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On an Inequality of Varopoulos for 2-Parameter Brownian Martingales

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Introduction

Recently, N. Th. Varopoulos ([10]) has shown that if X and Y are 2-parameter Brownian martingales, then

$$(*)\qquad \left\|\left(\int_0^\infty\int_0^\infty |\nabla_1 X_{st}|^2\,|\nabla_2 Y_{st}|^2 ds dt\right)^{1/2}\right\|_r \leqq C_{p,q}\|\sup_{s,t}|X_{st}|\,\|_p\|\sup_{s,t}|Y_{st}|\,\|_q\ ,$$

for all $1<p<2$, $2<q<\infty$ and $(1/r)=(1/p)+(1/q)$ with $(1/p)+(2/q)<1$.

In this paper we prove that the inequality (*) also holds true whenever $0<p<\infty$, $0<q\leqq\infty$ and $(1/r)=(1/p)+(1/q)$ (cf. Theorem 2.1). Moreover, as an application of this result we give a probabilistic extension of a result of E. M. Stein ([8]) on a variant of the area integral for the bi-disc (cf. Corollary 3.1).

After some preliminaries in section 1, in section 2 we prove our main result, Theorem 2.1. Corollary 3.1 will be proved in section 3.

§1. Preliminaries.

In this section we mention some probabilistic facts that will be used for the proofs of our results.

Let $n(1)$ and $n(2)$ be two natural numbers and let $B^j(t_j)=(x_1^j(t_j), \cdots, x_{n(j)}^j(t_j))$ be an $n(j)$ dimensional Brownian motion on a complete probability space (Ω_j, F^j, p^j) such that $p_j(B^j(0)=(0, \cdots, 0))=1$, $j=1, 2$. For every $t_j\geqq 0$, let us denote by $F^j(t_j)$ the σ-field generated by $\{B^j(s): 0\leqq s\leqq t_j\}$ and all p^j-null sets. Suppose $F^j=\vee_{t\geqq 0}F^j(t)$, $j=1, 2$. Let (Ω, F, P) be the completion of the product measure space $(\Omega_1\times\Omega_2, F^1\times F^2, p^1\times p^2)$, and put $F(s, t)=(F^1(s)\times F^2(t))\vee\{P\text{-null sets}\}$, $s, t\geqq 0$. Let $E[\cdot]$ denote the expectation with respect to P.

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Given $0<p\leqq\infty$, we will denote by H^p the space of all 2-parameter stochastic processes $X=(X_{st})$ satisfying

(1) $(X_{s0})_s$ and $(X_{0t})_t$ are 1-parameter local martingales adapted to $(F^1(s))_s$ and $(F^2(t))_t$ respectively,

(2) $\Delta X_{st}\equiv X_{st}-X_{s0}-X_{0t}+X_{00}$ has a unique stochastic integral representation

$$\Delta X_{st}=\sum_{j=1}^{n(1)}\sum_{k=1}^{n(2)}\int_0^t\int_0^s \Phi^{j,k}dx_j^1dx_k^2\ , \quad s,\, t\geqq 0\ , \tag{2.1}$$

where $\Phi^{j,k}$ is a 2-parameter predictable process with

$$E\Big[\Big(\int_0^\infty\int_0^\infty |\Phi^{j,k}|^2dsdt\Big)^{r/2}\Big]<\infty \quad \text{for some} \quad 0<r<\infty \quad \text{(cf. [3]), and}$$

(3) $X^*\equiv\sup_{s,t}|X_{st}|\in L^p(\Omega,\, F,\, P)$.

As we see in [3], for every $2\leqq p<\infty$, we can regard H^p as $L^p(\Omega,\, F,\, P)$ by identifying $f\in L^p(\Omega,\, F,\, P)$ with $(E[f\,|\,F(s,\,t)])_{st}\in H^p$.

