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Examples of Simply Connected Compact Complex 3-folds II

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Introduction

This note is the continuation of [2]. In [2], the first named author has constructed a series of compact complex manifolds $\{M_n\}_{n=1,2,3,\dots}$ of dimension 3 which are non-algebraic and non-Kaehler with the properties: $\pi_1(M_n)=0$, $\pi_2(M_n)=\boldsymbol{Z}$, $b_3(M_n)=4n$, $\dim H^1(M_n, \mathcal{O})\geqq n$, and $\dim H^1(M_n, \Omega^1)\geqq n$. The present note consists of two sections, § 5, § 6. In section 5, we shall show how to describe differentiable structures of $\{M_n\}$ in terms of connected sums, using a result of C. T. C. Wall [4]. We note, in particular, that M_1 is diffeomorphic to the connected sum of twice $S^3\times S^3$ and $S^2\times S^4$; $M_1\approx 2(S^3\times S^3)\#_t S^2\times S^4$ and that M_2 is diffeomorphic to that of 4 times $S^3\times S^3$ and $\boldsymbol{P}^3$; $M_2\approx 4(S^3\times S^3)\#_t P^3$. Here $\#_t$ indicates the usual connected sum in the category of differentiable topology. In section 6, we shall calculate all of their Hodge invariants. We have $\dim H^1(M_n, \mathcal{O})=n$ and $\dim H^1(M_n, \Omega^1)=n+1$, while $H_1(M_n, \boldsymbol{Z})=0$ and $H_2(M_n, \boldsymbol{Z})=\boldsymbol{Z}$.

In the following, we shall use the notation in [2].

§ 5. In this section, we shall study the differentiable structures of the compact complex manifolds of dimension 3 $\{M_n\}_{n=1,2,3,\dots}$, which were constructed in [2].

LEMMA 11.

(v)
$$H_q(M_n, \boldsymbol{Z})=\begin{cases}\boldsymbol{Z} & q\text{: even ,}\\ 0 & q=1, 5 ,\\ \boldsymbol{Z}^{4n} & q=3 .\end{cases}$$

(vi) *Let l be a projective line in $\Sigma\subset\boldsymbol{P}^3$. Then, for any $n\geqq 1$, $l_n:=i_1(l)$ $(\subset M_1^{n-1}\subset M_n)$ represents a generator of $H_2(M_n, \boldsymbol{Z})$, where M_1^0 is understood to be M_1.*

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PROOF. By (ii) in Theorem of [2], we have $H_2(M_n, Z)=Z$ and $b_3(M_n)=4n$. Hence (v) follows from the Poincaré duality and the universal coefficient theorem. (vi) is clear from the proof of (ii) in Theorem of [2].

LEMMA 12. *The bilinear form*

$$\mu: H^2(M_1, Z)\times H^2(M_1, Z) \longrightarrow H^4(M_1, Z)$$

defined by taking cup products is zero.

PROOF. Let S be a general fibre of $p_1: M_1\to P^1$. For a section l' of p_1, we have $S\cdot l'=1$. Hence the 1st Chern class $c_1([S])$ of the line bundle $[S]$ associated to S is a generator of $H^2(M_1, Z)$. Since S is a fibre of p_1, we have $c_1([S])^2=0$. Thus $\mu=0$ as desired. □

Since $H_2(M_n, Z)=Z$, we can define the dual element $\hat{l}_n$ in $H^2(M_n, Z)$ of l_n by $\hat{l}_n(l_n)=1$. In general, for a complex manifold M, we let $c_i(M)$ and $p_1(M)$ denote the i-th Chern class and the 1st Pontrjagin class, respectively.

PROPOSITION 6. $c_1(M_1)=4\hat{l}_1$ *and* $\hat{l}_1^2=c_2(M_1)=p_1(M_1)=0$.

PROOF. Let S be a general fibre of p_1, and $j: S\to M_1$ be the natural inclusion mapping. Let K_{M_1} denote the canonical line bundle of M_1. Since $\deg_{l_1} K_{M_1}=-4$, we have

$$c_1(M_1)=4\hat{l}_1 .$$

Let Θ_M denote the sheaf of germs of holomorphic vector fields on M. Since S is a Hopf surface, we have $c_1(S)=c_2(S)=0$. Therefore, from the exact sequence

$$0 \longrightarrow \Theta_S \longrightarrow j^*\Theta_{M_1} \longrightarrow \mathcal{O}_S \longrightarrow 0 .$$

it follows that

$$j^*c_1(M_1)=j^*c_2(M_1)=0 . \tag{22}$$

Consider the exact sequence

$$\cdots \longrightarrow H^4(M_1, Z) \xrightarrow{j^*} H^4(S, Z) \longrightarrow H^5(M_1; S, Z) \longrightarrow \cdots .$$

Note that $p_1^{-1}(0)$ is simply connected by Proposition 1 and is a deformation retract of M_1-S. Hence, by the Lefschetz duality, we have

$$\begin{aligned} H^5(M_1; S, \boldsymbol{Z}) &= H_1(M_1 - S, \boldsymbol{Z}) \\ &= H_1(p_1^{-1}(0), \boldsymbol{Z}) \\ &= 0 . \end{aligned}$$

Therefore the homomorphism

$$j^*: H^4(M_1, \boldsymbol{Z}) \longrightarrow H^4(S, \boldsymbol{Z})$$

is bijective, since we know that

$$H^4(M_1, \boldsymbol{Z}) = H^4(S, \boldsymbol{Z}) = \boldsymbol{Z} .$$

Hence we have

$$c_2(M_1) = 0$$

from (22). It follows from Lemma 12 that

$$\hat{l}_1^2 = c_1^2(M_1) = 0 .$$

Therefore we obtain

$$p_1(M_1) = c_1^2(M_1) - 2c_2(M_1) = 0 .$$

Thus the proposition is proved. □

PROPOSITION 7. *For* $n \geqq 2$, *we have* $c_1(M_n) = 4\hat{l}_n$, $c_2(M_n) = 6\hat{l}_n^2$, $p_1(M_n) = 4\hat{l}_n^2$, *and* $\hat{l}_n^3 = 1 - n$.

PROOF. For the Chern numbers, we have by Proposition 6 and [3, Proposition 2.2] that

$$c_1c_2[M_n] = (1-n)c_1c_2[\boldsymbol{P}^3] = 24(1-n) , \tag{23}$$

$$c_1^3[M_n] = (1-n)c_1^3[\boldsymbol{P}^3] = 64(1-n) . \tag{24}$$

Since $\deg_{l_n} K_{M_n} = -4$, we have easily

$$c_1(M_n) = 4\hat{l}_n . \tag{25}$$

Then it follows from $c_1^3[M_n] = 64\hat{l}_n^3$ and (24) that

$$\hat{l}_n^3 = 1 - n . \tag{26}$$

Put $c_2(M_n) = a\hat{l}_n^2$, $a \in \boldsymbol{Q}$. Then by (23), (25), (26) and the equality $c_1c_2[M_n] = 4a\hat{l}_n^3$, we obtain $a = 6$. Hence

$$c_2(M_n) = 6\hat{l}_n^2 .$$

Therefore we have

$$p_1(M_n)=c_1^2(M_n)-2c_2(M_n)=4\hat{l}_n^2 . \qquad \square$$

For any $n\geqq 1$, M_n is simply connected, and all its homology groups are torsion free. Moreover, by Propositions 6 and 7, the 2nd Whitney classes vanish. Therefore all M_n satisfy the condition (H) of C. T. C. Wall [4]. Hence M_n is determined completely by the data of Propositions 6 and 7. Let $X\#_t Y$ indicate the connected sum of differentiable manifolds X and Y in the usual sense in the differential topology. By virtue of [4, Theorem 5], we have the following immediately.

THEOREM 2. *For any $n\geqq 1$, there is a simply connected compact differentiable manifold L_n of real dimension 6 such that M_n is diffeomorphic to the connected sum (in the usual sense of differential topology) of $2n$ times $S^3\times S^3$ and L_n;*

$$M_n\cong 2n(S^3\times S^3)\#_t L_n .$$

Here L_n satisfies the following.

(1) $H_*(L_n, \boldsymbol{Z})=H_*(\boldsymbol{P}^3, \boldsymbol{Z})$

(2) $p_1(L_n)=4\lambda_n^2$, $\lambda_1^2=0$, $\lambda_n^3=n-1$;

where $\lambda_n\in H^2(L_n, \boldsymbol{Z})$ is a generator. In particular, we have

$$M_1\cong 2(S^3\times S^3)\#_t(S^2\times S^4) ,$$

and

$$M_2\cong 4(S^3\times S^3)\#_t\boldsymbol{P}^3 .$$

§6. In this section, we shall calculate Hodge invariants of M_n.

THEOREM 3. *For $n\geqq 1$, we have*

$$\dim H^q(M_n, \mathcal{O}_{M_n})=\begin{cases}1 & q=0 ,\\ n & q=1 ,\\ 0 & q=2, 3 ,\end{cases} \tag{27}$$

and

$$\dim H^q(M_n, \Omega^1_{M_n})=\begin{cases}0 & q=0, 3 ,\\ n+1 & q=1 ,\\ 2n & q=2 .\end{cases} \tag{28}$$

First we shall prove the theorem for $n=1$, i.e.,

(29) $$\dim H^q(M_1, \mathcal{O}_{M_1}) = \begin{cases} 1 & q=0, 1\ , \\ 0 & q=2, 3\ , \end{cases}$$

and

(30) $$\dim H^q(M_1, \Omega^1_{M_1}) = \begin{cases} 0 & q=0, 3\ , \\ 2 & q=1, 2\ . \end{cases}$$

As for the equality (29), the case $q=0$ is trivial, and the case $q=1$ was proved in Lemma 8. The case $q=3$ follows easily from [3, Proposition 2.3] using the Serre duality. The remaining case $q=2$ follows from Proposition 6 using the Riemann-Roch theorem. Thus (29) is proved.

