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Periodic Solutions on a Convex Energy Surface of a Hamiltonian System

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Introduction

Let $p=(p_1, \cdots, p_n)$, $q=(q_1, \cdots, q_n)$ be points of $\boldsymbol{R}^n$ and write $z=(p, q)\in \boldsymbol{R}^{2n}$. We consider a Hamiltonian system of $H\in C^2(\boldsymbol{R}^{2n}, \boldsymbol{R})$

$$(H) \qquad \dot{p}=-H_q , \quad \dot{q}=H_p$$

or equivalently

$$(H) \qquad \dot{z}=JH'(z) , \quad J=\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} ,$$

with I being the identity in $\boldsymbol{R}^n$.

On any compact energy surface for classical Hamiltonian, that is, $H=$"*kinetic energy*"$+$"*potential*", we have at least one periodic solution of (H) [6] [5].

For any star-shaped energy surface, there exists at least one periodic solution of (H) on it [7].

For a convex energy surface, Ekeland and Lasry [3] found n periodic solutions on it and Ambrosetti-Mancini [2] extended it to the following.

We define $[s]_+=[s]_-=s$ for $s\in \boldsymbol{Z}$ and $[s]_-=j$, $[s]_+=j+1$ for $s\in(j, j+1)$ with $j\in \boldsymbol{Z}$.

THEOREM 1. *Let C be a compact strictly convex subset of $\boldsymbol{R}^n$ with C^2 boundary S. For some $h\in \boldsymbol{R}$, $H^{-1}(h)=S$ and $H'(z)\neq 0$ for any $z\in S$.*

Assume further that there exist r, $R\in \boldsymbol{R}^+$ and $k\in \boldsymbol{Z}$, $2\leqq k\leqq n$, with

$$(0.1) \qquad R<\sqrt{k}\, r$$

such that

$$(0.2) \qquad rB\subset C\subset RB ,$$

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where B is the closed unit ball in $\boldsymbol{R}^{2n}$.

Then there exist at least $[n/(k-1)]_+$ distinct periodic solutions of (H) on S.

The case $k=2$ is the one by Ekeland-Lasry.

Let $0<\omega_1\leqq\omega_2\leqq\cdots\leqq\omega_n=1$.

We identify $z=(z_1,\cdots,z_n)\in\boldsymbol{C}^n$ and $(p,q)\in\boldsymbol{R}^{2n}$ by $z_j=p_j+iq_j$, $j=1,\cdots,n$, and furnish a norm in z-space

$$|z|_\omega^2=\sum_{j=1}^n\omega_j|z_j|^2 . \tag{0.3}$$

In this note, we have

THEOREM 2. *Let Q_ω be the ellipsoid*

$$\{z\in\boldsymbol{C}^n;\ |z|_\omega\leqq 1\} ,$$

$K\geqq 1$ and $N\in\boldsymbol{Z}$ with $N>K$.

Let C, S and h be as in Theorem 1, replacing (0.1) and (0.2) with

$$R<\sqrt{N/K}r , \tag{0.4}$$

and

$$rB\subset C\subset RQ_\omega . \tag{0.5}$$

Then we have at least $[\sum_{j=1}^n[\omega_jK]_-/(N-1)]_+$ distinct periodic solutions of (H) on S.

COROLLARY

(i) *In the case $\omega_1=\omega_2=\cdots=\omega_n=1$, we have Theorem 1.*

(ii) *In the case $0<\omega_1<\omega_2=\cdots=\omega_n=1$, taking $K=1/\omega_1$ and $N=[K]_-+1$, we have at least n periodic solutions on S.*

(iii) *Taking proper K with $N=[K]_-+1$, we have at least $[\sum_{j=1}^n\omega_j]_+$ periodic solutions on S.*

PROOF.

(i) Take $K=1$ and $N=k$.

(ii) $[\sum_{j=1}^n[\omega_jK]_-(N-1)]_+=[(1+(n-1)[K]_-/[K]_-]_+=n$

(iii) Because $\sum_{j=1}^n[\omega_jK]_-/[K]_-\to\sum_{j=1}^n\omega_j$ as $K\to\infty$. Q.E.D.

Remark that there is a case where we obtain more periodic solution than (iii) assures. For example, take $n=5$,

$$\omega_1=0.1 ,\quad \omega_2=\omega_3=\omega_4=0.6 ,\quad \omega_5=1$$

and $K=1.8$ with $N=[K]_-+1=2$. Then $[\sum_{j=1}^{5}\omega_j]_+=[2.9]_+=3$ but $[\sum^{5}[\omega_jK]_-/[K]_-]_+=4$.