For a process $X=(X_{st})\in\bigcup_{p>0}H^p$, we define stochastic gradients of X as follows: $\nabla_iX_{st}=((\partial X/\partial x_j^i)_{st})$, $i=1,\,2$, and $\nabla_{12}X_{st}=((\partial^2X/\partial x_j^1\partial x_k^2)_{st})$, where $\partial X/\partial x_j^i$ and $\partial^2X/\partial x_j^1\partial x_k^2$ are stochastic derivatives in the sense of [3] or [10]. Further, we set $|\nabla_iX_{st}|^2=\sum_{j=1}^{n(i)}|(\partial X/\partial x_j^i)_{st}|^2$, $i=1,\,2$, and $|\nabla_{12}X_{st}|^2=\sum_{j=1}^{n(1)}\sum_{k=1}^{n(2)}|(\partial^2X/\partial x_j^1\partial x_k^2)_{st}|^2$.

Throughout this paper we write $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(dP)}$, $0<p\leqq\infty$. The following was proved in [2]:

PROPOSITION 1.1. *Suppose* $0<p<\infty$. *Let* $Y\in H^\infty$. *If* $X\in H^p$, *then*

$$E\Big[\int_0^\infty\int_0^\infty |X_{st}|^p|\nabla_{12}Y_{st}|^2dsdt\Big]\leqq C\|Y_{\infty\infty}\|_\infty^2\|X^*\|_p^p\ ,$$

$$E\Big[\int_0^\infty |X_{s0}|^p|\nabla_1Y_{s0}|^2ds\Big]\leqq C\|Y_{\infty0}\|_\infty^2\|\sup_s|X_{s0}|\|_p^p$$

and

$$E\Big[\int_0^\infty |X_{0t}|^p|\nabla_2Y_{0t}|^2dt\Big]\leqq C\|Y_{0\infty}\|_\infty^2\|\sup_t|X_{0t}|\|_p^p\ ,$$

where C *is a constant independent of* X, Y *and* p.

REMARK. This result, in fact, holds for all $Y\in BMO$ (See [2, Lemma 3.1].), however we need only the above case for the proof of our main result. The space BMO was defined by H. Sato ([6]), and he proved that BMO is the dual of H^1.

We introduce here operators of holomorphic projection type. In this paragraph we assume that $n(1)$ and $n(2)$ are even numbers. Let H_j $(j=1, 2)$ be the Hilbert martingale transform with respect to j-th parameter in the sense of Varopoulos [9], that is, $H_j(dx^j_m)=dx^j_{(n(j)/2)+m}$, $H_j(dx^j_{(n(j)/2)+m})=-dx^j_m$ and $H_j(dx^k_l)=0$ $(m=1,\cdots,n(j)/2;\ l=1,\cdots,n(j);\ k=1,2$ with $k\neq j;\ j=1,2)$. Further let $T_j=(I+(-1)^{1/2}H_j)/2$ and $S_j=(I-(-1)^{1/2}H_j)/2$, $j=1, 2$, where I is the identity transform.

Now we define holomorphic projection type operators K^1, K^2, K^3 and K^4 as follows:

$$K^1=I\oplus T_1\oplus T_2\oplus T_1T_2\,, \quad K^2=0\oplus 0\oplus 0\oplus T_1S_2\,,$$
$$K^3=0\oplus 0\oplus 0\oplus S_1T_2 \quad \text{and} \quad K^4=0\oplus S_1\oplus S_2\oplus S_1S_2\,,$$

where 0 is the zero mapping, and where $A=A_1\oplus A_2\oplus A_3\oplus A_4$ means that $A(X)=A_1(X_{00})+A_2((X_{s0}-X_{00}))+A_3((X_{0t}-X_{00}))+A_4((\Delta X_{st}))$ for an $X\in\bigcup_{p>0}H^p$.

The following proposition was obtained in [1].

Proposition 1.2. *Given $0<p<\infty$, there exist constants c_p and C_p depending only on p such that*

$$c_p\,\|K^j(x)^*\|_p\leqq\sum_{j=1}^{4}\sup_{s,t}\|K^j(X)_{st}\|_p\leqq C_p\,\|X^*\|_p\,,$$

for every $X\in H^p$.

§2. Main theorem.

Our main theorem is the following

Theorem 2.1. *Suppose $0<p<\infty$, $0<q\leqq\infty$ and $(1/r)=(1/p)+(1/q)$. If $X\in H^p$ and $Y\in H^q$, then*

$$V(X, Y)=\left(\int_0^\infty\int_0^\infty|\nabla_1X_{st}|^2|\nabla_2Y_{st}|^2dsdt\right)^{1/2}\in L^r(\Omega, F, P)$$

and

$$(*)\qquad \|V(X, Y)\|_r\leqq C_{p,q}\|X^*\|_p\,\|Y^*\|_q\,,$$

where $C_{p,q}$ is a constant depending only on p and q.