Now we shall show the equality (30). Recall the construction of the 3-fold M in §2. Take two copies $\tilde{V}_1$, $\tilde{V}_2$ of C^3. Let (ξ_j, ζ_j, s_j) be a standard system of coordinates on $\tilde{V}_j$. Form the union $\tilde{V} = \tilde{V}_1 \cup \tilde{V}_2$ by identifying $(\xi_1, \zeta_1, s_1) \in \tilde{V}_1$ with $(\xi_2, \zeta_2, s_2) \in \tilde{V}_2$ if and only if

$$\begin{cases} \xi_1 = \xi_1 s_2^{-1} \\ \zeta_1 = \zeta_2 s_2^{-1} \\ s_1 = s_2^{-1}\ . \end{cases}$$

Put $l_0 = \{\xi_1 = \zeta_1 = 0\} \cup \{\xi_2 = \zeta_2 = 0\}$ and $\tilde{V}^* = \tilde{V} - l_0$. Let α be the holomorphic automorphism of $\tilde{V}^*$ defined by

(31) $$(\xi_j, \zeta_j, s_j) \longmapsto (\alpha\xi_j, \alpha\zeta_j, s_j)$$

on $\tilde{V}^* \cap \tilde{V}_j$, $j=1, 2$, where $\alpha \in C$ is a constant satisfying $0<|\alpha|<1$. Then M is defined to be the quotient space $\tilde{V}^*/\langle\alpha\rangle$ of $\tilde{V}^*$ factored by the action of the infinite cyclic group $\langle\alpha\rangle$ generated by α. Denote $\varpi: \tilde{V}^* \to M$ be the canonical projection. Taking a small positive constant δ, we consider the following subdomains $\tilde{V}^*$:

$$\tilde{V}_{j0} = \{(\xi_j, \zeta_j, s_j) \in \tilde{V}_j : (1-2\delta)|\alpha|^2(1+|s_j|^2) < |\xi_j|^2 + |\zeta_j|^2 < (1+\delta)|\alpha|^2(1+|s_j|^2)\}\ ,$$
$$\tilde{V}_{j1} = \{(\xi_j, \zeta_j, s_j) \in \tilde{V}_j : (1-\delta)|\alpha|^2(1+|s_j|^2) < |\xi_j|^2 + |\zeta_j|^2 < (1+\delta)(1+|s_j|^2)\}\ ,$$
$$\tilde{V}_{j2} = \{(\xi_j, \zeta_j, s_j) \in \tilde{V}_j : (1-\delta)(1+|s_j|^2) < |\xi_j|^2 + |\zeta_j|^2 < (1+2\delta)(1+|s_j|^2)\}\ .$$

Then the open subdomains $V_{j\nu} := \varpi(\tilde{V}_{j\nu})$, $j=1, 2$, $\nu=0, 1, 2$, cover M. On each $V_{j\nu}$, we define local coordinates $(u_{j\nu}, v_{j\nu}, t_{j\nu})$ by

$$(u_{j\nu}, v_{j\nu}, t_{j\nu}) = (\varpi \mid \tilde{V}_{j\nu})^{-1*}(\xi_j, \zeta_j, s_j)\ .$$

The projections

$$(u_{j\nu}, v_{j\nu}, t_{j\nu}) \longmapsto t_{j\nu} \quad \text{on} \quad V_{j\nu}$$

define the fibre bundle structure

$$\pi: M \longrightarrow \boldsymbol{P}^1 ,$$

whose fibre is biholomorphic to

$$S_\alpha = \boldsymbol{C}^2 - \{(0, 0)\} \Big/ \left\langle \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\rangle .$$

LEMMA 13. $R^q\pi_*\mathcal{O}_M \cong \mathcal{O}_{\boldsymbol{P}^1}$, $q=0, 1$.

PROOF. This is trivial for $q=0$. Suppose that $q=1$. By Leray's spectral sequence

$$E_2^{p,q} = H^p(\boldsymbol{P}^1, R^q\pi_*\mathcal{O}_M) \Longrightarrow H^{p+q}(M, \mathcal{O}_M)$$

and by Lemma 8, we have easily

$$\boldsymbol{C} \cong H^1(M, \mathcal{O}_M) \cong H^1(\boldsymbol{P}^1, R^0\pi_*\mathcal{O}_M) + H^0(\boldsymbol{P}^1, R^1\pi_*\mathcal{O}_M) . \tag{32}$$

Since the lemma holds for $q=0$, we have

$$H^1(\boldsymbol{P}^1, R^0\pi_*\mathcal{O}_M) = 0 .$$

Therefore we obtain from (32) that

$$H^0(\boldsymbol{P}^1, R^1\pi_*\mathcal{O}_M) \cong \boldsymbol{C} . \tag{33}$$

Then we can take a non-zero section s of $H^0(\boldsymbol{P}^1, R^1\pi_*\mathcal{O}_M)$. We form an exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\boldsymbol{P}^1} & \xrightarrow{\otimes s} & R^1\pi_*\mathcal{O}_M & \longrightarrow & \mathcal{S} & \longrightarrow & 0 , \\ & & \cup\!\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\!\shortmid & & & & \\ & & h & \longmapsto & h\otimes s & & & & \end{array} \tag{34}$$

on $\boldsymbol{P}^1$, where $\mathcal{S}$ is the cokernel of $\otimes s$. By the long exact sequence of cohomologies associated to (34), and by the fact $H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1})=0$, we have the exact sequence

$$0 \longrightarrow H^0(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}) \longrightarrow H^0(\boldsymbol{P}^1, R^1\pi_*\mathcal{O}_M) \longrightarrow H^0(\boldsymbol{P}^1, \mathcal{S}) \longrightarrow 0 .$$

Hence the equality

$$H^0(\boldsymbol{P}^1, \mathcal{S}) = 0 \tag{35}$$

follows from (33). Since $\dim H^1(\pi^{-1}(t), \mathcal{O}_{\pi^{-1}(t)})=1$ for any $t \in \boldsymbol{P}^1$, $R^1\pi_*\mathcal{O}_M$ is a locally free sheaf of rank 1 by a theorem of Grauert. Therefore, the support of $\mathscr{S}$ is a finite set of points. Hence (35) implies that $\mathscr{S}=0$. Thus the lemma is proved. □

Let $\rho: \pi_1(M) \to C^*$ be the group representation which sends the holomorphic automorphism α of (31) to the complex number α^{-1}. Denote by F the flat line bundle associated to ρ. Put

$$G = \mathcal{O}_{P^1}(-1) \oplus \mathcal{O}_{P^1}(-1) .$$

Then we have

Lemma 14. *There is an exact sequence of sheaves on M:*

$$0 \longrightarrow \pi^* \Omega^1_{P^1} \xrightarrow{i} \Omega^1_M \xrightarrow{\eta} \pi^* G \otimes F \longrightarrow 0 , \tag{36}$$

where i is the natural inclusion. The homomorphism η will be defined below.

Proof. The homomorphism η is defined by a collection of sheaf homomorphisms

$$\eta_{j\nu}: \Omega^1_M | V_{j\nu} \longrightarrow \mathcal{O}^2_{V_{j\nu}} \quad j=1, 2 , \quad \nu=0, 1, 2 .$$

Let ω be any given germ in $\Omega^1_{M,x}$, $x \in V_{j\nu}$, which is written as

$$\omega = a_{j\nu}(x) du_{j\nu} + b_{j\nu}(x) dv_{j\nu} + c_{j\nu}(x) dt_{j\nu} .$$

Then we define

$$\eta_{j\nu}(\omega) = (a_{j\nu}(x), b_{j\nu}(x)) .$$

Note that we have the relations

$$\begin{cases} a_{j\nu} = a_{j\,\nu+1} \\ b_{j\nu} = b_{j\,\nu+1} \end{cases} \text{on} \quad V_{j\nu} \cap V_{j\,\nu+1} , \quad \nu=0, 1 ,$$

$$\begin{cases} a_{j0} = \alpha^{-1} a_{j2} \\ b_{j0} = \alpha^{-1} b_{j2} \end{cases} \text{on} \quad V_{j0} \cap V_{j2} ,$$

and

$$\begin{cases} a_{1\nu} = s_2 a_{2\nu} \\ b_{1\nu} = s_2 b_{2\nu} \end{cases} \text{on} \quad V_{1\nu} \cap V_{2\nu} , \quad \nu=0, 1, 2 .$$

Hence the collection $\{\eta_{j\nu}\}$ gives the desired sheaf homomorphism

$$\eta: \Omega^1_M \longrightarrow \pi^* G \otimes F .$$

The exactness of the sequence follows from the definition. □

LEMMA 15. $R^q\pi_*F=0,\ q\geqq 0$.

PROOF. Let F_t denote the restriction of F to a fibre $\pi^{-1}(t)$, $t\in \boldsymbol{P}^1$. Since a fibre of π is a Hopf surface, we have

$$\dim H^0(\pi^{-1}(t), F_t)-\dim H^1(\pi^{-1}(t), F_t)+\dim H^2(\pi^{-1}(t), F_t)=0$$

by the Riemann-Roch theorem. Since the canonical line bundle of $\pi^{-1}(t)$ is F_t^2, we have by using the Serre duality and the equation above,

$$\begin{aligned}(37)\qquad 2\dim H^0(\pi^{-1}(t), F_t)&=\dim H^1(\pi^{-1}(t), F_t)\\ &=2\dim H^2(\pi^{-1}(t), F_t)\ .\end{aligned}$$

Suppose that φ is any section of $H^0(\pi^{-1}(t), F_t)$. Then φ defines a holomorphic function $\tilde{\varphi}$ on the universal covering $\boldsymbol{C}^2-\{(0, 0)\}$ of $\pi^{-1}(t)$ satisfying

$$\tilde{\varphi}(\alpha z, \alpha w)=\alpha^{-1}\tilde{\varphi}(z, w)\ ,$$

where (z, w) is a standard system of homogenous coordinates on $\boldsymbol{C}^2$. But this equation implies $\tilde{\varphi}=0$. Hence we have $\dim H^0(\pi^{-1}(t), F_t)=0$. Therefore $\dim H^q(\pi^{-1}(t), F_t)=0$ for $q\geqq 0$ by (37). This implies the lemma by a theorem of Grauert. □

LEMMA 16. $H^q(M, \pi^*G\otimes F)=0,\ q\geqq 0$.

PROOF. Since $\pi\colon M\to \boldsymbol{P}^1$ is a fibre bundle, we have

$$R^q\pi_*(\pi^*G\otimes F)=G\otimes R^q\pi_*F=0\ ,\quad q\geqq 0$$

by Lemma 15. Hence the lemma follows immediately. □

LEMMA 17. $H^1(M, \Omega_M^1)\cong \boldsymbol{C}$.

PROOF. By the long exact sequence of cohomologies associated to (36), and by Lemma 16, it suffices to show that

$$H^1(M, \pi^*\Omega_{P^1}^1)\cong \boldsymbol{C}\ .$$

But this follows immediately from Lemma 13 using Leray's spectral sequence

$$E_2^{p,q}=H^p(\boldsymbol{P}^1, R^q\pi_*(\pi^*\Omega_{P^1}^1))\Longrightarrow H^{p+q}(M, \pi^*\Omega_{P^1}^1)\ . \qquad \square$$

Recall that M has a structure of a fibre bundle of elliptic curves over R with the projection $\pi_M\colon M\to R$, where R is biholomorphic to $\boldsymbol{P}^1\times \boldsymbol{P}^1$ (§ 2).

LEMMA 18. *There is an exact sequence of sheaves on R:*

$$0 \longrightarrow \Omega^1_R \longrightarrow R^0\pi_{M*}\Omega^1_M \longrightarrow \mathcal{O}_R \longrightarrow 0 .$$

PROOF. It is easy to form the exact sequence

$$0 \longrightarrow \pi^*_M\Omega^1_R \xrightarrow{j'} \Omega^1_M \xrightarrow{j''} \mathcal{O}_M \longrightarrow 0 ,$$

where j' is the natural inclusion. From this we have the long exact sequence

$$0 \longrightarrow \Omega^1_R \xrightarrow{j'_*} R^0\pi_{M*}\Omega^1_M \xrightarrow{j''_*} \mathcal{O}_R \longrightarrow \cdots .$$

Since $\pi: M \to R$ is a fibre bundle of elliptic curves, the homomorphism j''_* is surjective. Hence we obtain the lemma. $\square$

Recall also that M_1 has a structure of fibre bundle of elliptic curves over R^1 with the projection $\pi_{M_1}: M_1 \to R_1$, where R_1 is the blown up $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ at one point (§ 2). Similarly to Lemma 18, we have

LEMMA 19. *There is an exact sequence of sheaves on R_1:*

(38) $$0 \longrightarrow \Omega^1_{R_1} \longrightarrow R^0\pi_{M_1^*}\Omega^1_{M_1} \longrightarrow \mathcal{O}_{R_1} \longrightarrow 0 .$$

LEMMA 20. $H^0(R, R^1\pi_{M*}\Omega^1_M)=0$.