In order to obtain n periodic solutions, there is a case in which a proper choice of K is necessary. For example, consider the case $n=3$, $\omega_1^2=1/3$, $\omega_2=5/6$, and $\omega_3=1$.

For $K=3$ and $N=4$, we have

$$\left(\left[\frac{1}{3}\times 3\right]_-+\left[\frac{5}{6}\times 3\right]_-+[1\times 3]_-\right)/[3]_-=2\ ,$$

hence only 2 solutions.

For $K=6$ and $N=7$, we have

$$\left(\left[\frac{1}{3}\times 6\right]_-+\left[\frac{5}{6}\times 6\right]_-+[1\times 6]_-\right)/[6]_-=2+\frac{1}{6}\ ,$$

thus at least 3 solutions.

A special case $\omega_1=\cdots=\omega_{n-1}=1/2$ and $\omega_n=1$ is treated in [8].

Theorems 1 and 2 are proved by *Dual Action Principle*, which will be explained in §1, counting the cohomological index developed by Fadell and Rabinowitz [4] of an invariant subset under the natural S^1-action.

§1. Dual action principle.

This method was developed in [1]. We explain it briefly and collect some facts for later use.

Let S be the C^2 boundary of a compact strictly convex subset C of $\boldsymbol{R}^{2n}$ (not necessarily satisfying (0.2) or (0.5)).

Take $\beta>2$ and determine the Hamiltonian $H=H(z)$: $\boldsymbol{R}^{2n}\to\boldsymbol{R}$ by

$$H^{-1}(1)=S \tag{1.1}$$

$$H\text{: }\beta\text{-homogeneous.}\quad (H(\lambda z)=\lambda^{\beta}H(z),\ \lambda>0)\ . \tag{1.2}$$

Then H is convex, so the Legendre transform $G=G(u)$ is obtained, which is α-homogeneous ($1/\alpha+1/\beta=1$, $1<\alpha<2$).

Put $E=\left\{u\in L^{\alpha}(0, 2\pi;\ \boldsymbol{R}^{2n})\Big|\int u\equiv\int_0^{2\pi}u(t)dt=0\right\}$ and define a C^1-function $f\colon E\to\boldsymbol{R}$ by

$$f(u)=-\frac{1}{2}\int u\cdot Lu+\int G(u)\ .$$

where $z=Lu$ is determined by $u=-J\dot{z}$ and $\int z=0$. We also consider u as a complex n-vector and $u\cdot Lu$ means the usual Euclidian inner product, in C^n as $2n$-dimensional vector space over $\boldsymbol{R}$.

Finally put

$$M=\left\{u\in E\backslash 0;\ \int u\cdot Lu=\alpha\int G(u)\right\}\ .$$

Then M is a C^1 Banach submanifold of E and $f\colon M\to \boldsymbol{R}$ satisfies $P-S$ condition.

And we have a one to one correspondence between critical points of f in M and periodic orbits on S.

Furthermore we have

$$m=\min\{f(u);\ u\in M\}>0\ , \tag{1.3}$$

and for $\mu\in \boldsymbol{Z}_+=\{1, 2, \cdots\}$

$$u\in M \Rightarrow u^\mu\equiv \mu^\delta u(\mu\cdot)\in M \tag{1.4}$$

$$f(u^\mu)=\mu^\vartheta f(u) \quad \text{for} \quad u\in M \tag{1.5}$$

where $\delta=1/(2-\alpha)$ and ϑ $\theta=\alpha/(2-\alpha)=\alpha\delta$.

And for $u\in E$ with $\int u\cdot Lu>0$, there is the unique $\lambda>0$ such that $\lambda u\in M$. λ is explicitly determined by

$$\lambda^{2-\alpha}=\alpha\fint G(u)/\fint u\cdot Lu\ , \tag{1.6}$$ ((5) in [2])

where $\fint$ means $1/2\pi\int$.

So

$$\lambda=\left[\alpha\fint G(u)\right]^\delta \quad \text{if} \quad \fint u\cdot Lu=1 \tag{1.7}$$

and because of

$$f(u)=\frac{\pi}{\delta}\fint G(u)=\frac{1}{2\vartheta}\int u\cdot Lu \quad \text{for} \quad u\in M \tag{1.8}$$ ((6) in [2])

we have

$$\begin{aligned} f(\lambda u)&=\frac{\pi}{\vartheta}\alpha\fint G(\lambda u)\\ &=\frac{\pi}{\vartheta}\lambda^\alpha\alpha\fint G(u)\\ &=\frac{\pi}{\vartheta}\left[\alpha\fint G(u)\right]^{2\delta} \quad \text{if} \quad \fint u\cdot Lu=1\ . \end{aligned} \tag{1.9}$$

§ 2. Harmonic oscillators.