Proof. By the same argument as in Stein [8], it is only necessary to verify this lemma in the following cases (I), (II), (III) and (IV): (I) $0<p<2$ and $0<q<2$; (II) $2<q<\infty$, $0<p<\infty$ and $r<2$; (III) $0<p<\infty$, $0<q<\infty$ and $r>4$; (IV) $0<p<2$ and $q=\infty$.

Proofs of cases (II), (III) and (IV) which we will describe later were inspired from [8] (and [10]). However, as will be shown below, a probabilistic method gives a drastic simplification of the proof of Case (I) due to Stein.

Now we begin by proving the lemma in the case of (I). By Hölder's inequality we have that

$$\|V(X,\ Y)\|_r \leqq \left\|\left(\int_0^\infty \sup_v |\nabla_1 X_{sv}|^2 ds\right)^{1/2} \left(\int_0^\infty \sup_u |\nabla_2 Y_{ut}|^2 dt\right)^{1/2}\right\|_r$$

$$\leqq \left\|\left(\int_0^\infty \sup_v |\nabla_1 X_{sv}|^2 ds\right)^{1/2}\right\|_p \left\|\left(\int_0^\infty \sup_u |\nabla_2 Y_{ut}|^2 dt\right)^{1/2}\right\|_q .$$

Here we recall an estimate of H^p-martingales:

LEMMA 2.2 (Brossard-Chevalier [3, Corollary 2 and Theorem 3]). *For every $0<\alpha\leqq 2$, there exists a constant C_α depending only on α such that*

$$\left\|\left(\int_0^\infty \sup_v |\nabla_1 Z_{sv}|^2 ds\right)^{1/2}\right\|_\alpha \leqq C_\alpha \|Z^*\|_\alpha$$

and

$$\left\|\left(\int_0^\infty \sup_u |\nabla_2 Z_{ut}|^2 dt\right)^{1/2}\right\|_\alpha \leqq C_\alpha \|Z^*\|_\alpha ,$$

for every $Z\in H^2$.

In order to apply Lemma 2.2 to our proof we need the following

LEMMA 2.3. *Let $T(j, L)=\inf\{t_j: |B^j(t_j)|>L\}$ $(j=1, 2)$ and $U(L)=\{(t_1, t_2): 0<t_j<T(j, L),\ j=1, 2\}$. Let $S\equiv\{X_{00}+X_{T(1,L),0}+X_{0,T(2,L)}+\Delta X^{U(L)}: X\in H^2;\ (\nabla_1 X_{s0}),\ (\nabla_2 X_{0t})$ and $(\nabla_{12} X_{st})$ are simple processes* (cf. [3])*\}. If there exists a constant $C_{p,q}$ depending only on p and q such that*

$$(**)\qquad \|V(X,\ Y)\|_r \leqq C_{p,q} \|X^*\|_p \|Y^*\|_q ,$$

for every $X, Y\in S$, then the inequality (**) holds for all $X\in H^p$ and all $Y\in H^q$.

The proof of this lemma will be given in Appendix. From Lemma 2.2 and Lemma 2.3 it follows the Case (I).

Case (II). In order to prove the Theorem in this case it suffices to verify it when $n(1)$ and $n(2)$ are even numbers, because by using technique in [9, p. 104], we can show it for every two natural numbers $n(1)$ and $n(2)$. Hence we suppose that $n(1)$ and $n(2)$ are even numbers.

Let $i\in\{1, 2, 3, 4\}$. Let $X^i=K^i(X)$ and $|X^i|_\varepsilon^\sigma=(|X^i|^2+\varepsilon)^{\sigma/2}$, $0<\sigma<2$. By Ito's formula we have

$$(\partial/\partial s)(|X^i_{st}|^\sigma_\varepsilon)=(1/4)(\sigma^2+\varepsilon\sigma(2-\sigma))(|X^i_{st}|^2+\varepsilon)^{(\sigma/2)-2}\times|\nabla_1 X^i_{st}|^2 \geqq(\sigma^2/4)(|X^i_{st}|^2+\varepsilon)^{(\sigma/2)-2}|\nabla_1 X^i_{st}|^2 ,$$