PROOF. By Lemma 18, we have the exact sequence

(39) $$\begin{aligned} 0 &\longrightarrow H^0(R, \Omega^1_R) \longrightarrow H^0(R, R^0\pi_{M*}\Omega^1_M) \longrightarrow H^0(R, \mathcal{O}_R) \\ &\longrightarrow H^1(R, \Omega^1_R) \longrightarrow H^1(R, R^0\pi_{M*}\Omega^1_M) \longrightarrow H^1(R, \mathcal{O}_R) \\ &\longrightarrow H^2(R, \Omega^1_R) \longrightarrow H^2(R, R^0\pi_{M*}\Omega^1_M) \longrightarrow H^2(R, \mathcal{O}_R) \longrightarrow \cdots . \end{aligned}$$

It is easy to check the following facts:

$$H^q(R, \Omega^1_R) = \begin{cases} 0 & q \neq 1 , \\ \boldsymbol{C}^2 & q=1 , \end{cases}$$

$$H^q(R, \mathcal{O}_R) = \begin{cases} 0 & q \neq 0 , \\ \boldsymbol{C} & q=0 . \end{cases}$$

Hence we have by (39)

(40) $$\dim H^1(R, R^0\pi_{M*}\Omega^1_M) = 1 + \dim H^0(R, R^0\pi_{M*}\Omega^1_M)$$

and

(41) $$\dim H^2(R, R^0\pi_{M*}\Omega^1_M) = 0 .$$

From the inclusion

$$H^0(R, R^0\pi_{M*}\Omega_M^1) \subset H^0(M, \Omega_M^1)$$

and from [3, Proposition 2.3], it follows that

$$H^0(R, R^0\pi_{M*}\Omega_M^1) = 0 .$$

Therefore, by (40), we get

$$\dim H^1(R, R^0\pi_{M*}\Omega_M^1) = 1 . \tag{42}$$

By Leray's spectral sequence

$$E_2^{p,q} = H^p(R, R^q\pi_{M*}\Omega_M^1) \Longrightarrow H^{p+q}(M, \Omega_M^1)$$

and by (41), we have

$$H^1(M, \Omega_M^1) \cong H^0(R, R^1\pi_{M*}\Omega_M^1) + H^1(R, R^0\pi_{M*}\Omega_M^1) .$$

Then, by (42) and Lemma 17, we obtain

$$H^0(R, R^1\pi_{M*}\Omega_M^1) = 0 . \qquad \square$$

LEMMA 21. $H^0(R_1, R^1\pi_{M_1^*}\Omega_{M_1}^1) = 0$.

PROOF. By Proposition 4, we have a homomorphism

$$H^0(R_1 - l, R^1\pi_{M_1^*}\Omega_{M_1}^1) \longrightarrow H^0(R - P, R^1\pi_{M*}\Omega_M^1) . \tag{43}$$

Moreover we have the homomorphisms defined by restrictions:

$$H^0(R_1, R^1\pi_{M_1^*}\Omega_{M_1}^1) \longrightarrow H^0(R_1 - l, R^1\pi_{M_1^*}\Omega_{M_1}^1) \tag{44}$$

and

$$H^0(R, R^1\pi_{M*}\Omega_M^1) \longrightarrow H^0(R - P, R^1\pi_{M*}\Omega_M^1) . \tag{45}$$

Both $R^1\pi_{M_1^*}\Omega_{M_1}^1$ and $R^1\pi_{M*}\Omega_M^1$ are locally free sheaves by a theorem of Grauert. Therefore the homomorphisms (43) and (44) are injective, and the homomorphism (45) is bijective. Hence the lemma follows from Lemma 20. $\square$

PROOF OF (30). The case $q=0$ follows from [3, Proposition 2.3]. The case $q=3$ follows from [3, Proposition 2.3] using the Serre duality. Suppose that $q=1$. From the long exact sequence of cohomologies associated to (38) and from the facts

$$H^q(R_1, \Omega^1_{R_1}) = \begin{cases} 0 & q \neq 1 , \\ C^3 & q = 1 , \end{cases}$$

$$H^q(R_1, \mathcal{O}_{R_1}) = \begin{cases} 0 & q \neq 0 , \\ C & q = 0 , \end{cases}$$

it follows that

$$\dim H^1(R_1, R^0\pi_{M_1^*}\Omega^1_{M_1}) = 2 + \dim H^0(R_1, R^0\pi_{M_1^*}\Omega^1_{M_1}) \tag{46}$$

and

$$\dim H^2(R_1, R^0\pi_{M_1^*}\Omega^1_{M_1}) = 0 . \tag{47}$$

By the inclusion

$$H^0(R_1, R^0\pi_{M_1^*}\Omega^1_{M_1}) \subset H^0(M_1, \Omega^1_{M_1})$$

and by [3, Proposition 2.3], we have

$$H^0(R_1, R^0\pi_{M_1^*}\Omega^1_{M_1}) = 0 .$$

Hence by (46) we obtain

$$\dim H^1(R_1, R^0\pi_{M_1^*}\Omega^1_{M_1}) = 2 . \tag{48}$$

By Leray's spectral sequence

$$E_2^{p,q} = H^p(R_1, R^q\pi_{M_1^*}\Omega^1_{M_1}) \Longrightarrow H^{p+q}(M_1, \Omega^1_{M_1}) ,$$

and by (47), we have

$$H^1(M_1, \Omega^1_{M_1}) = H^1(R_1, R^0\pi_{M_1^*}\Omega^1_{M_1}) + H^0(R_1, R^1\pi_{M_1^*}\Omega^1_{M_1}) .$$

Hence it follows that

$$H^1(M_1, \Omega^1_{M_1}) = C^2$$

from Lemma 21 and (48). Thus the case $q=1$ is proved. The remaining case $q=2$ follows from the Riemann-Roch theorem together with Proposition 6. □

PROOF OF (27) AND (28) FOR $n \geqq 1$. Recall that M_n contains Hopf surfaces $H_1, H_2, \cdots, H_n$, which are the copies of the surface S_0 in M_1 (§ 4, pp. 354–355). Note that, in each inductive step of constructing M_n, the image of the inclusion mapping i_ν: $U_{\varepsilon_\nu} \to M_\nu$ ($\nu = 1, 2, \cdots, n-1$) does not intersect $H_1, H_2, \cdots, H_\nu$. Namely, i_ν is a mapping of U_{ε_ν} into $M_\nu - \bigcup_{\mu=1}^{\nu} H_\mu$. Therefore, we can replace all the Hopf surfaces $H_1, H_2, \cdots, H_n$ in M_n

with elliptic curves $E_1, E_2, \cdots, E_n$, respectively, to obtain a compact 3-fold $M_{(n)}$ (Proposition 4). The 3-fold $M_{(n)}$ is nothing but the manifold obtained by connecting n-copies of M by using the above inclusion mappings

$$i_\nu\colon U_{\varepsilon_\nu} \longrightarrow M_\nu - \bigcup_{\mu=1}^{\nu} H_\mu \cong M_\nu - \bigcup_{\mu=1}^{\nu} E_\mu .$$

We describe another method of constructing $M_{(n)}$. Put

$$N_1 = U_b - K_1 , \quad K_1 = \overline{U_{b'}} ,$$
$$N_2 = U_{b/(|\alpha|^2)} - \overline{U_{b'/(|\alpha|^2)}} , \quad K_2 = \boldsymbol{P}^3 - U_{b/(|\alpha|^2)} ,$$

where b and b' are positive constants satisfying $b'<|\alpha|<b<1/|\alpha|$ with $|\alpha|-b'$ and $b-|\alpha|$ very small. Let g_1 be the isomorphism induced by g. Put

$$W_1 = \boldsymbol{P}^3 - K_1 - K_2 .$$

Note that $N_1 \subset W_1$, $N_2 \subset W_1$, and that $M_{(1)}$ is obtained from W_1 by identifying N_1 and N_2 by g_1. Here we can assume that $b-|\alpha|$ is so small that $i_1(U_{\varepsilon_1}) \cap (N_1 \cup N_2) = \varnothing$. We regard i_1 as an open embedding of U_{ε_1} into W_1. Put

$$W_2 = M(W_1', W_1, i_1', i_1) ,$$

where W_1' and i_1' are copies of W_1 and i_1, respectively. Denote by N_3 and N_4 the subsets in W_1' corresponding to N_1 and N_2 in W_1, respectively. Let g_2 denote the biholomorphic map of N_3 onto N_4 corresponding to g_1. Then it is easy to see that $M_{(2)}$ is obtained from W_2 identifying N_1 with N_2 by g_1, and N_3 with N_4 by g_2. By our definition,

$$\begin{aligned} W_2 &= M((\boldsymbol{P}^3 - K_1 - K_2)', \boldsymbol{P}^3 - K_1 - K_2, i_1', i_1) \\ &= \boldsymbol{P}^3 - K_1 - K_2 - K_3 - K_4 . \end{aligned}$$

where K_3 and K_4 are the new "holes" of $\boldsymbol{P}^3$ corresponding to K_1 and K_2 of the first component in the connecting operation. Then, W_1' identified naturally with $\boldsymbol{P}^3 - K_3 - K_4$. For general $n \geqq 3$, we put

$$W_n = M(W_1', W_{n-1}, i_1' | U_{\varepsilon_{n-1}}, i_{n-1}) .$$

Then we can find the subsets $N_1, N_2, \cdots, N_{2n-1}, N_{2n}$ of W_n, and the biholomorphic maps $g_\nu\colon N_{2\nu-1} \to N_{2\nu}$, $\nu = 1, 2, \cdots, n$, which are copies of N_1, N_2 and g_1 of W_1, such that $M_{(n)}$ is constructed from W_n by identifying $N_{2\nu-1}$ and $N_{2\nu}$ by g_ν for all ν. Moreover there are compact subsets $K_1, K_2, \cdots, K_{2n-1}, K_{2n}$ in $\boldsymbol{P}^3$ such that

$$W_n = \boldsymbol{P}^3 - \bigcup_{\mu=1}^{2n} K_\mu ,$$

$$W_{n-1} = \boldsymbol{P}^3 - \bigcup_{\mu=1}^{2n-2} K_\mu ,$$

and

$$W_1' = \boldsymbol{P}^3 - (K_{2n-1} \cup K_{2n}) .$$

By the construction $N_\mu \cup K_\mu$ is a connected open neighborhood of K_μ biholomorphic to U. Let

$$\pi_n : W_n \longrightarrow M_{(n)}$$

be the canonical projection.

LEMMA 22. *Let $\mathscr{E}$ be the sheaf of germs of a holomorphic covariant tensor field on a complex manifold such that $H^q(\boldsymbol{P}^3, \mathscr{E}) = 0$ for $q = 1, 2$. Then the induced homorphism*

$$\pi_n^* : H^1(M_{(n)}, \mathscr{E}) \longrightarrow H^1(W_n, \mathscr{E})$$

is zero.