We consider the Hamiltonian

$$H_2(z)=\frac{1}{2}\sum_{j=1}^{n}\omega_j|z_j|^2 , \tag{2.1}$$

where $0<\omega_1\leqq\omega_2\leqq\cdots\leqq\omega_n=1$ are angular frequencies.

Since the complex version of (H) is $\dot{z}=2i(\partial/\partial\bar{z})H(z)$ and $2(\partial/\partial\bar{z}_j)H_2(z)=\omega_j z_j$, (H) becomes componentwisely

$$\dot{z}_j=i\omega_j z_j , \qquad j=1, 2, \cdots, n . \tag{2.2}$$

Hence the j-th periodic solution with multiplicity $\mu\in \boldsymbol{Z}_+$ is

$$c_j e^{i\omega_j t}a_j;\ c_j\in \boldsymbol{C}\backslash 0 , \quad 0\leqq t\leqq 2\mu\pi/\omega_j \tag{2.3}$$

where a_j is the j-th vector of the usual orthogonal basis of $\boldsymbol{C}^n$.

We put

$$H_\beta(z)=\frac{1}{\beta}|z|_\omega^\beta \tag{2.4}$$

for $\beta>2$.

H_β is β-homogeneous and satisfies (1.1) if S is

$$\{z\in \boldsymbol{C}^n;\ |z|_\omega=\beta^{1/\beta}\} \tag{2.5}$$

The Legendre transform $G(u)$ of $H_\beta(z)$ is

$$G(u)=\frac{1}{\alpha}|u|_\tau^\alpha \tag{2.6}$$

where $|u|_\tau^2=\sum_{j=1}^n \tau_j|u_j|^2$, $\tau_j=1/\omega_j$.

An elementary calculation gives

LEMMA 1. *The corresponding critical point of f in M to* (2.3) *is*

$$v_j^\mu(t)\equiv\mu^\vartheta\tau_j^{\vartheta/2}e^{\mu i t}a_j \tag{2.7}$$

and, writing v_j^1 as v_j, we have

$$f(v_j^\mu)=\mu^\vartheta f(v_j)=(\mu\tau_j)^\vartheta\frac{\pi}{\vartheta} . \tag{2.8}$$

We also have

$$Lv_j^\mu = \frac{1}{\mu} v_j^\mu \tag{2.9}$$

$$\int v_j^\mu L v_k^\nu = \delta^{\mu\nu}\delta_{jk}\mu^\vartheta \tau_j^\vartheta \ . \tag{2.10}$$

Thus, for S defined by (2.5), *m in* (1.3) *is given by*

$$m = f(v_n) = \frac{\pi}{\vartheta}\ ,$$

hence

$$f(v_j) = \tau_j^\vartheta m \quad \textit{and} \quad f(v_j^\mu) = (\mu\tau_j)^\vartheta m\ . \tag{2.11}$$

We put $\nu_j = [\omega_j K]_-$, then we have

$$\nu_j \tau_j \leqq K \quad \text{for} \quad j = 1, 2, \cdots, n\ . \tag{2.12}$$

Also put $a = \nu_1 + \nu_2 + \cdots + \nu_n$.

For $\zeta = (\zeta_{j\mu})_{j=1,2,\cdots,n;\mu=1,2,\cdots,\nu_j} \in C^a$, we define

$$u_\zeta = \sum_{j=1}^{n} \sum_{\mu=1}^{\nu_j} \zeta_{j\mu} v_j^\mu \in E\ . \tag{2.13}$$

Then, from (2.10), we have

$$\begin{aligned} \int u_\zeta \cdot L u_\zeta &= \sum_{j=1}^{n} \sum_{\mu=1}^{\nu_j} (\mu\tau_j)^\vartheta |\zeta_{j\mu}|^2 \\ &\equiv \|\zeta\|^2\ . \end{aligned} \tag{2.14}$$

We put $\Sigma = \{\zeta \in C^a;\ \|\zeta\| = 1\}$ and for $\zeta \in \Sigma$ we define

$$\lambda(\zeta) = \left[\alpha \int G(u_\zeta)\right]^\delta\ , \tag{2.15}$$

then, (1.7) implies $\varphi(\zeta) \equiv \lambda(\zeta) u_\zeta \in M$ and we have

$$f \circ \varphi(\zeta) = m\left[\alpha \int G(u_\zeta)\right]^{2\delta}\ , \qquad \text{(by (1.9))} \tag{2.16}$$

where $m = \pi/\vartheta = \min\{f(u);\ u \in M\}$ in the situation of this section.