for every $0<\sigma<2$. Consequently,

$$(\partial/\partial s)(|X^i|^\sigma_\varepsilon|\nabla_2 A|^2)+C|X^i|^\sigma_\varepsilon|\nabla_1\nabla_2 A|^2 \geqq C_1|X^i|^{\sigma-2}_\varepsilon|\nabla_1 X^i|^2|\nabla_2 A|^2 , \tag{2.1}$$

for every $A\in\bigcup_{d>0}H^d$ and every $0<\sigma<2$, where C and C_1 are constants depending only on σ. Hence from (2.1) and the argument in [10, p. 200–p. 201] it follows that

$$\|V(X^i,\ Y)\|_r\leqq C'_{p,q}\|X^{i*}\|_p\|Y^*\|_q .$$

Thus we get that

$$\|V(X,\ Y)\|_r\leqq\sum_{i=1}^4\|V(X^i,\ Y)\|_r\leqq C'_{p,q}\sum_{i=1}^4\|X^{i*}\|_p\|Y^*\|_q \leqq C_{p,q}\|X^*\|_p\|Y^*\|_q \quad \text{(Proposition 1.2)} .$$

Case (III). For the proof of Theorem 2.1 in this case we use a duality argument with Case (I). Let m be the exponent dual of $r/2$. Let $H^\infty\ni\phi\geqq 0$ with $\|\phi_{\infty\infty}\|_m\leqq 1$. Suppose that $\|X^*\|_p\leqq 1$, $\|Y^*\|_q\leqq 1$ and $\|V(X,\ Y)\|_r\ (\equiv A)<\infty$. Let $S(Y)^2=\int_0^\infty\int_0^\infty|\nabla_1\nabla_2 Y|^2 dsdt$. For the simplicity, let $L[\Pi]=E\left[\int_0^\infty\int_0^\infty \Pi_{st}dsdt\right]$ for a process Π. We put

$$B=L[(\partial/\partial s)(\phi|\nabla_1 X|^2|\nabla_2 Y|^2)]-L[\phi|\nabla_1 X|^2|\nabla_2 Y|^2] .$$

Since the first term in the right-hand side of the above equality is estimated by $C_1\|\phi\|_m\|X^*\|_p\|Y^*\|_q\ (\leqq C_1)$, we have

$$E[\phi_{\infty\infty}V(X,\ Y)^2]\leqq L[\phi|\nabla_1 X|^2|\nabla_2 Y|^2]\leqq C_1+|B| ,$$

where C_1 is a constant depending only on p and q. From Ito's formula it follows that

$$\begin{aligned}|B|&\leqq L[\phi|X|^2(\partial|\nabla_2 Y|^2/\partial s)]+\sum_{k=1}^{n(1)}\{L[(\partial\phi/\partial x^1_k)(\partial|X|^2/\partial x^1_k)|\nabla_2 Y|^2]\\&\quad+L[(\partial\phi/\partial x^1_k)(\partial|\nabla_2 Y|^2/\partial x^1_k)|X|^2]+L[\phi(\partial|X|^2/\partial x^1_k)(\partial|\nabla_2 Y|^2/\partial x^1_k)]\}\\&\leqq E[\phi^*X^{*2}S(Y)^2]+2E[X^*V(\phi,\ Y)V(X,\ Y)]+E[X^{*2}V(\phi,\ Y)S(Y)]\\&\quad+2E[\phi^*X^*V(X,\ Y)S(Y)]\end{aligned}$$

Since $r>4$, we may assume that $q>4$ without loss of generality.

Let $(1/n)=(1/m)+(1/q)$ $(>1/2)$. By the above inequality, Hölder's inequality and [3, Theorem 3] we get that

$$E[\phi_{\infty\infty}V(X, Y)^2]\leqq C_1+C_2+C_3A\|V(\phi, Y)\|_n+C_4\|V(\phi, Y)\|_n+C_5A ,$$

where $C_2, \cdots, C_5$ are constants depending only on p and q. Hence if we apply Case (I) to the above $V(\phi, Y)$, we obtain $A^2\leqq cA+c'$, where c and c' are constants depending only on p and q. This guarantees that $A\leqq C_{p,q}$, namely, $\|V(X, Y)\|_r\leqq C_{p,q}\|X^*\|_p\|Y^*\|_q$ if $\|V(X, Y)\|_r<\infty$. Further we can easily remove this assumption by Lemma 2.3, because an easy calculation implies that $\|V(X, Y)\|_r<\infty$, for every $X, Y\in S(\subset\bigcap_{p<\infty}H^p)$.