PROOF. Let us consider the following commutative diagram;

$$\begin{array}{ccccccccc} \longrightarrow H^1(W_n, \mathscr{E}) & \xrightarrow{\tilde{\rho}} & \bigoplus_{\nu=1}^{2n} H^1(N_\nu, \mathscr{E}) & \xrightarrow{\tilde{\delta}} & H_L^2(W_n, \mathscr{E}) \longrightarrow & & \\ \uparrow & & \uparrow & & \cong\uparrow & & \\ \longrightarrow H^1(\boldsymbol{P}^3, \mathscr{E}) & \longrightarrow & \bigoplus_{\nu=1}^{2n} H^1(N_\nu \cup K_\nu, \mathscr{E}) & \xrightarrow{\delta} & H_L^2(\boldsymbol{P}^3, \mathscr{E}) & \longrightarrow & H^2(\boldsymbol{P}^3, \mathscr{E}) \longrightarrow . \end{array} \tag{49}$$

Here the horizontal sequences are the exact sequence of local cohomologies with the restriction map $\tilde{\rho}$ and $L = \boldsymbol{P}^3 - \bigcup_{\nu=1}^{2n} (N_\nu \cup K_\nu)$. Let $\theta \in H^1(M_{(n)}, \mathscr{E})$ be any element. Put $\tilde{\theta} = \pi_n^* \theta$ and

$$\tilde{\rho}(\tilde{\theta}) = \sum_{\nu=1}^{2n} \tilde{\theta}_\nu , \quad \text{where} \quad \tilde{\theta}_\nu \in H^1(N_\nu, \mathscr{E}) . \tag{50}$$

By the assumption on $\mathscr{E}$, using Mayer-Vietoris exact sequence for $\boldsymbol{P}^3 = (N_\nu \cup K_\nu) \cup (\boldsymbol{P}^3 - K_\nu)$, we can find $\tilde{\alpha}_\nu \in H^1(N_\nu \cup K_\nu, \mathscr{E})$ and $\tilde{\beta}_\nu \in H^1(\boldsymbol{P}^3 - K_\nu, \mathscr{E})$ such that

$$\tilde{\theta}_\nu = \tilde{\alpha}_\nu + \tilde{\beta}_\nu \quad \text{on} \quad N_\nu = (N_\nu \cup K_\nu) \cap (\boldsymbol{P}^3 - K_\nu) . \tag{51}$$

Since $\tilde{\theta}$ is the lifting of an element of $H^1(M_{(n)}, \mathscr{E})$, we have the relations;

$$g_\nu^*(\tilde{\alpha}_{2\nu} + \tilde{\beta}_{2\nu}) = \tilde{\alpha}_{2\nu-1} + \tilde{\beta}_{2\nu-1} , \quad \nu = 1, 2, \cdots, n .$$

Hence we have

$$g_\nu^*\tilde{\alpha}_{2\nu}-\tilde{\beta}_{2\nu-1}=\tilde{\alpha}_{2\nu-1}-g_\nu^*\tilde{\beta}_{2\nu} .$$

The right hand side of this equation is defined on $N_{2\nu-1}\cup K_{2\nu-1}$, and the left hand side is defined on $P^3-K_{2\nu-1}$. Since $(N_{2\nu-1}\cup K_{2\nu-1})\cup(P^3-K_{2\nu-1})=P^3$, and since $H^1(P^3, \mathscr{E})=0$, this implies that

$$g_\nu^*\tilde{\alpha}_{2\nu}=\tilde{\beta}_{2\nu-1} \quad \text{and} \quad g_\nu^*\tilde{\beta}_{2\nu}=\tilde{\alpha}_{2\nu-1} . \tag{52}$$

Since $\tilde{\delta}(\tilde{\rho}(\tilde{\theta}))=0$, $\delta(g_\nu^*\tilde{\alpha}_{2\nu})=0$, and $\delta(\tilde{\beta}_{2\nu})=0$, we have

$$\sum_{\nu=1}^{n}\delta(\tilde{\alpha}_{2\nu})+\sum_{\nu=1}^{n}\delta(g_\nu^*\tilde{\beta}_{2\nu})=0 .$$

Recall that $\tilde{\alpha}_{2\nu}\in H^1(N_{2\nu}\cup K_{2\nu}, \mathscr{E})$ and $g_\nu^*\tilde{\beta}_{2\nu}\in H^1(N_{2\nu-1}\cup K_{2\nu-1}, \mathscr{E})$. Therefore we obtain $\tilde{\alpha}_{2\nu}=0$ and $\tilde{\beta}_{2\nu}=0$ for $\nu=1, 2, \cdots, n$, since δ is bijective. Then it follows from (50), (51) and (52) that $\tilde{\rho}(\tilde{\theta})=0$. By the Mayer-Vietoris exact sequence for $P^3=W_n\cup(\cup_{\nu=1}^{2n}(N_\nu\cup K_\nu))$, we infer that $\tilde{\rho}(\tilde{\theta})$ extends to an element of $H^1(P^3, \mathscr{E})$, which is equal to zero. Thus $\tilde{\theta}=0$. This proves the lemma. $\square$

In general, we let X_1 and X_2 be compact 3-folds of Class L and ι_ν: $U_\varepsilon\to X_\nu$, $\nu=1, 2$, be open holomorphic embeddings. Define X to be the manifold $M(X_1, X_2, \iota_1, \iota_2)$. Put $K_\nu=\overline{\iota_\nu(U_{1/\varepsilon})}$ and $X_\nu^\sharp=X_\nu-K_\nu$. Let j_ν: $X_\nu^\sharp\to X_\nu$ and h_ν: $X_\nu^\sharp\to X$ be the natural inclusions. Let s_ν: $N(\varepsilon)\to X_\nu^\sharp$ and ι: $N(\varepsilon)\to X$ be the open holomorphic embeddings defined by $s_\nu=\iota_\nu|N(\varepsilon)$ and $\iota=h_1\cdot s_1=h_2\cdot s_2\cdot\sigma$, respectively.

LEMMA 23. *If the induced homomorphisms*

$$\iota^*: H^1(X, \mathscr{E}) \longrightarrow H^1(N(\varepsilon), \mathscr{E})$$
$$\iota_\nu^*: H^1(X_\nu, \mathscr{E}) \longrightarrow H^1(U_\varepsilon, \mathscr{E})$$

are zero for the sheaf $\mathscr{E}$ of germs of a covariant holomorphic tensor field, then the equality

$$\dim H^1(X_1, \mathscr{E})+\dim H^1(X_2, \mathscr{E})=\dim H^1(X, \mathscr{E})$$

holds.

PROOF. Consider the following diagram of cohomologies with the coeficient $\mathscr{E}$;

$$\begin{array}{ccccccc}
\longrightarrow & H^1(X) & \xrightarrow{h_1^*\oplus h_2^*} & H^1(X_1^\sharp)\oplus H^1(X_2^\sharp) & \xrightarrow{s_1^*-(s_2\sigma)^*} & H^1(N(\varepsilon)) & \longrightarrow \\
 & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\cong} & & & \\
\longrightarrow & H^1(X_1)\oplus H^1(X_2) & \xrightarrow{j_1^*\oplus j_2^*} & H^1(X_1^\sharp)\oplus H^1(X_2^\sharp) & \xrightarrow{\delta_1\oplus\delta_2} & H^2_{K_1}(X_1)\oplus H^2_{K_2}(X_2) & \longrightarrow .
\end{array} \tag{53}$$

Here the map α will be defined below. The first horizontal sequence is the Mayer-Vietoris exact sequence for $X=X_1^\sharp \cup X_2^\sharp$. The second horizontal sequence is the direct sum of exact sequences for the pairs $(X_\nu, X_\nu^\sharp)$, $\nu=1, 2$. Let $u_\nu \in H^1(X_\nu, \mathscr{E})$, $\nu=1, 2$, be any elements. By the assumption that ι_ν^*, $\nu=1, 2$, are zero, it follows that $s_1^* \cdot j_1^*(u_1)=0$ and $(s_2 \cdot \sigma)^* \cdot j_2^*(u_2)=0$. Therefore we can find an element $u \in H^1(X, \mathscr{E})$ such that $h_1^*(u)=j_1^*(u_1)$ and $h_2^*(u)=j_2^*(u_2)$. Since $\mathscr{E}$ is the sheaf of germs of a holomorphic covariant tensor field, $H^0(N(\varepsilon), \mathscr{E})=0$ and $H^1_{K_\nu}(X_\nu, \mathscr{E})=0$ hold. Hence the map α: $(u_1, u_2) \mapsto u$ is well-defined by the injectivity of $j_1^* \oplus j_2^*$ and $h_1^* \oplus h_2^*$. It is easy to see that α is injective. To prove the surjectivity take any element $u \in H^1(X, \mathscr{E})$. Since ι^* is zero, both $h_1^*(u)$ and $h_2^*(u)$ extends to elements of $H^1(X_1, \mathscr{E})$ and $H^1(X_2, \mathscr{E})$, respectively, by the Mayer-Vietoris exact sequences. □

LEMMA 24. $\dim H^1(M_{(n)}, \mathcal{O})=n$.

PROOF. By Lemma 22, we see that the assumptions of Lemma 23 is satisfied, if we substitute $M_{(n)}$, $M_{(1)}$, $M_{(n-1)}$, $i_1|U_{\varepsilon_{n-1}}$ and i_{n-1} for X, X_1, X_2, ι_1, and ι_2. Therefore Lemma 24 follows easily from Lemmas 8 and 23 by the induction on n.

LEMMA 25. *The natural homomorphism* $H^1(M_{(n)}, \boldsymbol{C}) \to H^1(M_{(n)}, \mathcal{O})$ *is an isomorphism.*

PROOF. It is easy to see that $b_1(M_{(n)})=n$. Since $H^0(X, d\mathcal{O})=0$ for any 3-fold X of Class L, the lemma follows easily from Lemma 24 and the exact sequence

$$0 \longrightarrow \boldsymbol{C} \longrightarrow \mathcal{O} \longrightarrow d\mathcal{O} \longrightarrow 0 . \tag{54}$$ □

Let us study neighborhoods of the Hopf surfaces H_ν in M_n. Put $\tilde{V}=(\boldsymbol{C}^2-\{0\}) \times \boldsymbol{C}$. Let V be the quotient manifold of $\tilde{V}$ by the action of the holomorphic automorphism

$$\beta\colon ((x, y), z) \longmapsto ((\beta_0 x, \beta_0 y), \beta_0^{-1} z) ,$$

where β_0 is the constant defined on page 341 in § 1. Denote by π_V the canonical projection $\tilde{V} \to V$. Let S be the submanifold in V defined by $z=0$. Then by the construction of X in § 1, we see that S_0 has a neighborhood which is biholomorphic to that of S in V. We shall prove

LEMMA 26. $\dim H^1_S(V, d\mathcal{O})=1$.

