TOKYO J. MATH.
VOL. 12, NO. 1, 1989

Certain Random Motion of a Ball Colliding with Infinite Particles of Jump Type

Hideki TANEMURA

Keio University

(Communicated by M. Obata)

§0. Introduction.

In this paper we consider a system consisting of a hard ball with radius r and of infinitely many point particles moving in $\boldsymbol{R}^d$ according to the following rules:

(i) Let $x(t)$ be the center, the position, of the hard ball at time t. Then, there are no particles in $B_r(x(0))$ at time 0, where $B_r(x)$ denotes the r-neighborhood of x.

(ii) The ball or a particle at x waits an exponential holding time with mean one which is independent of the motion of the other particles. It jumps to the position y where y is distributed according to $p_x(dy)=p(|x-y|)dy$ independently of the holding time and the motion of the other particles, except that the jump is suppressed, if it causes a collision, that is, if there comes to lie a particle within the region occupied by the hard ball.

To give a precise description of the model we denote the position of infinite particles at time t by $\{y^i(t)\}_{i=1}^{\infty}$. We construct a Markov process describing an infinite particle system $\eta_t=\{z^i(t)\}_{i=1}^{\infty}$, where $z^i(t)=y^i(t)-x(t)$, which describes the entire configuration of particles seen from $x(t)$. We construct $x(t)$ as a functional of η_t. Let ν_0 be a Poisson distribution on $\boldsymbol{R}^d\backslash B_r(0)$ with intensity measure dx. Then, ν_0 is a stationary measure for η_t. The ergodicity of the stationary process is easily obtained. The main result of this paper is Theorem 2.1 which states that $\varepsilon x(t/\varepsilon^2)\to\sigma B(t)$ as $\varepsilon\to 0$, in the sense of distribution in $\boldsymbol{D}[0,\infty)$, where $B(t)$ is a d-dimensional Brownian motion and σ is a positive constant. We employ a method of Kipnis and Varadhan [2].

In §1 we construct a Markov process η_t and then $x(t)$ as a process driven by η_t. In §2 and §3 we prove the central limit theorem.

Received February 26, 1988

§1. Construction of the process η_t.

Let $(X, \mathscr{B}(X), \lambda)$ be a σ-finite measure space. Denote by $\mathscr{M}(X)$ the family of all integer valued (including $+\infty$) measures on X of the form $\sum_{i=1}^{\infty}\delta_{x_i}$, where δ_x denotes the δ-measure at x. $\mathscr{M}(X)$ is equipped with $\mathscr{B}(\mathscr{M}(X))$ the σ-field which is generated by $\{\xi\in\mathscr{M}(X): \xi(A)=n\}$, $n\geqq 0$, $A\in\mathscr{B}(X)$. For any $\xi\in\mathscr{M}(X)$ and $x\in X$ we denote $\xi+\delta_x$ by $\xi\cdot x$. $\xi-\delta_z$ is denoted by $\xi\backslash z$ in case $\xi(z)\geqq 1$.

DEFINITION 1.1. A probability measure μ on $(\mathscr{M}(X), \mathscr{B}(\mathscr{M}(X)))$ is called a Poisson distribution on X with intensity measure λ if for any disjoint system $\{A_1, A_2, \cdots, A_m\}\subset\mathscr{B}(X)$ such that $\lambda(A_i)<\infty$, $i=1, 2, \cdots, m$, $\xi(A_1), \cdots, \xi(A_m)$ are independent random variables on the probability space $(\mathscr{M}(X), \mathscr{B}(\mathscr{M}(X)), \mu)$ and

$$\mu(\xi(A_i)=n)=\frac{\lambda(A_i)^n}{n!}\exp(-\lambda(A_i)) , \qquad i=1, 2, \cdots, m .$$