Case (IV). As we see in Case (II), it suffices to show that the inequality (*) is true when $n(1)$ and $n(2)$ are even and $X=K^d(X)$ for some $d\in\{1, 2, 3, 4\}$. Since $\|\sup_t|Y_{\infty t}|\|_\infty\leqq\|Y^*\|_\infty<\infty$, we have by Proposition 1.1 that

$$\begin{aligned}&\lim_{\varepsilon\to0+}L[(\partial/\partial s)(|X|^2+\varepsilon)^{p/2}|\nabla_2Y|^2]\\&\leqq\lim_{\varepsilon\to0+}E\left[\int_0^\infty(|X_{\infty t}|^2+\varepsilon)^{p/2}|\nabla_2Y_{\infty t}|^2dt\right]\\&\leqq C_pE[\sup_t|X_{\infty t}|^p]\|Y^*\|_\infty^2\leqq C_p\|X^*\|_p^p\|Y^*\|_\infty^2 .\end{aligned}$$

Proposition 1.1 and (2.1) tell us

$$L[|X|^{p-2}|\nabla_1X|^2|\nabla_2Y|^2]\leqq C_p\|X^*\|_p^p\|Y^*\|_\infty^2 .$$

Thus by using the argument in [10, p. 201] we obtain

$$\|V(X, Y)\|_r\leqq C_{p,\infty}\|X^*\|_p\|Y^*\|_\infty .$$

§3. An application to Stein's area integrals.

Let $D=D_1\times D_2$ be a direct product of two bounded planer domains with C^2 boundaries, and let $\sigma=\sigma_1\times\sigma_2$ be the product induced Euclidean measure on $\partial D_1\times\partial D_2$ (cf. [7]). As usual, we denote by $H^p(D)$ the space of bi-harmonic functions u on D, whose nontangential maximal function $N(u)(x)=\sup\{|u(z)|: z\in\Gamma(x)\}$ is in the Lebesgue space $L^p(\partial D_1\times\partial D_2, d\sigma)$, $0<p\leqq\infty$. Here $\Gamma(x)$ is the product domain $\Gamma^1(x_1)\times\Gamma^2(x_2)$, if $x=(x_1, x_2)$, with $\Gamma^j(x_j)=\{y_j\in D_j: |y_j-x_j|<2\,\mathrm{dist}(y_j, \partial D_j)\}$, $j=1, 2$. We give in $H^p(D)$ the norm as follows: $\|u\|_{H^p}=\|N(u)\|_{L^p(d\sigma)}$.

In [8], E. M. Stein defined the area integral of two bi-harmonic functions u and v on D as follows:

$$B(u, v)(x)=\left(\iint_{\Gamma(x)}|\nabla_1u(z_1, z_2)|^2|\nabla_2v(z_1, z_2)|^2dz_1d\bar z_1dz_2d\bar z_2\right)^{1/2} ,$$

$x \in \partial D_1 \times \partial D_2$. As the main result in [8], Stein showed the following theorem when D is the bi-disc.

THEOREM S. *Suppose* $0<p<\infty$, *and* $0<q\leqq\infty$, *with* $(1/r)=(1/p)+(1/q)$. If $u \in H^p(D)$, $v \in H^q(D)$, then $B(u, v) \in L^r(d\sigma)$ *and*

$$\|B(u, v)\|_r \leqq C_{p,q}\|u\|_{H^p}\|v\|_{H^q},$$

where $C_{p,q}$ is a constant depending only on D, p and q.