PROOF. Naturally, V has the structure of a line bundle on S. Attaching the infinite section S_∞ to V, we get a compact 3-fold $\bar{V}$. $\bar{V}$ is a $\boldsymbol{P}^1$-bundle over S. On the other hand, $\bar{V}-S$ is biholomorphic to the complement $W-C$ of an elliptic curve C of a 3-dimensional Hopf manifold W. Hence $H^0(W-C, d\mathcal{O})=H^0(W, d\mathcal{O})=0$. We claim that the holomorphic 1-form idz/z on $(\bar{V}-S)\cap(\bar{V}-S_\infty)$ defines a non-zero cocycle in $H^1(\bar{V}, d\mathcal{O})$, but is cohomologous to zero in $H^1(\bar{V}-S, d\mathcal{O})$. In fact, we have the equation $idz/z=i-\bar{w}dw/(|w|^2+|x|^2+|y|^2)\}+\{(|x|^2+|y|^2)\bar{z}dz/(1+|xz|^2+|yz|^2)\}$, where $w=z^{-1}$. Since $H^0(\bar{V}, d\Omega^1)\subset H^0(\bar{V}-S, d\Omega^1)\cong H^0(W-C, d\Omega^1)=H^0(W, d\Omega^1)=0$, $H^1(\bar{V}, d\mathcal{O})$ and $H^1(\bar{V}-S, d\mathcal{O})$ can be regarded as subspaces of $H^1(\bar{V}, \Omega^1)$ and $H^1(\bar{V}-S, \Omega^1)$, respectively. Regarding the 1-cocycle $\{idz/z\}$ as an element of $H^1(\bar{V}, \Omega^1)$, we see that its Dolbeault cohomology class is represented by the $\bar{\partial}$-closed form

$$\begin{aligned}\omega&=-i\bar{\partial}(\bar{w}dw/(|w|^2+|x|^2+|y|^2)) \quad \text{on} \quad \bar{V}-S\\ &=-i\bar{\partial}((|x|^2+|y|^2)\bar{z}dz/(1+|xz|^2+|yz|^2)) \quad \text{on} \quad \bar{V}-S_\infty\,.\end{aligned}$$

The triviality of the class $\{idz/z\}$ in $H^1(\bar{V}-S, d\mathcal{O})$ follows immediately from this. By a direct calculation, we have $\int_F\omega>0$, where F is a fibre of the $\boldsymbol{P}^1$-bundle $\bar{V}$ over S. This implies that ω is not $\bar{\partial}$-exact, since, if $\omega=\bar{\partial}\varphi$ for some smooth (1, 0)-form φ on $\bar{V}$, then we have $\int_F\omega=\int_F\bar{\partial}\varphi=\int_F d\varphi=0$ by the fact that the integration of (2, 0)-form on F vanishes. By the exact sequence (54) on $\bar{V}$ and Leray's spectral sequence applied to the $\boldsymbol{P}^1$-bundle structure of $\bar{V}$, we have $\dim H^1(\bar{V}, d\mathcal{O})=1$. Then we have the lemma by the exact sequence of local cohomologies;

$$\begin{aligned}\longrightarrow H^0(\bar{V}-S, d\mathcal{O})\longrightarrow H^1_S(\bar{V}, d\mathcal{O})\longrightarrow H^1(\bar{V}, d\mathcal{O})&\\ \longrightarrow H^1(\bar{V}-S, d\mathcal{O})\longrightarrow .&\end{aligned} \qquad \square$$

Lemma 27. $\dim H^1_S(V, \mathcal{O})=0$.

Proof. It is easy to check that the restriction

$$\boldsymbol{C}\cong H^1(\bar{V}, \mathcal{O})\longrightarrow H^1(\bar{V}-S, \mathcal{O})$$

is injective. Note that $H^0(\bar{V}, \mathcal{O})=H^0(W-C, \mathcal{O})=\boldsymbol{C}$. Therefore the lemma follows from the exact sequence

$$\begin{aligned}\longrightarrow H^0(\bar{V}, \mathcal{O})\longrightarrow H^0(\bar{V}-S, \mathcal{O})\longrightarrow H^1_S(\bar{V}, \mathcal{O})&\\ \longrightarrow H^1(\bar{V}, \mathcal{O})\longrightarrow H^1(\bar{V}-S, \mathcal{O})\longrightarrow .&\end{aligned} \qquad \square$$

Lemma 28. *The natural homomorphism* $H^2_S(V, \boldsymbol{C})\to H^2_S(V, \mathcal{O})$ *is zero.*

Proof. Consider the exact sequence of local cohomologies;

$$\longrightarrow H^1_S(\bar{V}, \mathcal{O}) \longrightarrow H^1_S(\bar{V}, d\mathcal{O}) \longrightarrow H^2_S(\bar{V}, \boldsymbol{C}) \longrightarrow H^2_S(\bar{V}, \mathcal{O}) \longrightarrow .$$

We see easily that $\dim H^2_S(\bar{V}, \boldsymbol{C})=1$. Then the lemma follows from Lemmas 26 and 27. $\square$

Let

$$\mu: G \longrightarrow \boldsymbol{C}^2$$

be the blowing up at the origin $0=(0, 0)$, where (u, v) is a standard system of coordinates on $\boldsymbol{C}^2$. G is covered by 2 copies G_1 and G_2 of $\boldsymbol{C}^2$. Let (u_i, v_i), $i=1, 2$, be their standard systems of coordinates such that $u=u_1$, $v=u_1v_1$ and $u=u_2v_2$, $v=u_2$. Then, on $G_1\cap G_2$, we have the relations $u_1=v_2u_2$, $v_1=v_2^{-1}$. We define a holomorphic map

$$\lambda: V=\tilde{V}/\langle\beta\rangle \longrightarrow G$$

by

$$[(x, y), z] \longmapsto (u_1, v_1)=(xz, y/x)\ , \quad \text{if} \quad x\neq 0$$
$$[(x, y), z] \longmapsto (u_2, v_2)=(yz, x/y)\ , \quad \text{if} \quad y\neq 0\ .$$

Here, for any point $((x, y), z)\in \tilde{V}=(\boldsymbol{C}^2-\{0\})\times \boldsymbol{C}$, we indicate by $[(x, y), z]$ the corresponding point on the quotient space V. Similarly, for any point $z\in \boldsymbol{C}^*$, we indicate by $[z]$ the corresponding point on the quotient space $\Delta=\boldsymbol{C}^*/\langle\beta_0\rangle$. Put $\tilde{S}=\{((x, y), z)\in \tilde{V}: z=0\}$. We define biholomorphic maps

$$\tilde{\nu}: \tilde{V}-\tilde{S} \longrightarrow (\boldsymbol{C}^2-\{0\})\times \boldsymbol{C}^*$$

by

$$((x, y), z) \longmapsto (xz, yz, z)\ ,$$

and

$$\nu: V-S \longrightarrow (\boldsymbol{C}^2-\{0\})\times \Delta$$

by

$$[(x, y), z] \longmapsto (xz, yz, [z])\ .$$

Let $p_1: (\boldsymbol{C}^2-\{0\})\times\Delta \to \boldsymbol{C}^2-\{0\}$ be the projection to the 1st component. Then we have

$$(\mu\cdot\lambda)\,|\,(V-S)=p_1\cdot\nu\ .$$

Denote by π_V (resp. π_B) the canonical projection to the quotient space $\tilde{V}\to V$ (resp. $(\boldsymbol{C}^2-\{0\}\times \boldsymbol{C}^* \to (\boldsymbol{C}^2-\{0\})\times\Delta)$. Let c be a small positive constant. Put

$$B_c=\{(u, v)\in C^2; |u|^2+|v|^2<c^2\},$$
$$G_c=\mu^{-1}(B_c),$$
$$\tilde{V}_c=\{((x, y), z)\in\tilde{V}: |xz|^2+|yz|^2<c^2\}, \quad\text{and}$$
$$V_c=\tilde{V}_c/\langle\beta\rangle=\lambda^{-1}(G_c).$$

Put

$$\pi_{B_c}=\pi_B|(B_c-\{0\})\times C^*, \qquad \pi_{V_c}=\pi_V|\tilde{V}_c,$$
$$\tilde{\nu}_c=\tilde{\nu}|(\tilde{V}_c-\tilde{S}), \quad\text{and}\quad \nu_c=\nu|(V_c-S).$$

Now we borrow an idea of Douady from [1]. Let $V=\{V_i\}$ be a covering of V_c such that each V_i is a simply connected Stein subdomain. Put $\tilde{V}_i=\pi_V^{-1}(V_i)$. Then $\tilde{V}=\{\tilde{V}_i\}$ is a covering of $\tilde{V}_c$. Each $\tilde{V}_i$ is β-invariant, and is a disjoint union of Stein domains. Let $\mathscr{E}$ denote the sheaf of germs of a holomorphic covariant tensor field on a complex manifold. Then the automorphism β induces an automorphism β^* of the cochain group $C^*(\tilde{V}, \mathscr{E})$. There is the following exact sequence

$$0\longrightarrow C^*(V, \mathscr{E})\xrightarrow{\pi_V^*} C^*(\tilde{V}, \mathscr{E})\xrightarrow{1-\beta^*} C^*(\tilde{V}, \mathscr{E})\longrightarrow 0. \tag{55}$$

In fact, $1-\beta^*$ is surjective. To prove this, for any $(i_0, i_1, \cdots, i_q)$, we let $V'_{i_0,i_1,\cdots,i_q}$ denote the open subset of $\tilde{V}_c$ such that

$$\pi_V\colon V'_{i_0,i_1,\cdots,i_q}\longrightarrow V_{i_0,i_1,\cdots,i_q}$$

is a homeomorphism, where $V_{i_0,i_1,\cdots,i_q}=\bigcap_{s=0}^q V_{i_s}$. Then $\tilde{V}_{i_0,i_1,\cdots,i_q}=\pi_V^{-1}(V_{i_0,i_1,\cdots,i_q})$ is a disjoint union of $\beta^p(V'_{i_0,i_1,\cdots,i_q})$, $p\in Z$. Any $\gamma\in C^q(\tilde{V}, \mathscr{E})$ can be written as

$$\gamma=-\gamma_1+\gamma_2$$

with

$$\gamma_1=0 \quad\text{on}\quad \beta^p(V'_{i_0,i_1,\cdots,i_q}) \quad\text{for}\quad p<0,$$
$$\gamma_2=0 \quad\text{on}\quad \beta^p(V'_{i_0,i_1,\cdots,i_q}) \quad\text{for}\quad p\geqq 0.$$

Put

$$\varphi=\sum_{p<0}(\beta^*)^p\gamma_1+\sum_{p\geqq 0}(\beta^*)^p\gamma_2 \quad\text{(locally finite sum)}.$$

Then

$$\beta^*\varphi=\sum_{p\leqq 0}(\beta^*)^p\gamma_1+\sum_{p>0}(\beta^*)^p\gamma_2.$$

Therefore we have $\varphi-\beta^*\varphi=-\gamma_1+\gamma_2=\gamma$. Thus $1-\beta^*$ is surjective. From (55), we have the long exact sequence

$$
\begin{aligned}
(56)\qquad & 0 \longrightarrow H^0(V_c, \mathscr{E}) \xrightarrow{\pi_V^*} H^0(\tilde{V}_c, \mathscr{E}) \xrightarrow{1-\beta^*} H^0(\tilde{V}_c, \mathscr{E}) \\
& \xrightarrow{\delta_V} H^1(V_c, \mathscr{E}) \xrightarrow{\pi_V^*} H^1(\tilde{V}_c, \mathscr{E}) \xrightarrow{1-\beta^*} H^1(\tilde{V}_c, \mathscr{E}) \longrightarrow .
\end{aligned}
$$

