (see Remark 1) is maximal cyclically monotone operator in $H$, i.e., there is a proper lower-semicontinuous convex functional $\phi: H\to(-\infty, \infty]$ such that $A=\partial\phi$.*

PROOF. $A$ is a cyclically monotone operator in $H$, by Remark 1. On the assumption of this corollary, $A$ is maximal monotone in $H$ (see [1, Example 2.3.7]). Hence, $A$ is maximal cyclically monotone in $H$.

REMARK 3. The functional $\phi: H \to (-\infty, \infty]$ defined in Remark 2 satisfies the conclusion of Corollary 2 also.

In fact, the functional $\tilde{\phi}: Y \to (-\infty, \infty]$ defined by $\tilde{\phi}(x) = \int_0^1 (x-x_0, \tilde{A}(x_0 - t(x-x_0)))dt$ for $x \in Y$ is proper, convex and $A = \partial\phi$ in $Y \times Y'$, from the proof of Theorem. Hence, we easily have that $A = \partial\phi$ in $H \times H$ and $\phi$ is a proper lower-semicontinuous convex functional from $H$ to $(-\infty, \infty]$.

## §3. Example.

In this section, we give an example of Corollary 2.

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded damain with smooth boundary $\partial\Omega$. $(\partial/\partial x_i)$, $i=1, \cdots, n$, denote distributional derivatives. $\mathring{H}^1(\Omega)$ is the usual Sobolev space which consists of $\{u \in L^2(\Omega); (\partial/\partial x_i)u \in L^2(\Omega)\ i=1, \cdots, n, u=0 \text{ on } \partial\Omega\}$. $H^{-1}(\Omega)$ denotes the dual space of $\mathring{H}^1(\Omega)$. Let $\tilde{A}: \mathring{H}^1(\Omega) \to H^{-1}(\Omega)$ be an operator such that

$$\tilde{A}u = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_j(x, u_1, \cdots, u_n) \quad (u \in \mathring{H}^1(\Omega)),$$

where

$$u_i = \frac{\partial}{\partial x_i} u, \quad i=1, \cdots, n,$$

$$a_j(x, u_1, \cdots, u_n): (u_1, \cdots, u_n) \in (L^2(\Omega))^n \longrightarrow L^2(\Omega),$$

(3.1) $$a_j(x, \cdot, \cdots, \cdot) \in C^1(\boldsymbol{R}^n) \quad \text{for each fixed} \quad x \in \Omega,$$

(3.2) $$\frac{\partial}{\partial u_k} a_j = \frac{\partial}{\partial u_j} a_k (\equiv a_{jk}),$$

(3.3) $$|a_{jk}(x, y_1, \cdots, y_n)| \leqq M \quad \text{for} \quad {}^\forall x \in \Omega,\ {}^\forall y_i \in \boldsymbol{R} \quad (i=1, \cdots, n),$$

(3.4) $$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \geqq \alpha \sum_{j=1}^{n} \xi_j^2 \quad ({}^\exists \alpha > 0) \quad \text{(uniformly elliptic)}.$$

$A = \tilde{A}_H: L^2(\Omega) \to L^2(\Omega)$ is an operator defined by

$$D(A) = \{u \in \mathring{H}^1(\Omega); \tilde{A}u \in L^2(\Omega)\}, \qquad Au = \tilde{A}u \quad \text{for} \quad u \in D(A).$$

Then $A$ is a maximal cyclically monotone operator in $H$.

PROOF. We set $H = L^2(\Omega)$ with norm $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and $Y = \mathring{H}^1(\Omega)$ with norm $|||\cdot||| = \|\cdot\|_{\mathring{H}^1(\Omega)}$. We have only to show that the hypothesis of

Corollary 2 are satisfied. We put $\partial u=(u_1, \cdots, u_n)$. It is well-known that $Y$ is reflexive.

1) First, we show that $\tilde{A}$ is $w$-Fréchet differentiable on $Y$ (and therefore $\tilde{A}$ is $w$-Gâteaux differentiable and $w$-continuous on $Y$) with

$$\delta\tilde{A}(u)v=-\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}(a_{jk}(x, \partial u)v_k) .$$

Let $u, w \in Y$ be any fixed elements. It suffices to show that

$$\frac{1}{|||v|||}(w, \tilde{A}(u+v)-\tilde{A}u+\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}(a_{jk}(x, \partial u)v_k)) \longrightarrow 0$$

as $|||v|||\to 0$. By (3.1), it holds that

$$\begin{aligned}(w, \tilde{A}(u+v)-\tilde{A}u)&=\sum_{j=1}^{n}(w_j, a_j(x, \partial(u+v))-a_j(x, \partial u))\\ &=\sum_{j=1}^{n}\left(w_j, \sum_{k=1}^{n}\frac{\partial}{\partial u_k}a_j(x, \partial u+\theta_{x,v}\partial v)v_k\right)\end{aligned}$$