By Theorem 2.1 we have a probabilistic extension of Theorem S. Before stating our result, we prepare some notations. Let $n(1)=n(2)=2$. Suppose that $B^1(t_1)$, $B^2(t_2)$, (Ω, F, P), $\cdots$ are established by a similar way as in section 1. For every $\xi=(\xi_1, \xi_2) \in D$, we put $X^j(\xi_j, t_j)=B^j(t_j)+\xi_j$ and $\tau(\xi_j)=\inf\{t_j\colon X^j(\xi_j, t_j) \notin D_j\}$, $j=1, 2$. If u is a function defined on D, then we denote by $M^\xi u$ the Brownian maximal function

$$\sup\{|u(X^1(\xi_1, t_1), X^2(\xi_2, t_2))|\colon 0\leqq t_j<\tau(\xi_j),\ j=1, 2\}.$$

Further, we define a probabilistic analogue of $B(u, v)$ as follows:

$$\tilde{B}^\xi(u, v)=\left(\int_0^{\tau(\xi_1)}\int_0^{\tau(\xi_2)}|\nabla_1 u|^2|\nabla_2 v|^2(X^1(\xi_1, s), X^2(\xi_2, t))dtds\right)^{1/2}.$$

As a corollary of Theorem 2.1 we have the following

COROLLARY 3.1. *Let $D=D_1\times D_2$ be a direct product of two bounded simply connected C^2 domains in the plane. Suppose that $0<p<\infty$, $0<q\leqq\infty$ and $(1/r)=(1/p)+(1/q)$. If $u \in H^p(D)$ and $v \in H^q(D)$, then*

$$\begin{aligned}\|B(u, v)\|_r &\leqq C'_{p,q}\|M^\xi u\|_{L^p(dP)}\|M^\xi v\|_{L^q(dP)}\\ &\leqq C_{p,q}\|u\|_{H^p}\|v\|_{H^q},\end{aligned}$$

for every $\xi \in D$, where $C'_{p,q}$ and $C_{p,q}$ are constants depending only on ξ, p, q and D.

In order to prove Corollary 3.1 we need the following

LEMMA 3.1. *Let $\xi \in D$. If u and v are bi-harmonic functions on D, then*

(1) $P(M^\xi u>\lambda)\leqq C'\sigma(N(u)>\lambda)$, $\lambda>0$ *and*

(2) $\sigma(B(u, v)>\lambda)\leqq C^{-1}\sum\{P(\tilde{B}^\xi(K_\varepsilon u, K_\delta v)>C\lambda)\colon \varepsilon, \delta=1, 2, 3, 4\}$, $\lambda>0$, *where C' and C are constants depending only on ξ and D, and K_ε is a projection on $H^p(D)$ defined in a similar way of the definition of the operator K^ε.*

The inequality (1) is proved by a same way as the proof of [5, Lemma 4′], and (2) follows from modification of proofs of [4, Theorem 1] and [5, Theorem 14]. We omit a detail.

Now we prove Corollary 3.1.

PROOF OF COROLLARY 3.1. Since the second inequality in Corollary 3.1 are immediate consequence of Lemma 3.1, we need to prove the first inequality. For every small $\varepsilon>0$, we put $T(j, \varepsilon)=\inf\{t_j: \mathrm{dist}(X^j(\xi_j, t_j), \partial D_j)\leqq\varepsilon\}$, $j=1, 2$, $X_{st}^{(\varepsilon)}=u(X^1(\xi_1, T(1, \varepsilon)\wedge s), X^2(\xi_2, T(2, \varepsilon)\wedge t))$ and $Y_{st}^{(\varepsilon)}=v(X^1(\xi_1, T(1, \varepsilon)\wedge s, X^2(\xi_2, T(2, \varepsilon)\wedge t))$, $s, t\geqq 0$. Then $X^{(\varepsilon)}, Y^{(\varepsilon)}\in H^\infty$. Hence Theorem 2.1, Lemma 3.1 and the monotone convergence theorem yield the desired inequality.

APPENDIX.