Similarly, for the infinite cyclic coverings

$$
\begin{aligned}
&\pi_{V'}: \tilde{V}_c - \tilde{S} \longrightarrow V_c - S\ , \\
&\pi_B: B_c \times C^* \longrightarrow B_c \times \Delta\ , \quad \text{and} \\
&\pi_{B'}: (B_c - \{0\}) \times C^* \longrightarrow (B_c - \{0\}) \times \Delta\ ,
\end{aligned}
$$

we have the similar exact sequences as (56). Moreover, there is the following commutative diagram of cohomologies with coefficient $\mathscr{E}$:

$$
(57)\qquad
\begin{array}{ccccccccc}
\longrightarrow & H^0(\tilde{V}_c) & \xrightarrow{\delta_V} & H^1(V_c) & \xrightarrow{\pi_V^*} & H^1(\tilde{V}_c) & \xrightarrow{1-\beta^*} & H^1(\tilde{V}_c) & \longrightarrow \\
 & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle r_3} & & \downarrow{\scriptstyle r_4} & \\
\longrightarrow & H^0(\tilde{V}_c - \tilde{S}) & \xrightarrow{\delta_{V'}} & H^1(V_c - S) & \xrightarrow{\pi_{V'}^*} & H^1(\tilde{V}_c - \tilde{S}) & \xrightarrow{1-\beta^*} & H^1(\tilde{V}_c - \tilde{S}) & \longrightarrow \\
 & \uparrow{\scriptstyle \tilde{\nu}_c^*}\cong & & \uparrow{\scriptstyle \nu_c^*}\cong & & \uparrow{\scriptstyle \tilde{\nu}_c^*}\cong & & \uparrow{\scriptstyle \tilde{\nu}_c^*}\cong & \\
\longrightarrow & H^0(B_c' \times C^*) & \xrightarrow{\delta_{B'}} & H^1(B_c' \times \Delta) & \xrightarrow{\pi_{B'}^*} & H^1(B_c' \times C^*) & \xrightarrow{1-\beta_1^*} & H^1(B_c' \times C^*) & \longrightarrow \\
 & \uparrow{\scriptstyle r_5} & & \uparrow{\scriptstyle r_6} & & \uparrow{\scriptstyle r_7} & & \uparrow{\scriptstyle r_8} & \\
\longrightarrow & H^0(B_c \times C^*) & \xrightarrow{\delta_B} & H^1(B_c \times \Delta) & \xrightarrow{\pi_B^*} & H^1(B_c \times C^*) & \xrightarrow{1-\beta_1^*} & H^1(B_c \times C^*) & \longrightarrow .
\end{array}
$$

Here the homomorphisms r_j, $1 \leqq j \leqq 8$, are restrictions, and B_c' indicates $B_c - \{0\}$. β_1^* is the homomorphism induced by the automorphism of $B_c \times C^*$ defined by $((u, v), z) \mapsto ((u, v), \beta_0^{-1} z)$. For simplicity, we denote by π_V^*, $\pi_{V'}^*$, π_B^* and $\pi_{B'}^*$, the homomorphisms induced by $\pi_V | \tilde{V}_c$, $\pi_V | (\tilde{V}_c - \tilde{S})$, $\pi_B | B_c \times C^*$ and $\pi_B | B_c' \times C^*$, respectively.

LEMMA 29: *There is an isomorphism*

$$
\tilde{\nu}_c^*: H^1(B_c \times \Delta, \mathscr{O}) \longrightarrow H^1(V_c, \mathscr{O})
$$

which makes the diagram

$$
(58)\qquad
\begin{array}{ccc}
H^1(B_c \times \Delta, \mathscr{O}) & \xrightarrow{\tilde{\nu}_c^*} & H^1(V_c, \mathscr{O}) \\
\downarrow{\scriptstyle r_B} & & \downarrow{\scriptstyle r_V} \\
H^1((B_c - \{0\})) \times \Delta, \mathscr{O}) & \xrightarrow[\nu_c^*]{} & H^1(V_c - S, \mathscr{O})
\end{array}
$$

commutative. Here r_B and r_V are restrictions, and ν_c^* is the isomorphism induced by ν_c.

SUBLEMMA 1. *For $\mathscr{E}=\mathscr{O}$, the homomorphisms π_V^* and π_B^* are zero.*

PROOF. It is enough to show that the homomorphisms $1-\beta^*$ and $1-\beta_1^*$ of the first cohomology groups are injective. First we shall show that $1-\beta^*$ is injective. Let

$$\tilde{W}_1=\{(x, y, z)\in \boldsymbol{C}^3 : |xz|^2+|yz|^2<c^2,\ x\neq 0\} \quad \text{and}$$
$$\tilde{W}_2=\{(x, y, z)\in \boldsymbol{C}^3 : |xz|^2+|yz|^2<c^2,\ y\neq 0\} .$$

Then $\{\tilde{W}_1, \tilde{W}_2\}$ is a Stein open covering of $\tilde{V}_c$. Therefore we have an isomorphism

$$H^1(\tilde{V}_c, \mathscr{O})\cong\Gamma(\tilde{W}_1\cap \tilde{W}_2, \mathscr{O})/(\Gamma(\tilde{W}_1, \mathscr{O})+\Gamma(\tilde{W}_2, \mathscr{O})) .$$

Every element $\gamma\in H^1(\tilde{V}_c, \mathscr{O})$ can be represented uniquely by a Laurent series of the following form:

$$\varphi=\sum_{\substack{m<0,\, n<0\\ p\geq 0}} a_{mnp}x^m y^n z^p ,$$

which is convergent on $\tilde{W}_1\cap \tilde{W}_2$. γ is in the kernel of $1-\beta^*$ if and only if the equality $\varphi=\beta^*\varphi$ holds. This is equivalent to the equalities $a_{mnp}(1-\beta_0^{m+n-p})=0$ for all $m<0$, $n<0$ and $p\geqq 0$. But these imply $a_{mnp}=0$. Therefore $1-\beta^*$ is injective. Next we have to show that the homomorphism $1-\beta_1^*$ is injective. Let

$$\tilde{W}_1=\{((u, v), z)\in B_c\times \boldsymbol{C}^* : u\neq 0\} \quad \text{and}$$
$$\tilde{W}_2=\{((u, v), z)\in B_c\times \boldsymbol{C}^* : v\neq 0\} .$$

Then $\{\tilde{W}_1, \tilde{W}_2\}$ is a Stein open covering of $B_c\times\boldsymbol{C}^*$. Hence the injectivity of $1-\beta_1^*$ follows by the similar calculation as above. $\square$

PROOF OF LEMMA 29. Take any $\gamma\in H^1(B_c\times\Delta, \mathscr{O})$. By Sublemma 1 and (57), there is an element $\xi\in H^0(B_c\times \boldsymbol{C}^*, \mathscr{O})$ such that $\gamma=\delta_B(\xi)$. Put

$$\xi=\sum_{\substack{m\geq 0,\, n\geq 0\\ -\infty<p<\infty}} a_{mnp}u^m v^n z^p .$$

Let

$$\zeta=\sum_{m\geq 0,\, n\geq 0}\{\sum_{p\neq 0}(a_{mnp}/(1-\beta_0^{-p}))u^m v^n z^p\} .$$

Then ζ is convergent on $B_c\times\boldsymbol{C}^*$ and satisfies the functional equation

$$\zeta-\beta_1^*\zeta=\sum_{\substack{m\geq 0,\, n\geq 0\\ -\infty<p<\infty,\, p\neq 0}} a_{mnp}u^m v^n z^p .$$

Therefore, replacing ξ by $\xi-(\zeta-\beta_1^*\zeta)$, we can assume that ξ is of the form

$$\xi=\sum_{m\geq 0,\ n\geq 0} a_{m\,n\,0}u^m v^n .$$

Then

$$\tilde{\nu}_c^*\cdot r_5(\xi)=\sum_{m\geq 0,\ n\geq 0} a_{m\,n\,0}x^m y^n z^{m+n} . \tag{59}$$

This shows that $\tilde{\nu}_c^*\cdot r_5(\xi)$ extends to an element $\xi_V\in H^0(\tilde{V}_c, \mathcal{O})$. Define the homomorphism $\bar{\nu}_c^*$ by $\bar{\nu}_c^*(\gamma)=\delta_V(\xi_V)$. Then it is easy to see that $\bar{\nu}_c^*$ is injective and makes the diagram (58) commutative. It remains to show that every element of $H^1(V_c, \mathcal{O})$ is represented by an element of the form (59). Let

$$\rho=\sum_{\substack{m\geq 0,\ n\geq 0\\ p\geq 0}} a_{m\,n\,p}x^m y^n z^p$$

by any element of $H^0(\tilde{V}_c, \mathcal{O})$. Put

$$\tau=\sum_{\substack{m\geq 0,\ n\geq 0\\ m+n-p\neq 0}} (a_{m\,n\,p}/(1-\beta_0^{m+n-p}))x^m y^n z^p .$$

Then τ is convergent on $\tilde{V}_c$ and satisfies the functional equation

$$\tau-\beta^*\tau=\sum_{\substack{m\geq 0,\ n\geq 0\\ m+n-p\neq 0}} a_{m\,n\,p}x^m y^n z^p .$$

Since $\delta_V(\rho)=\delta_V(\rho-(\tau-\beta^*\tau))$, every element of $H^1(V_c, \mathcal{O})$ is represented by an element of the form (59) by Sublemma 1. This completes the proof of the lemma. □

For $\mathcal{E}=\Omega^1$, we have the following

LEMMA 30. *For any element* $a\in H^1(V_c, \Omega^1)$, *there is an element* $b\in H^1(B_c\times\Delta, \Omega^1)$ *such that* $r_2(a)=\nu_c^*\cdot r_6(b)$.

SUBLEMMA 2. *For* $\mathcal{E}=\Omega^1$, *the homomorphism* π_V^* *is zero.*

PROOF. It is enough to show that the homomorphism $1-\beta^*$ of the first cohomology group is injective. We use the Stein open covering of $\tilde{V}_c$ used in the proof of Sublemma 1. Then we have an isomorphism

$$H^1(\tilde{V}_c, \Omega^1)\cong\Gamma(\tilde{W}_1\cap\tilde{W}_2, \Omega^1)/(\Gamma(\tilde{W}_1, \Omega^1)+\Gamma(\tilde{W}_2, \Omega^1)) .$$

Every element γ of $H^1(\tilde{V}_c, \Omega^1)$ can be represented uniquely by a Laurent series of the following form:

$$\varphi=\sum_{\substack{m<0,\ n<0\\ p\geq 0}} \{a_{m\,n\,p}x^m y^n z^p dx+a'_{m\,n\,p}x^m y^n z^p dy+a''_{m\,n\,p}x^m y^n z^p dz\} ,$$

which is convergent on $\tilde{W}_1 \cap \tilde{W}_2$. γ is in the kernel of $1-\beta^*$ if and only if the equality $\varphi=\beta^*\varphi$ holds. This is equivalent to the equalities $a_{m\,n\,p}(1-\beta_0^{m+n-p+1})=a'_{m\,n\,p}(1-\beta_0^{m+n-p+1})=a''_{m\,n\,p}(1-\beta_0^{m+n-p-1})=0$ for all $m<0$, $n<0$ and $p\geqq 0$. These equalities imply that φ is zero. Therefore $1-\beta^*$ is injective.