REMARK 1.1 ([1]). For any $\mathscr{B}(\mathscr{M}(X))\times\mathscr{B}(X)$-measurable bounded function F and for any $A\in\mathscr{B}(X)$ such that $\lambda(A)<\infty$, the following equation holds:

$$\int_{\mathscr{M}(X)}\mu(d\xi)\int_A\lambda(dx)\,F(\xi\cdot x, x)=\int_{\mathscr{M}(X)}\mu(d\xi)\int_A\xi(dx)\,F(\xi, x) .$$

We shall construct a stochastic process $x(t)$ describing the motion of a ball colliding with infinitely many particles. First, following [6], we construct a Markov process ξ_t in equilibrium of non-interacting particles in X_0, where $X_0=\boldsymbol{R}^d\backslash B_r(0)$. Let $\mathscr{B}_0$ be the topological Borel field of X_0. Denote by W the space $\boldsymbol{D}((-\infty, \infty)\to X_0)$ of all X_0-valued right continuous functions with left limits defined on $(-\infty, \infty)$ with Skorohod topology and by $\mathscr{B}(W)$ the σ-field generated by all measurable cylindrical subsets of W.

Given a non-negative measurable function $p(\cdot)$ on $[0, \infty)$ satisfying

$$\int_{\boldsymbol{R}^d}dx\,p(|x|)=1 , \tag{1.1}$$

$$\int_{\boldsymbol{R}^d}dx\,p(|x|)|x|^2<\infty , \tag{1.2}$$

$$\operatorname*{essinf}_{z\in[0,h]} p(z)=\kappa>0 \qquad \text{for some } h>0 , \tag{1.3}$$

we put

$$L\phi(x)=\int_{X_0} dy\, p(|x-y|)\{\phi(y)-\phi(x)\}\ , \qquad \phi\in C_b(X_0)\ ,$$

where $C_b(X_0)$ is the set of all bounded continuous functions on X_0. Clearly L generates a unique Feller semigroup U_t on $C_b(X_0)$. We denote the associated transition function by $u(t, x, A)$, $t\geqq 0$, $x\in X_0$, $A\in\mathscr{B}_0$. Then,

$$U_t\phi(x)=\int_{X_0} u(t, x, dy)\phi(y)\ , \qquad \text{for} \quad \phi\in C_b(X_0)\ .$$

The Lebesgue measure dx is a stationary measure for the Markov process with semigroup U_t. We define the σ-finite measure Q on $(W, \mathscr{B}(W))$ by

$$\begin{aligned} &Q(w(t_1)\in A_1, w(t_2)\in A_2, \cdots, w(t_m)\in A_m) \\ &\quad =\int_{A_1} dx_1\int_{A_2} u(t_2-t_1, x_1, dx_2)\cdots\int_{A_m} u(t_m-t_{m-1}, x_{m-1}, dx_m)\ , \end{aligned}$$

for $-\infty<t_1<t_2<\cdots<t_m<\infty$, $A_1, A_2, \cdots, A_m\in\mathscr{B}_0$, $m\in \boldsymbol{N}$. Since dx is also a reversible measure, for $A_1, A_2\in\mathscr{B}_0$ and for $t_1, t_2\in(-\infty, \infty)$

$$Q(w(t_1)\in A_1, w(t_2)\in A_2)=Q(w(t_2)\in A_1, w(t_1)\in A_2)\ . \tag{1.4}$$

Denote by ν_0 a Poisson distribution on X_0 with intensity measure dx and by P_{ν_0} a Poisson distribution on W with intensity measure Q. Put $\Omega=\mathscr{M}(W)$ and $\mathscr{M}_0=\mathscr{M}(X_0)$. We define an $\mathscr{M}_0$-valued process ξ_t on $(\Omega, \mathscr{B}(\Omega), P_{\nu_0})$ by

$$\xi_t(\omega)=\sum \delta_{w^i(t)} \qquad \text{for} \quad \omega=\sum \delta_{w^i(\cdot)}\ .$$