Then we have

LEMMA 2. *For small $\varepsilon > 0$, choosing α properly, we have*

$$\text{Max}\ f \circ \varphi(\Sigma) \leqq (1+\varepsilon)^\vartheta K^\vartheta m\ .$$

PROOF.

For $\zeta \in \Sigma$, we have

$$\alpha G(u_\zeta)=\left(\sum_{j=1}^{n} \tau_j \left|\sum_{\mu=1}^{\nu_j} \zeta_{j\mu} v_j^\mu\right|^2\right)^{\alpha/2} \tag{2.17}$$
$$=\left(\sum_{j=1}^{n} \tau_j^{\vartheta+1} \left|\sum_{\mu=1}^{\nu_j} \zeta_{j\mu} \mu^\vartheta e^{\mu i t}\right|^2\right)^{\alpha/2}$$
$$\equiv F(t)^{\alpha/2} .$$

Then

$$\int F(t)dt=\sum_{j=1}^{n} \sum_{\mu=1}^{\nu_j} (\mu\tau_j)^{\vartheta+1} |\zeta_{j\mu}|^2 \tag{2.18}$$
$$\leqq K \sum_j \sum_\mu (\mu\tau_j)^\vartheta |\zeta_{j\mu}|^2 \qquad \text{(by (2.12))}$$
$$=K\|\zeta\|^2$$
$$=K$$

First for given $\varepsilon>0$, we choose $\varepsilon_1>0$ so small that

$$(1+\varepsilon_1)^2(1+2\varepsilon_1)\leqq 1+\varepsilon \quad \text{and} \quad K\varepsilon_1<1 . \tag{2.19}$$

Then choose α so near 2 that

$$s/(1+\varepsilon_1)\leqq s^{\alpha/2}\leqq(1+\varepsilon_1)s \quad \text{for} \quad (K\varepsilon_1)^2\leqq s\leqq 2K . \tag{2.20}$$

Then we have

$$\int \alpha G(u_\zeta)=\int F(t)^{\alpha/2} \tag{2.21}$$
$$=\int_{F(t)\leqq(K\varepsilon_1)^2}+\int_{(K\varepsilon_1)^2<F(t)}$$
$$\leqq(K\varepsilon_1)^{2\alpha/2}+\int(1+\varepsilon_1)F(t) \qquad \text{(by (2.20))}$$
$$\leqq K\varepsilon_1+(1+\varepsilon_1)K \qquad \text{(by (2.18))}$$
$$=(1+2\varepsilon_1)K$$
$$\leqq(1+\varepsilon_1)[(1+2\varepsilon_1)K]^{\alpha/2} \qquad \text{(by (2.20))}$$
$$\leqq[(1+\varepsilon_1)^2(1+2\varepsilon_1)K]^{\alpha/2}$$
$$\leqq[(1+\varepsilon)K]^{\alpha/2} \qquad \text{(by (2.19))}$$

Therefore

$$f\circ\varphi(\zeta)=m\left[\alpha\int G(u_\zeta)\right]^{2\delta}$$

$$\leqq m[(1+\varepsilon)K]^{(\alpha/2)2\delta}$$
$$\leqq(1+\varepsilon)^{\vartheta}K^{\vartheta}m\ ,$$

proving the lemma. Q.E.D.

§3. Proof of Theorem 2.

We attach the tilde $\tilde{}$ to the notations as $\tilde{G}$, $\tilde{f}$, $\tilde{M}$ and $\tilde{\varphi}$ derived from C in Theorem 2, E and Σ unchanged and $\tilde{\varphi}: \Sigma \to \tilde{M}$ with

$$\tilde{f}\circ\tilde{\varphi}(\zeta)=m\left[\alpha\int\tilde{G}(u_\zeta)\right]^{2\delta},\quad \zeta\in\Sigma\ , \tag{3.1}$$

by (1.9). Recall that $m=\pi/\vartheta=\min\{\tilde{f}(u); u\in M\}$.