PROOF OF LEMMA 30. By Sublemma 2, $a \in H^1(V_c, \Omega^1)$ is represented by an element of $H^0(\tilde{V}_c, \Omega^1)$, which is of the form

$$\rho=\sum_{\substack{m\geqq 0,\ n\geqq 0\\ p\geqq 0}}\{a_{m\,n\,p}x^m y^n z^p dx+a'_{m\,n\,p}x^m y^n z^p dy+a''_{m\,n\,p}x^m y^n z^p dz\} .$$

We define

$$\tau=\sum_{\substack{m\geqq 0,\ n\geqq 0\\ m+n-p+1\neq 0}}\{b_{m\,n\,p}x^m y^n z^p dx+b'_{m\,n\,p}x^m y^n z^p dy\}+\sum_{\substack{m\geqq 0,\ n\geqq 0\\ m+n-p-1\neq 0}} b''_{m\,n\,p}x^m y^n z^p dz ,$$

where

$$b_{mnp}=a_{m\,n\,p}/(1-\beta_0^{m+n-p+1}) ,$$
$$b'_{mnp}=a'_{m\,n\,p}/(1-\beta_0^{m+n-p+1}) ,$$
$$b''_{mnp}=a''_{m\,n\,p}/(1-\beta_0^{m+n-p-1}) .$$

Then τ is convergent on $\tilde{V}_c$ and satisfies the functional equation

$$\rho-(\tau-\beta^*\tau)=\sum_{m\geqq 0,\ n\geqq 0}\{a_{m\,n\,m+n+1}x^m y^n z^{m+n+1}dx+a'_{m\,n\,m+n+1}x^m y^n z^{m+n+1}dy\}$$
$$+\sum_{\substack{m\geqq 0,\ n\geqq 0\\ m+n>0}} a''_{m\,n\,m+n-1}x^m y^n z^{m+n-1}dz .$$

Put

$$\rho'=\sum_{m\geqq 0,\ n\geqq 0} a_{m\,n\,m+n+1}u^m v^n(du-udz/z)+\sum_{m\geqq 0,\ n\geqq 0} a'_{m\,n\,m+n+1}u^m v^n(dv-vdz/z)$$
$$+\sum_{\substack{m\geqq 0,\ n\geqq 0\\ m+n>0}} a''_{m\,n\,m+n-1}u^m v^n dz/z .$$

Then ρ' is an element of $H^0(B_c\times C^*, \Omega^1)$ such that $\tilde{\nu}_c^*\cdot r_5(\rho')=r_1(\rho-(\tau-\beta^*\tau))$. Put $b=\delta_B(\rho')$. Then we have $\nu_c^*\cdot r_6(b)=r_2\cdot\delta_V(\rho)=r_2(a)$. This completes the proof of the lemma. □

LEMMA 31. $\dim H^1(M_n, \mathcal{O})=n$.

PROOF. Consider the following diagram of cohomologies with the coefficient $\mathcal{O}$;

$$
(60)\qquad
\begin{array}{ccccccccc}
\longrightarrow \bigoplus_{\nu=1}^{n} H^1_{H_\nu}(M_n) & \xrightarrow{j_3} & H^1(M_n) & \xrightarrow{j_4} & H^1\Big(M_n-\bigcup_{\nu=1}^{n} H_\nu\Big) & \xrightarrow{\delta_1} & \bigoplus_{\nu=1}^{n} H^2_{H_\nu}(M_n) \longrightarrow \\
 & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle r}\,\cong & & \\
0=\bigoplus_{\nu=1}^{n} H^1_{E_\nu}(M_{(n)}) & \xrightarrow{j_5} & H^1(M_{(n)}) & \xrightarrow{j_6} & H^1\Big(M_{(n)}-\bigcup_{\nu=1}^{n} E_\nu\Big) & \xrightarrow{\delta_2} & \bigoplus_{\nu=1}^{n} H^2_{E_\nu}(M_{(n)}) \longrightarrow .
\end{array}
$$

Here the homomorphism r is the natural isomorphism. The homomorphism α will be defined now. Let θ be any element of $H^1(M_n)$. Then by Lemma 29, there is an element $\eta \in H^1(M_{(n)})$ such that $j_6(\eta)=r\cdot j_4(\theta)$. By Lemma 27, j_3 is zero. Since $\operatorname{codim} E_\nu>1$, we have $H^1_{E_\nu}(M_{(n)})=0$. Therefore the correspondence $\theta\mapsto\eta$ is a well-defined homomorphism and injective, which is denoted by α. Thus we have the inequality

$$
(61)\qquad \dim H^1(M_n, \mathcal{O}) \leqq \dim H^1(M_{(n)}, \mathcal{O}) .
$$

On the other hand, consider the commutative diagram

$$
(62)\qquad
\begin{array}{ccc}
H^1\Big(M_n-\bigcup_{\nu=1}^{n} H_\nu, \boldsymbol{C}\Big) & \longrightarrow & \bigoplus_{\nu=1}^{n} H^2_{H_\nu}(M_n, \boldsymbol{C}) \\
\Big\downarrow{\scriptstyle j_7} & & \Big\downarrow \\
H^1\Big(M_n-\bigcup_{\nu=1}^{n} H_\nu, \mathcal{O}\Big) & \longrightarrow & \bigoplus_{\nu=1}^{n} H^2_{H_\nu}(M_n, \mathcal{O}) .
\end{array}
$$

Note that $\dim H^1(M_n-\bigcup_\nu H_\nu, \boldsymbol{C})=n$. Since $H^0(M_n-\bigcup_\nu H_\nu, d\mathcal{O})=0$, j_7 is injective. Hence by Lemma 28 and by the first row of the exact sequence (60), we have the inequality

$$
(63)\qquad \dim H^1(M_n, \mathcal{O}) \geqq n .
$$

Then, combining (61), (63) and Lemma 24, we obtain the lemma. $\square$

Lemma 32. $\dim H^2(M_n, \mathcal{O})=0$.

Proof. We know that $\dim H^0(M_n, \mathcal{O})=1$, $\dim H^1(M_n, \mathcal{O})=n$, and $\dim H^3(M_n, \mathcal{O})=0$. Moreover, all Chern numbers of M_n are known by Proposition 7. Therefore the lemma follows immediately from the Riemman-Roch theorem. $\square$

Thus (27) is proved completely.

In [2], the proofs of Lemmas 9 and 10 were not clear. Note that these two lemmas have been essentially reproved here.

It remains to prove (28).

Lemma 33. *The image of the homomorphism*

$$\pi_n^*: H^1(M_{(n)}, \Omega^1) \longrightarrow H^1(W_n, \Omega^1)$$

is contained in the image of the restriction map

$$H^1(\boldsymbol{P}^3, \Omega^1) \longrightarrow H^1(W_n, \Omega^1) .$$

PROOF. We use the commutative diagram (49) with $\mathscr{E}=\Omega^1$. Let $\omega \in H^1(M_{(n)}, \Omega^1)$ be any element. Put $\tilde{\omega}=\pi_n^*\omega$ and

$$\tilde{\rho}(\tilde{\omega})=\sum_{\nu=1}^{2n} \tilde{\omega}_\nu , \quad \text{where} \quad \tilde{\omega}_\nu \in H^1(N_\nu, \Omega^1) . \tag{64}$$

Using Mayer-Vietoris exact sequence for $\boldsymbol{P}^3=(N_\nu \cup K_\nu) \cup (\boldsymbol{P}^3-K_\nu)$, we can find $\tilde{\alpha}_\nu \in H^1(N_\nu \cup K_\nu, \Omega^1)$ and $\tilde{\beta}_\nu \in H^1(\boldsymbol{P}^3-K_\nu, \Omega^1)$ such that

$$\tilde{\omega}_\nu=\tilde{\alpha}_\nu+\tilde{\beta}_\nu \quad \text{on} \quad N_\nu=(N_\nu \cup K_\nu) \cap (\boldsymbol{P}^3-K_\nu) . \tag{65}$$

Since $\tilde{\omega}$ is the lifting of an element of $H^1(M_{(n)}, \Omega^1)$, we have the relations;

$$g_\nu^*(\tilde{\alpha}_{2\nu}+\tilde{\beta}_{2\nu})=\tilde{\alpha}_{2\nu-1}+\tilde{\beta}_{2\nu-1} , \quad \nu=1, 2, \cdots, n .$$

Hence we have

$$g_\nu^*\tilde{\alpha}_{2\nu}-\tilde{\beta}_{2\nu-1}=\tilde{\alpha}_{2\nu-1}-g_\nu^*\tilde{\beta}_{2\nu} .$$

The left hand side of this equation is defined on $\boldsymbol{P}^3-K_{2\nu-1}$, and the right hand side is defined on $K_{2\nu-1} \cup N_{2\nu-1}$. Since $(K_{2\nu-1} \cup N_{2\nu-1}) \cup ((\boldsymbol{P}^3-K_{2\nu-1})=\boldsymbol{P}^3$, this implies that

$$g_\nu^*\tilde{\alpha}_{2\nu}-\tilde{\beta}_{2\nu-1}=\tilde{\omega}'_\nu \quad \text{and} \quad g_\nu^*\tilde{\beta}_{2\nu}-\tilde{\alpha}_{2\nu-1}=-\tilde{\omega}'_\nu \tag{66}$$

for some element $\tilde{\omega}'_\nu \in H^1(\boldsymbol{P}^3, \Omega^1)$. Since $\tilde{\delta}(\tilde{\omega}'_\nu)=0$, $\tilde{\delta}(\tilde{\rho}(\tilde{\omega}))=0$, $\tilde{\delta}(g_\nu^*\tilde{\alpha}_{2\nu})=0$, and $\tilde{\delta}(\tilde{\beta}_{2\nu})=0$, we obtain from (64), (65) and (66) the equality

$$\sum_{\nu=1}^{n} \delta(\tilde{\alpha}_{2\nu})+\sum_{\nu=1}^{n} \delta(g_\nu^*\tilde{\beta}_{2\nu})=0 .$$

Then, since $\dim H^1(\boldsymbol{P}^3, \Omega^1)=1$, by the exact sequence (49), there is an element $\tilde{\omega}'_0 \in H^1(\boldsymbol{P}^3, \Omega^1)$ such that

$$\tilde{\alpha}_{2\nu}=a_{2\nu}\tilde{\omega}'_0 \quad \text{and} \quad g_\nu^*\tilde{\beta}_{2\nu}=b_{2\nu}\tilde{\omega}'_0 \tag{67}$$

for some complex numbers $a_{2\nu}$ and $b_{2\nu}$, $\nu=1, 2, \cdots, n$. Since every g_ν entends to an automorphism of $\boldsymbol{P}^3$, we infer from (64), (65), (66) and (67) that $\tilde{\rho}(\tilde{\omega})$ is defined on the total space $\boldsymbol{P}^3$. This implies the lemma, since $H_L^1(W_n, \Omega^1)=0$. □

LEMMA 34. $\dim H^1(M_{(n)}, \Omega^1)=1$ *and a generator is represented by a* *smooth d-closed real* (1, 1)*-form* ω_n *with the properties*

(P.1) $$\int_l \omega_n > 0 \quad \textit{for any line } l \textit{ in } M_{(n)}\ ,$$

and

(P.2) $$\pi_n^* \omega_n - \tilde{\omega}_0 = i\partial\bar{\partial} F_n$$

for some smooth real valued function F_n *on* W_n, *where* $\tilde{\omega}_0$ *is the d-closed real* (1, 1)*-form associated with the Fubini-Study metric on* $\boldsymbol{P}^3$.