Let a genetic element ξ of $\mathscr{M}_0$ be expressed as $\xi=\sum_{i=1}^{\infty}\delta_{x_i}$. Let Y_i, $i\geqq 1$, be independent random variables with distributions $u(t, x_i, \cdot)$, $i\geqq 1$, and put

$$p(t, \xi, \Gamma)=P\left\{\sum_{i=1}^{\infty}\delta_{Y_i}\in\Gamma\right\}\ , \qquad \Gamma\in\mathscr{B}(\mathscr{M}_0)\ .$$

Then, from Proposition 1.5 and Proposition 2.1 in [5] we have the following

PROPOSITION 1.1. *ξ_t is an ergodic stationary Markov process with transition function $p(t, \xi, \Gamma)$ such that $P_{\nu_0}(\xi_t\in\cdot)=\nu_0(\cdot)$.*

Since ν_0 is an invariant measure for $p(t, \xi, \Gamma)$, we can define a strongly continuous contraction semigroup on $L^2(\mathscr{M}_0, \nu_0)$ by

$$S_t f(\xi)=\int_{\mathscr{M}_0} p(t,\xi,d\eta)f(\eta)\,, \qquad f\in L^2(\mathscr{M}_0,\nu_0)\,, \quad t\geqq 0\,.$$

From Proposition 1.7 in [5] we have for $\phi\geqq 0$

$$\log S_t \exp(-\langle\phi,\cdot\rangle)(\xi)=\langle \log U_t e^{-\phi},\xi\rangle\,, \tag{1.5}$$

where $\langle\phi,\xi\rangle$ is the integral of the function ϕ with respect to the measure ξ. (1.4) implies the following lemma.

LEMMA 1.1. *ξ_t is a reversible Markov process, i.e.*

$$(S_t f, g)_{\nu_0}=(f,S_t g)_{\nu_0} \qquad \textit{for any} \quad f,g\in L^2(\mathscr{M}_0,\nu_0)\,,$$

where $(\cdot,\cdot)_{\nu_0}$ is an L^2 inner product with respect to ν_0.

We denote the generator of S_t by $\mathscr{L}_1$. We define a subspace $\mathfrak{A}$ of $L^2(\mathscr{M}_0,\nu_0)$ by

$$\begin{aligned}\mathfrak{A}=\{\Psi(\langle\phi_1,\xi\rangle,\langle\phi_2,\xi\rangle,\cdots,\langle\phi_m,\xi\rangle)\colon \Psi \text{ is a polynomial,}\\ \phi_i\in C_b(X_0)\cap L^1(X_0,dx),\ 1\leqq i\leqq m,\ m\in \boldsymbol{N}\}\,.\end{aligned}$$

Then, $\mathfrak{A}$ is dense in $L^2(\mathscr{M}_0,\nu_0)$. From (1.5) we have

$$\mathscr{L}_1 f(\xi)=\int_{X_0}\xi(dx)\int_{X_0} dy\, p(|x-y|)\{f(\xi^{x,y})-f(\xi)\}\,, \tag{1.6}$$

for $f\in\mathfrak{A}$, where

$$\xi^{x,y}=\begin{cases}(\xi\backslash x)\cdot y\,, & \text{if} \quad y\in X_0,\ \xi(x)>0\,,\\ \xi\,, & \text{otherwise}\,.\end{cases}$$

Note that, for any ϕ_1, $\phi_2\in C_b(X_0)\cap L^1(X_0,dx)$ and $t\geqq 0$, $\phi_1\phi_2\in C_b(X_0)\cap L^1(X_0,dx)$ and $U_t\phi_1\in C_b(X_0)\cap L^1(X_0,dx)$. By using (1.4) together with these facts it is not hard to prove that $S_t\mathfrak{A}\subset\mathfrak{A}$ for any $t\geqq 0$, from which it follows that $\mathfrak{A}$ is a core for $\mathscr{L}_1$.

