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On Seminormal Underrings

Dedicated to Professor Masayoshi Nagata on his 60th birthday

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§1. Introduction and notation.

Let R be a (commutative integral) domain with quotient field K. One theme of enduring interest has been the study of R by analyzing properties of its overrings (that is, the rings contained between R and K). It seems remarkable that analogous "dual" studies have not been done in terms of the behavior of the underrings of R. (We shall say that B is an *underring* of R in case B is a subring of R also having quotient field K.) In [2], one took a first step by characterizing the R such that each *underring* of R is a Euclidean domain. These domains R were actually studied earlier by Gilmer [3] as the domains each of whose *subrings* is a Euclidean domain. In [1, Proposition 2.11], it was shown that the same domains R are characterized by requiring that each *subring* of R is seminormal. (As noted in [4, Theorem 1.1], a domain D, with quotient field L, is seminormal if and only if, whenever $u\in L$ satisfies $u^2\in D$ and $u^3\in D$, then $u\in D$.) One is naturally led to ask if the same domains R are characterized by requiring that each *underring* of R is seminormal. In [2], this was answered in the affirmative in the special case $R=K$. Our main result, Theorem 2.2, answers the general question in the affirmative. Its proof is independent of, and somewhat easier than, the work in [2].

R, K retain the above meanings throughout, all subrings contain the 1 of the larger ring, ch denotes characteristic, and $\boldsymbol{F}_p$ denotes the prime field of characteristic $p>0$. Any unexplained material is standard, as in [5].

§2. Results.

In any study of domains via behavior of their underrings, certain

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domains are catalogued by default. These are the domains having no proper underrings. They are characterized in Proposition 2.1, whose proof follows immediately from [2, Corollary] (or Theorem 2.2 below). Recall, by way of contrast, that R has just R and K as *overrings* if and only if R is a valuation domain of (Krull) dimension at most 1.

PROPOSITION 2.1. *The following conditions on R are equivalent:*
(1) *R is the only underring of R;*
(2) *Either $R \cong \boldsymbol{Z}$ or $R=K$ is a field algebraic over some $\boldsymbol{F}_p$.*

We come next to our main result. The equivalence (1)$\Leftrightarrow$(6) was obtained in case $R=K$ in [2, Theorem]; for arbitrary domains R, [2, Corollary] showed that (6) is equivalent to "each underring of R is a Prüfer (resp., Euclidean) domain."

THEOREM 2.2. *The following conditions on R are equivalent:*
(1) *Each underring of R is seminormal;*
(2) *Each subring of R is seminormal;*
(3) *Each underring of R is integrally closed;*
(4) *Each subring of R is integrally closed;*
(5) *Each subring of R is a Euclidean domain;*
(6) *Either R is isomorphic to an overring of $\boldsymbol{Z}$ or $R=K$ is a field algebraic over some $\boldsymbol{F}_p$.*

PROOF. (4)$\Leftrightarrow$(5)$\Leftrightarrow$(6): Apply [3, Theorem 1].
(2)$\Leftrightarrow$(6): Apply [1, Proposition 2.11].
(4)$\Rightarrow$(3): Trivial.
(3)$\Rightarrow$(1): Note that each integrally closed domain is seminormal.
(1)$\Rightarrow$(6): Assume (1). We consider first the case $\mathrm{ch}(R)=p>0$. We claim that $R=K$ is algebraic over $\boldsymbol{F}_p$. Indeed, choose a transcendence basis $\{X_i\}$ of K over $\boldsymbol{F}_p$, such that $\{X_i\}\subset R$. Set $L=\boldsymbol{F}_p(\{X_i\})$ and $A=\boldsymbol{F}_p[\{X_i\}]$; let T denote the integral closure of A in R. Since K is algebraic over L, the usual "clearing denominators" trick gives $R\subset T_{A\setminus\{0\}}$; in particular, K is the quotient field of T. Then $D=\boldsymbol{F}_p+u^2T$ is an underring of R, for each nonunit $u\in T\setminus\{0\}$. By (1), $u\in D$; write $u=e+u^2t$, with $e\in\boldsymbol{F}_p$ and $t\in T$. As $u^{-1}\notin T$, it follows easily that $e\neq 0$. Hence, e is a unit of T. It follows from $e=u(1-ut)$ that u is a unit of T, contrary to hypothesis. So no such u exists; that is, T is a field. By integrality (cf. [5, Theorem 48]), A is also a field. Hence $\{X_i\}$ is empty, and K is algebraic over $\boldsymbol{F}_p$. Moreover, since $T\subset R\subset T_{A\setminus\{0\}}$, $R=K$, and the claim has been proved.