PROOF. The property (P.1) implies that ω_n is *not* $\bar{\partial}$-exact. In fact, if $\omega_n = \bar{\partial}\varphi$ for some smooth (1, 0)-form φ, then we would have

$$\int_l \omega_n = \int_l \bar{\partial}\varphi = \int_l d\varphi = 0\ ,$$

since the integration of (2, 0)-form on a line vanishes. This contradicts (P.1). Now we shall prove the lemma by the induction on n. For $n=1$, we put

$$\omega_1 = (i/2)\partial\bar{\partial}\{\log(|z_0|^2 + |z_1|^2) + \log(|z_2|^2 + |z_3|^2)\}\ .$$

Then ω_1 is a well-defined smooth d-closed real (1, 1)-form on $M_{(1)}$. It is easy to check (P.1). Let

$$F_1 = (1/2)\log(|z_0|^2 + |z_1|^2)(|z_2|^2 + |z_3|^2)/(|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2)^2\ .$$

This is a smooth real valued function on W_1 and satisfies (P.2). Since we know $\dim H^1(M_{(1)}, \Omega^1) = 1$ by Lemma 17, we obtain the lemma for $n=1$. Consider the case $n>1$. Let k be a natural number such that $k<n$. By the induction assumption, $H^1(M_{(k)}, \Omega^1)$ is generated by the Dolbeault cohomology class represented by ω_k. We denote by $[u]$ the Dolbeault cohomology class represented by a smooth $\bar{\partial}$-closed form u. By the property (P.1), it is easy to see that the restriction mapping

(68) $$H^1(M_{(k)}, \Omega^1) \longrightarrow H^1(M_{(k)}^{\sharp}, \Omega^1) \quad \text{is injective.}$$

Consider the diagram;

(69)
$$\begin{array}{ccc}
H^1(M_{(n)}, \Omega^1) & \xrightarrow{h_1^* \oplus h_2^*} & H^1(M_{(1)}^{\sharp}, \Omega^1) \oplus H^1(M_{(n-1)}^{\sharp}, \Omega^1) \\
\downarrow \alpha & & \downarrow \cong \\
H^1(M_{(1)}, \Omega^1) \oplus H^1(M_{(n-1)}, \Omega^1) & \xrightarrow{j_1^* \oplus j_2^*} & H^1(M_{(1)}^{\sharp}, \Omega^1) \oplus H^1(M_{(n-1)}^{\sharp}, \Omega^1)
\end{array}$$

(cf. (53)). Here all horizontal homomorphisms are induced by the natural inclusions. The first row is the Mayer-Vietoris exact sequence, and the second row is the direct sum of exact sequences of local cohomologies. The homomorphism α will be defined below. By (68), j_1^* and j_2^* are injective. Since $H^0(N(\varepsilon), \Omega^1)=0$, $h_1^*\oplus h_2^*$ is injective. Let ξ be any element of $H^1(M_{(n)}, \Omega^1)$. By the defininition of $M_{(n)}$, we claim that both $h_1^*(\xi)$ and $h_2^*(\xi)$ extend to elements of $H^1(M_{(1)}, \Omega^1)$ and $H^1(M_{(n-1)}, \Omega^1)$, respectively. In fact, $\pi_n^*(\xi)\in H^1(W_n, \Omega^1)$ extends to an element $\tilde{\xi}$ of $H^1(\boldsymbol{P}^3, \Omega^1)$ by Lemma 33. Recall that $W_1'=\boldsymbol{P}^3-K_{2n-1}-K_{2n}$ and $W_{n-1}=\boldsymbol{P}^3-K_1-K_2-\cdots-K_{2(n-1)}$. Put $\tilde{\xi}_1=\tilde{\xi}|W_1'$ and $\tilde{\xi}_2=\tilde{\xi}|W_{n-1}$. Then, since $\tilde{\xi}$ is an extension of the lifting of an element of $H^1(M_{(n)}, \Omega^1)$, both $\tilde{\xi}_1$ and $\tilde{\xi}_2$ define $\xi_1\in H^1(M_{(1)}, \Omega^1)$ and $\xi_2\in H^1(M_{(n-1)}, \Omega^1)$, respectively, such that $\pi_1'^*(\xi_1)=\tilde{\xi}_1$ and $\pi_{n-1}^*(\xi_2)=\tilde{\xi}_2$, where $\pi_1'\colon W_1'\to M_{(1)}$ is the canonical projection. This proves our claim. Since $h_1^*\oplus h_2^*$ and $j_1^*\oplus j_2^*$ are injective, the correspondence $\xi\mapsto(\xi_1, \xi_2)$ defines the desired homomorphism α. It is easy to see that α is injective. By (P.1), $(-j_1^*([\omega_1]), j_2^*[\omega_{n-1}])\in H^1(M_{(1)}^\sharp, \Omega^1)\oplus H^1(M_{(n-1)}^\sharp, \Omega^1)$ cannot be in the image space of $h_1^*\oplus h_2^*$. Hence we have the inequality

$$\dim H^1(M_{(n)}, \Omega^1)\leqq 1 . \tag{70}$$

By (P.2), we have

$$\pi_1'^*\omega_1-\tilde{\omega}_0=i\partial\bar{\partial}F_1' \quad \text{on} \quad W_1' ,$$

and

$$\pi_{n-1}^*\omega_{n-1}-\tilde{\omega}_0=i\partial\bar{\partial}F_{n-1} \quad \text{on} \quad W_{n-1}' ,$$

for some smooth real functions F_1' on W_1' and F_{n-1} on W_{n-1}. Put $N=h_1(M_{(1)}^\sharp)\cap h_2(M_{(n-1)}^\sharp)$, $\tilde{N}=\pi_n^{-1}(N)$, and $\varphi=(\pi_n|\tilde{N})^{-1*}(F_1'-F_{n-1})$. Take a real non-negative smooth function ρ on $M_{(n)}$ which is equal to 1 on a neighborhood of $h_1(M_{(1)}^\sharp)-N$, equal to 0 on a neighborhood of $h_2(M_{(n-1)}^\sharp)-N$, and which satisfies $0\leqq\rho\leqq 1$ on N. We define

$$\omega_n=\begin{cases}\omega_1-i\partial\bar{\partial}((1-\rho)\varphi) & \text{on} \quad h_1(M_{(1)}^\sharp) ,\\ \omega_{n-1}+i\partial\bar{\partial}(\rho\varphi) & \text{on} \quad h_2(M_{(n-1)}^\sharp) .\end{cases}$$

Then ω_n is a smooth d-closed real (1, 1)-form on $M_{(n)}$. Since any line in $M_{(n)}$ is homologous to a line in $h_1(M_{(1)}^\sharp)$, ω_n satisfies (P.1). Put $\tilde{\rho}=\pi_n^*\rho$. Define a smooth real-valued function F_n on W_n by

$$F_n=\tilde{\rho}F_1'+(1-\tilde{\rho})F_{n-1} .$$

Then we have the equality

$$\pi_n^*\omega_n-\tilde{\omega}_0=i\partial\bar{\partial}F_n ,$$

which shows that (P.2) is satisfied. Thus we obtain the inequality

$$\dim H^1(M_{(n)}, \Omega^1) \geqq 1 . \tag{71}$$

Combining (70) and (71), we have the lemma. □

REMARK. ω_n is not a positive form. In fact, the integrations of ω_n on elliptic curves defined by $z_0 = az_1$, $z_2 = bz_3$, $(a, b \in \boldsymbol{C})$ in $h_1(M^{\sharp}_{(1)})$ vanish.

LEMMA 35. $\dim H^1_S(V, \Omega^1) = 1$.

PROOF. By Lemma 26 and by the exact sequence

$$0 \longrightarrow d\mathcal{O} \longrightarrow \Omega^1 \longrightarrow d\Omega^1 \longrightarrow 0 ,$$

it is enough to show that $\dim H^1_{S_0}(M_1, \Omega^1) \leqq 1$. Let $b \in H^1(M_{(1)}, \Omega^1)$ be the generator represented by ω_1 in the proof of Lemma 34. In view of the defining equation of ω_1, we see that b is equal to zero on a neighborhood of E_1. Therefore $b' := b \,|\, (M_{(1)} - E_1)$ extends to an element b'' of $H^1(M_1, \Omega^1)$. Now consider the exact sequence

$$0 \longrightarrow H^1_{S_0}(M_1, \Omega^1) \longrightarrow H^1(M_1, \Omega^1) \xrightarrow{\ r\ } H^1(M_1 - S_0, \Omega^1) \longrightarrow ,$$

where r is the restriction. Since $r(b'') = b' \neq 0$, and since $\dim H^1(M_1, \Omega^1) = 2$ by (30), we obtain the lemma. □

LEMMA 36. $\dim H^1(M_n, \Omega^1) = n+1$.

PROOF. We use the diagram (60) and its notation with the coefficient Ω^1. Note that $\dim H^1_{H_\nu}(M_n, \Omega^1) = 1$ by Lemma 35, $\dim H^1(M_{(n)}, \Omega^1) = 1$ by Lemma 34, and that j_3 is injective. Hence $H^1(M_n, \Omega^1)$ contains the n-dimensional subspace generated by the images of j_3. By the construction of the d-closed (1, 1)-form ω_n in Lemma 33, the Dolbeault cohomology class $[\omega_n]$ is trivial on a neighborhood of each elliptic curve E_ν. Therefore $[\omega_n] \,|\, (M_{(n)} - \cup E_\nu)$ extends to an element $b_n \in H^1(M_n, \Omega^1)$. That $b_n \neq 0$ follows from the property (P.1). Hence we have

$$\dim H^1(M_n, \Omega^1) \geqq n+1 .$$

On the other hand, let $a \in H^1(M_n, \Omega^1)$ be any element such that $j_4(a) \neq 0$. Then, by Lemma 30, $r \cdot j_4(a)$ extends to an element of $H^1(M_{(n)}, \Omega^1)$, which is of dimension 1 by Lemma 34. Therefore we have

$$\dim H^1(M_n, \Omega^1) \leqq n+1 .$$

Thus we have the lemma. □

PROOF OF (28). $\dim H^0(M_n, \Omega^1)=0$ holds, since M_n is of Class L. $\dim H^1(M_n, \Omega^1)=n+1$ was proved by Lemma 36. By the Serre duality, we have $\dim H^3(M_n, \Omega^1)=\dim H^0(M_n, \Omega^2)=0$, since M_n is of Class L. Therefore the Euler-Poincaré characteristic $\chi(M_n, \Omega^1)$ is equal to $-n-1+\dim H^2(M_n, \Omega^1)$. Hence $\dim H^2(M_n, \Omega^1)=2n$ follows from the Riemann-Roch theorem using Proposition 7. Thus (28) is proved completely.

References

[1] A. DOUADY, Déformations Régullières, Seminaire Henri Cartan, 13e année, 1960/61, n° 3.
[2] MA. KATO, Examples of simply connected compact complex 3-folds, Tokyo J. Math., **5** (1982), 341–364.
[3] MA. KATO, On compact complex 3-folds with lines, Japan. J. Math., **11** (1985), 1–58.
[4] C. T. C. WALL, Classification problems in differential topology V, On certain 6-manifolds, Invent. Math., **1** (1966), 355–374.
[5] A. YAMADA, Small deformations of certain compact Class L manifolds, Tôhoku Math. J., **38** (1986), 99–122.

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