for some $\theta_{x,v}$ with $0<\theta_{x,v}<1$. Hence we have that

$$\begin{aligned}&\frac{1}{|||v|||}\left(w, \tilde{A}(u+v)-\tilde{A}u+\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}(a_{jk}(x, \partial u)v_k)\right)\\ &=\frac{1}{|||v|||}\sum_{j,k}(w_j, \{a_{jk}(x, \partial u+\theta_{x,v}\partial v)-a_{jk}(x, \partial u)\}v_k)\\ &=\left(\sum_{j,k}\frac{v_k}{|||v|||}, w_j\{a_{jk}(x, \partial u+\theta_{x,v}\partial v)-a_{jk}(x, \partial u)\}\right)\\ &\leqq\sum_{j,k}\|w_j\{a_{jk}(x, \partial u+\theta_{x,v}\partial v)-a_{jk}(x, \partial u)\}\| .\end{aligned}$$

We put $g_{jk,v}(x)=w_j\{a_{jk}(x, \partial u+\theta_{x,v}\partial v)-a_{jk}(x, \partial u)\}$. Then we only need to show that $\|g_{jk,v}\|\to 0$ as $|||v|||\to 0$ for $j, k=1, \cdots, n$. If not, for some $j$, $k$, there are a sequence $\{v^{(m)}\}\subset Y$ and an $\varepsilon_0>0$ such that

(3.5) $$|||v^{(m)}|||\longrightarrow 0 \quad \text{as} \quad m\longrightarrow\infty \quad \text{and}$$

(3.6) $$\|g_m\|\geqq\varepsilon_0 ,$$

where $g_m=g_{jk,v^{(m)}}$. By (3.3), it holds that

(3.7) $$|g_m(x)|\leqq 2M|w_j(x)| .$$

(3.5) implies that

$$\|v_i^{(m)}\|\longrightarrow 0 \quad \text{as} \quad m\longrightarrow\infty , \quad i=1, \cdots, n .$$

Thus we can extract a subsequence $\{v^{(l)}\}$ of $\{v^{(m)}\}$ such that

$$v_i^{(l)}(x) \longrightarrow 0 \quad \text{a.e. } x \text{ on } \Omega \text{ as } \quad l \longrightarrow \infty\,, \quad i=1, \cdots, n\,.$$

By (3.1), this convergence yields that

$$(3.8) \qquad g_{jk,v^{(l)}}(x)^2 \longrightarrow 0 \quad \text{a.e. } x \text{ on } \Omega \text{ as } \quad l \longrightarrow \infty\,, \quad j, k=1, \cdots, n\,.$$

From (3.7) and (3.8), we have by Lebesgue's convergence theorem that $\|g_l\| \to 0$ as $l \to \infty$, which contradicts (3.5).

2) Secondly, we prove that $\tilde{A}$: $Y \to Y'$ is coercive. Let $u \in Y$. Then we have

$$\begin{aligned}(u, \tilde{A}u - \tilde{A}0) &= \sum_{j=1}^{n} \int_{\Omega} u_j(a_j(x, u) - a_j(x, 0))dx \\ &= \sum_{j,k=1}^{n} \int_{\Omega} a_{jk}(x, \theta_{x,u}\partial u)u_k u_j dx\,,\end{aligned}$$

for some $\theta_{x,u}$ with $0 < \theta_{x,u} < 1$, by (3.1). Thus we have by (3.4) that

$$(u, \tilde{A}u - \tilde{A}0) \geqq \alpha \sum_{j=1}^{n} \int_{\Omega} u_j^2 dx \geqq \alpha C |||u|||^2$$

for some constant $C > 0$. In the last inequality, we used Poincaré's inequality, since $\Omega$ is bounded. Therefore we have that

$$\frac{1}{|||u|||}(u, \tilde{A}u) \geqq \frac{1}{|||u|||}(u, \tilde{A}0) + \alpha C |||u||| \geqq -\|\tilde{A}0\| + \alpha C |||u|||\,.$$

This yields that $\lim_{|||u||| \to \infty} (1/|||u|||)(u, \tilde{A}u) = \infty$, i.e., $\tilde{A}$ is coercive.

3) Finally we show that $\delta\tilde{A}(u)$ is positive symmetric for each $u \in Y$. Let $u$, $v$, $w$ be any elements of $Y$. Then

$$\begin{aligned}(w, \delta\tilde{A}(u)v) &= \Big(w, -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}(a_{jk}(x, \partial u)v_k)\Big) \\ &= \sum_{j,k=1}^{n} \int_{\Omega} a_{jk}(x, \partial u) w_j v_k dx\,.\end{aligned}$$

Hence, by (3.2), we have that $(w, \delta\tilde{A}(u)v) = (v, \delta\tilde{A}(u)w)$, i.e., $\delta\tilde{A}(u)$ is symmetric. And positivity follows from (3.4).

Consequently, $A$ satisfies the hypothesis of Corollary 2, and hence $A$ is a maximal cyclically monotone operator in $H$.

REMARK 4. This example is dealt with by Y. Kōmura and Y. Konishi [3] without proof.

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