Next modifying ξ_t we construct a Markov process η_t which describes the time evolution of the entire configurations of the infinitely many particles seen from the ball. We introduce some notation. For $\eta=\sum\delta_{x_i}\in\mathscr{M}_0$ and $u\in\boldsymbol{R}^d$ we put

$$\tau_u\eta=\begin{cases}\sum\delta_{x_i+u}\,, & \text{if} \quad \sum\delta_{x_i+u}\in\mathscr{M}_0\,,\\ \eta\,, & \text{otherwise}\,,\end{cases}$$

$$\chi(u\,|\,\eta)=\begin{cases}1\,, & \text{if} \quad \eta(B_r(u))=0\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

We define a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ as follows. Put $\Omega^+ = \mathscr{M}(\boldsymbol{D}([0, \infty) \to X_0))$ with ω^+ indicating a genetic element of it, let P_η be the probability measure on Ω^+ defined by

$$P_\eta(\omega^+(t_1) \in \Gamma_1,\ \omega^+(t_2) \in \Gamma_2,\ \cdots,\ \omega^+(t_m) \in \Gamma_m)$$
$$= \int_{\Gamma_1} p(t_1, \eta, d\xi_1) \int_{\Gamma_2} p(t_2 - t_1, \xi_1, d\xi_2) \cdots \int_{\Gamma_m} p(t_m - t_{m-1}, \xi_{m-1}, d\xi_m)$$

for $0 \leqq t_1 < t_2 < \cdots < t_m < \infty$, $\Gamma_1, \Gamma_2, \cdots, \Gamma_m \in \mathscr{B}(\mathscr{M}_0)$, $m \in \boldsymbol{N}$, and let $(\Omega_i, \mathscr{B}(\Omega_i), P_\eta^i)$, $i = 1, 2, \cdots$, be copies of $(\Omega^+, \mathscr{B}(\Omega^+), P_\eta)$. Let v_i, $i = 1, 2, \cdots$, be i.i.d. $[0, \infty)$-valued random variables on some probability space $(Z, \mathscr{B}(Z), P')$ with exponential distribution of mean 1. Put $t_n = \sum_{i=1}^n v_i$ for $n = 1, 2, \cdots$, $t_0 = 0$ and

$$\hat{\Omega} = Z \times \prod_{i=1}^\infty \Omega_i\ , \quad \mathscr{F} = \mathscr{B}(Z) \otimes \prod_{i=1}^\infty \otimes \mathscr{B}(\Omega_i)\ .$$

We define a probability measure $\hat{P}$ on $(\hat{\Omega}, \mathscr{F})$ as the projective limit of P_n, $n = 1, 2, \cdots$, where

$$P_1(dz d\omega_1) = P'(dz) \int_{\mathscr{M}_0} \nu_0(d\eta) P_\eta^1(d\omega)\ ,$$

$$P_{n+1}(dz d\omega_1 \cdots d\omega_{n+1})$$
$$= P_n(dz d\omega_1 \cdots d\omega_n) \int_{\boldsymbol{R}^d} du\, p(|u|) P^{n+1}_{\tau_{-u}\omega_n(v_n(z))}(d\omega_{n+1})\ .$$

We denote by $\hat{\mathscr{F}}$ the $\hat{P}$-completion of $\mathscr{F}$ and define an $\mathscr{M}_0$-valued Markov process η_t on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ by

$$\begin{aligned} \eta_t(\hat{\omega}) &= \eta_t(z, \omega_1, \omega_2, \cdots) \\ &= \omega_{n+1}(t - t_n(z))\ , \qquad \text{for} \quad t_n(z) \leqq t < t_{n+1}(z)\ . \end{aligned} \tag{1.7}$$