In the proof of Theorem 2.1, we used Lemma 2.3. Here we prove Lemma 2.3. Throughout the appendix we put $\|X\|_{H^p}=\|X^*\|_p$ for every $X\in H^p$. We first prove the following

LEMMA. *Let $X^{(n)}\in H^p$ $(n=1, 2, \cdots)$ converge to an $X\in H^p$ in the topology of H^p, $0<p<\infty$. Then for every $M>1$,*
$(^*M)$ there exists a subsequence $\{n(j)\}$ of $\{n\}$ such that

$$\lim_{j\to\infty}|\nabla_1 X_{st}^{(n(j))}(\omega)-\nabla_1 X_{st}(\omega)|=0 \quad \text{a.e.} \ ds|_{[0,M[}\otimes dt\otimes dP\,.$$

PROOF. Let $p'=\min(p, 2)$. By [3, Corollary 2] we have

$$E\left[\left(\int_0^\infty \sup_t |\nabla_1 X_{st}^{(n)}-\nabla_1 X_{st}^{(m)}|^2 ds\right)^{p'/2}\right]\leqq C_p\|X^{(n)}-X^{(m)}\|_{H^p}\longrightarrow 0\,,$$

as $n, m\to\infty$.

This yields that for every $\varepsilon>0$,

$$\lim_{n\to\infty}\nu_M\otimes P((s, \omega): \sup_t |\nabla_1 X_{st}^{(n)}(\omega)-\nabla_2 X_{st}(\omega)|>\varepsilon)=0\,,$$

where $d\nu_M=ds|_{[0,M[}$. Hence we obtain the lemma.

By this lemma we have the following

ASSERTION 1. Let $X\in H^p$ and $Y\in H^q$. If $X^{(n)}\in S$, $\lim_{n\to\infty}\|X^{(n)}-X\|_{H^p}=0$, $\lim_{n,m\to\infty}V(X^{(n)}-X^{(m)}, Y)=0$ a.e., and $\|V(X^{(n)}-X^{(m)}, Y)\|_r\leqq C_{p,q}\|X^{(n)}-X^{(m)}\|_{H^p}\|Y\|_{H^q}$ then $\|V(X, Y)\|_r\leqq C_{p,q}\|X\|_{H^p}\|Y\|_{H^q}$.

PROOF. Put $\Omega_0\equiv\{\omega\in\Omega: \lim_{n,m\to\infty}V(X^{(n)}-X^{(m)}, Y)=0\}$, and $\Omega_M\equiv\{\omega\in\Omega: [0, M[\times[0, \infty[=\{(s, t): (s, t, \omega) \text{ satisfies } (^*M)\} \text{ a.e.}\}$, $M=1, 2, 3, \cdots$. By Lemma and Fubini's theorem we have $P(\Omega_0)=P(\Omega_M)=1$. Let $\Omega'\equiv\left\{\omega\in\Omega: \int_0^\infty\int_0^M |\nabla_2 Y_{st}(\omega)|^2 dsdt<\infty, \text{ for every } M=1, 2, \cdots\right\}$. Corollary 2 in [3] implies

$P(\Omega')=1$. Let $\Omega''=\Omega'\cap\Omega_0\cap\Omega_1\cap\Omega_2\cap\cdots$. For every $\omega\in\Omega''$, we set

$$d\mu^{\omega}_{M,N}=|\nabla_2 Y_{st}(\omega)|^2 dsdt_{[0,M[\times[0,N[}\quad (M,\ N=1,\ 2,\ \cdots,\ +\infty)\ .$$

Then there exists an $F^{\omega}\in \underbrace{L^2(d\mu^{\omega}_{\infty\infty})\times\cdots\times L^2(d\mu^{\omega}_{\infty\infty})}_{n(1)\ \text{fold}}$ such that

$$\lim_{n\to\infty}\iint|\nabla_1 X^{(n)}_{st}(\omega)-F^{\omega}(s,\ t)|^2 d\mu^{\omega}_{\infty\infty}(s,\ t)=0\ .$$

Hence $\nabla_1 X^{(n)}(\omega)$ converges to F^{ω} in measure $\mu^{\omega}_{M,N}$ $(M,\ N=1,\ 2,\ \cdots)$. Consequently, by $(*M)$ we get that $F^{\omega}(s,\ t)=\nabla_1 X_{st}(\omega)$ a.s. $d\mu_{M,N}$, $M,\ N=1,\ 2,\ \cdots$. Thus $F^{\omega}=\nabla_1 X_{st}$ a.e. $d\mu^{\omega}_{\infty\infty}$. From this it follows that $\lim_{n\to\infty} V(X^{(n)},\ Y)(\omega)=V(X,\ Y)(\omega)$ $(\omega\in\Omega'')$. Since $P(\Omega'')=1$ and $\lim_{m,n\to\infty}\|V(X^{(n)},Y)-V(X^{(m)},Y)\|_r=0$, we have $\|V(X,Y)\|_r\leqq C_{p,q}\|X^*\|_p\|Y^*\|_q<\infty$.