We turn now to the case $\mathrm{ch}(R)=0$. We shall show first that K is algebraic over $\boldsymbol{Q}$. Deny. Choose a transcendence basis $\{X_i\}$ of K over $\boldsymbol{Q}$ such that $\{X_i\}\subset R$; let X be one of the X_i. Set $L=\boldsymbol{Q}(\{X_i\})$, $A=\boldsymbol{Z}[\{X_i\}]$, and B the integral closure of A in R. Clearing denominators reveals that R is contained in the quotient field of B; thus, B is an underring of R. The preceding paragraph now may be reinterpreted for the present context. Using $D=\boldsymbol{Z}+X^2B$, we find $X=e+X^2b_1$, with $0\neq e\in\boldsymbol{Z}$ and $b_1\in B$. Let q be a rational prime not dividing e, and use integrality to produce $P\in\operatorname{Spec}(B)$ such that $P\cap A=(q, \{X_i\})$. It follows from $e=X(1-Xb_1)$ that $e\in P\cap\boldsymbol{Z}=q\boldsymbol{Z}$, the desired contradiction. Hence, K is algebraic over $\boldsymbol{Q}$.

It remains only to show that $K=\boldsymbol{Q}$. Deny. Hence, one may choose $\gamma\in R\backslash\boldsymbol{Q}$. Clearing denominators gives $0\neq m\in\boldsymbol{Z}$ such that $\delta=m\gamma$ is integral over $\boldsymbol{Z}$. Letting E denote the integral closure of $\boldsymbol{Z}$ in R, we see as above that K is the quotient field of E. Thus E is an underring of R, and so is $\boldsymbol{Z}+q^2E$ for each rational prime q. Put $M=\boldsymbol{Q}(\delta)$ and $S=E\cap M$. Note that S is integral over $\boldsymbol{Z}$ and, since $\delta\in S$, M is the quotient field of S. Also, since $\boldsymbol{Z}$ is Noetherian and S is a subring of the ring of algebraic integers of M, S is a finitely generated $\boldsymbol{Z}$-module.

We can now complete the proof of Theorem 2.2. Let q be a rational prime. By (1), $\boldsymbol{Z}+q^2E$ is seminormal, and it follows easily that $qE\subset\boldsymbol{Z}+q^2E$. Thus, if $u\in E$, then $u=v+qw$, with $w\in E$ and $v\in\boldsymbol{Z}q^{-1}\cap E=\boldsymbol{Z}$. It follows that $E=\boldsymbol{Z}+qE$. Now, for each $y\in S$, we have $y=m+qe$, for some $m\in\boldsymbol{Z}$, $e\in E$. Observe that $e=(y-m)q^{-1}\in E\cap M=S$. It follows easily that $S=\boldsymbol{Z}+qS$. As S is finitely generated over $\boldsymbol{Z}$, the standard determinant trick (essentially Nakayama's Lemma) gives $n\in q\boldsymbol{Z}$ such that $(1-n)S\subset\boldsymbol{Z}$. Then $S\subset\boldsymbol{Q}$, contradicting $\delta\in S\backslash\boldsymbol{Q}$.

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