LEMMA 3. (0.5) *with* (0.4) *implies*

$$\operatorname{Max}\tilde{f}\circ\tilde{\varphi}(\Sigma)<N^{\vartheta}\tilde{m}$$

where $\tilde{m}=\min\{\tilde{f}(u); u\in\tilde{M}\}$.

PROOF. First we claim

$$\rho_1^\alpha\cdot\frac{1}{\alpha}|u|^\alpha\leqq\tilde{G}(u)\leqq\rho_2^\alpha G(u)\quad\text{for any}\quad u\in\boldsymbol{R}^{2n}\ , \tag{3.2}$$

where $\rho_1=r\beta^{-1/\beta}$, $\rho_2=R\beta^{-1/\beta}$ and $G(u)$ is given by (2.6). This is obtained from (0.5) and the fact that $\rho_1^\alpha(1/\alpha)|u|^\alpha$ (or $\rho_2^\alpha G(u)$) is $G(u)$ in §1 derived from $C=rB$ (or RQ_ω respectively).

Since (3.2) implies $\min_{|u|=1}\tilde{G}(u)\geqq(1/\alpha)\rho_1^\alpha$, we have

$$\tilde{m}\geqq m\rho_1^{2\vartheta}\ , \tag{3.3}$$

by (15) in [2] (b in (15) is equal to 2π).

Now, in Lemma 2, we choose $\varepsilon>0$ so small that

$$\rho_2/\rho_1<\sqrt{N/(1+\varepsilon)K}\ . \tag{3.4}$$

For $\zeta\in\Sigma$, we have

$$\begin{aligned}\tilde{f}\circ\tilde{\varphi}(\zeta)&=m\left[\alpha\int\tilde{G}(u_\zeta)\right]^{2\delta}\\ &\leqq m\left[\alpha\int G(u_\zeta)\right]^{2\delta}\rho_2^{2\alpha\delta} && \text{(by (3.2))}\\ &=f\circ\varphi(\zeta)\cdot\rho_2^{2\vartheta} && \text{(by (2.16))}\\ &\leqq(1+\varepsilon)^{\vartheta}K^{\vartheta}\cdot m\rho_2^{2\vartheta} && \text{(by Lemma 2)}\end{aligned}$$

$$
\begin{aligned}
&= m(\sqrt{(1+\varepsilon)K}\rho_2)^{2\vartheta} \\
&< m(\sqrt{N}\rho_1)^{2\vartheta} && \text{(by (3.4))} \\
&= N^{\vartheta} m \rho_1^{2\vartheta} \\
&\leqq N^{\vartheta}\tilde{m} && \text{(by (3.3))}
\end{aligned}
$$

Q.E.D.

Now we use the index theory developed by Fadell and Rabinowitz [4]. We refer Lemma 1.13 of [7] for the definitions and properties.

We put $\tilde{M}_N=\{u\in\tilde{M};\ \tilde{f}(u)<N^{\vartheta}\tilde{m}\}$.

Then $\tilde{\varphi}(\Sigma)\subset\tilde{M}_N$ by Lemma 3.

And $\tilde{\varphi}\colon\Sigma\to\tilde{M}$ is equivariant under the S^1 $(=\boldsymbol{R}/2\pi\boldsymbol{Z})$-action: $A_s\zeta=(e^{i\mu s}\zeta_{j\mu})_{j=1,2,\cdots,n;\mu=1,2,\cdots,\nu_j}$ for $\zeta=(\zeta_{j\mu})_{j;\mu}\in\boldsymbol{C}^a$ and $A_s u(t)=u(t+s)$ for $u\in\tilde{M}$.

We can prove that index $\Sigma=a$, because Σ is embedded onto $F\cap\mathscr{S}$ in $(E_2^0)^{\perp}$, where F is a $2a$ dimensional invariant subspace of $(E_2^0)^{\perp}$ and $\mathscr{S}$ is the unit sphere of E_2 (see 6° in Lemma 1.13 in [7]. E_2 is written as E in the lemma). F is concretely given by

$$
span_{C}\{e^{\mu i t}a_j;\ j=1,\cdots,n;\ \mu=1,\cdots,\nu_j\}\ .
$$

Thus 2° in the lemma implies index $(\tilde{M}_N)\geqq a$. So Lusternik-Schnirelmann theory gives at least a critical points (up to S^1-action). But we cannot avoid the possibility that functions in $\tilde{M}_N$ has multiplicity less than N. Therefore we have at least $[a/(N-1)]_+$ distinct periodic solutions, as in the proof of [2].

This proves the theorem. Q.E.D.

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