By a similar way as the proof of Assertion 1 we have the following

ASSERTION 2. Let $X\in H^p$ and $Y\in H^q$. If $Y^{(n)}\in S$, $\lim_{n\to\infty}\|Y^{(n)}-Y\|_{H^q}=0$, $\lim_{m,n\to\infty} V(X,\ Y^{(n)}-Y^{(m)})=0$ a.e., and $\|V(X,\ Y^{(n)}-Y^{(m)})\|_r\leqq C_{p,q}\|X\|_{H^p}\|Y^{(n)}-Y^{(m)}\|_{H^q}$, then $\|V(X,\ Y)\|_{H^p}\leqq C_{p,q}\|X\|_{H^p}\|Y\|_{H^q}$.

Now we prove Lemma 2.3.

Let $X\in H^p$ and $Y\in H^q$. There exist $X^{(n)}\in S$, $Y^{(n)}\in S$ such that $\lim_{n\to\infty}\|X^{(n)}-X\|_{H^p}=0$ and $\lim_{m\to\infty}\|Y^{(m)}-Y\|_{H^q}=0$.

By the assumption of Lemma 2.3 we have that for every $m\in N$ there exists a subsequence $\{n(j)\}\subset\{n\}$ such that

$$\lim_{n\to\infty}\|X^{(n(j))}-X\|_{H^p}=0\ ,\quad \lim_{j,k\to\infty} V(X^{(n(j))}-X^{(n(k))},\ Y^{(m)})=0 \text{ a.e., and}$$
$$\|V(X^{(n(j))}-X^{(n(k))},\ Y^{(m)}\|\leqq C_{p,q}\|X^{(n(j))}-X^{(n(k))}\|_{H^p}\|Y^{(m)}\|_{H^q}\ .$$

Hence by Assertion 1 we obtain $\|V(X,\ Y^{(m)})\|_r\leqq C_{p,q}\|X\|_{H^p}\|Y^{(m)}\|_{H^q}$.

From this it follows that $\|V(X,\ Y^{(m)}-Y^{(n)})\|_r\to 0$ (as $m,\ n\to\infty$). Thus Assertion 2 tell us that $\|V(X,\ Y)\|_r\leqq C_{p,q}\|X\|_{H^p}\|Y\|_{H^q}$.

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References

[1] H. ARAI, Hardy spaces of 2-parameter Brownian martingales, Tokyo J. Math., **8** (1985), 355–375.

[2] H. ARAI, Carleson measures on product domains and 2-parameter Brownian martingales, Arch. Math., **46** (1986), 343–352.

[3] J. BROSSARD and L. CHEVALIER, Calcul stocastiques et inégalités de norme pour les martingales bi-browniennes; application aux fonctions bi-harmoniques, Ann. Inst. Fourier, vol. **30**, no. 4, (1980), 97–120.

[4] B. DAVIS, An inequality for the distribution of Brownian gradient function, Proc. Amer. Math. Soc., **37** (1973), 189–194.

[5] R. F. GUNDY, Inegalites pour martingales a un et deux indeces; l'espace H^p, Lecture Notes in Math., **774**, Springer Verlag Berlin-Heidelberg-New York, 1980, 252–334.

[6] H. SATO, Caractérisation par les transformations de Riesz de la classe Hardy H^1 de fonctions bi-harmoniques sur $R_+^{m+1}\times R_+^{n+1}$, Thése de doctrat de 3e cycle, Grenoble, 1979.

[7] E. M. STEIN, Boundary Behaviour of Holomorphic Functions of Several Complex Variables, Princeton Univ. Press, Princeton, 1972.

[8] E. M. STEIN, A variant of the area integral, Bull. Sc. Math. 2e série, **103** (1979), 449–461.

[9] N. TH. VAROPOULOS, The Helson-Szegö theorem and A_p-functions for Brownian motion and several variables, J. Funct. Anal., **39** (1980), 85–121.

[10] N. TH. VAROPOULOS, Probabilistic approach to some problems in complex analysis, Bull. Sc. Math. 2e série, **105** (1981), 181–224.

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