The transition function $q(t, \xi, \Gamma)$, $t \geqq 0$, $\xi \in \mathscr{M}_0$, $\Gamma \in \mathscr{B}(\mathscr{M}_0)$, for the process η_t is given by

$$\begin{aligned} q(t, \xi, \Gamma) &= e^{-t} p(t, \xi, \Gamma) \\ &\quad + e^{-t} \sum_{n=1}^\infty \int_{[0,t]} ds_1 \cdots \int_{[0,t]} ds_n\, \mathbf{1}(s_1 + s_2 + \cdots + s_n \leqq t) \\ &\quad \cdot \int_{\boldsymbol{R}^d} du_1 \cdots \int_{\boldsymbol{R}^d} du_n\, p(|u_1|) \cdots p(|u_n|) \\ &\quad \cdot \int_{\mathscr{M}_0} p(s_1, \xi, d\eta_1) \int_{\mathscr{M}_0} p(s_2, \tau_{-u_1}\eta_1, d\eta_2) \cdots \int_{\mathscr{M}_0} p(t - s_n, \tau_{-u_n}\eta_n, \Gamma)\ . \end{aligned} \tag{1.8}$$

We define a strongly continuous semigroup on $L^2(\mathscr{M}_0, \nu_0)$ by

$$T_t f(\xi)=\int_{\mathscr{M}_0} q(t,\xi,d\eta)f(\eta)\,, \qquad f\in L^2(\mathscr{M}_0,\nu_0)\,, \quad t\geqq 0\,,$$

and denote the generator of T_t by $\mathscr{L}$. Then, by (1.8) we have

$$\mathscr{L}=\mathscr{L}_1+\mathscr{L}_2 \quad \text{on} \quad \mathscr{D}(\mathscr{L}_1)\,, \tag{1.9}$$

$$T_t f=S_t f+\int_{[0,t]} S_{t-s}\mathscr{L}_2 T_s f\,ds\,, \tag{1.10}$$

where $\mathscr{L}_2$ is a bounded linear operator on $L^2(\mathscr{M}_0,\nu_0)$ defined by

$$\mathscr{L}_2 f(\eta)=\int_{R^d} du\, p(|u|)\{f(\tau_{-u}\eta)-f(\eta)\}\,. \tag{1.11}$$

Let ν denote a Poisson distribution on R^d with intensity measure dx. Then, for $f,\ g\in L^2(\mathscr{M}_0,\nu_0)$, $\tilde{f},\ \tilde{g}\in L^2(\mathscr{M}(R^d),\nu)$ with $\tilde{f}=f,\ \tilde{g}=g$ on $\mathscr{M}_0$ and for $u\in R^d$, we have

$$\int_{\mathscr{M}_0}\nu_0(d\eta)f(\tau_{-u}\eta)g(\eta)\chi(u\,|\,\eta)=\frac{1}{\nu(\mathscr{M}_0)}\int_{\mathscr{M}(R^d)}\nu(d\eta)\tilde{f}(\tilde{\tau}_{-u}\eta)\tilde{g}(\eta)\chi(0\,|\,\eta)\chi(u\,|\,\eta)\,,$$

where $\tilde{\tau}_u$ is defined by $\tilde{\tau}_u(\sum\delta_{x_i})=\sum\delta_{x_i+u}$. From the shift invariance of ν we have the following:

Lemma 1.2. *$(\mathscr{L}_2 f,g)_{\nu_0}=(f,\mathscr{L}_2 g)_{\nu_0}$ for any $f,\ g\in L^2(\mathscr{M}_0,\nu_0)$.*

From Lemma 1.1 and Lemma 1.2 we have the following lemma by simple calculation.

Lemma 1.3. *For any $f\in\mathfrak{A}_0$,*

(i) $(\mathscr{L}_1 f,f)_{\nu_0}=-\dfrac{1}{2}\displaystyle\int_{\mathscr{M}_0}\nu_0(d\eta)\int_{X_0}\eta(dx)\int_{X_0}dy\,p(|x-y|)\{f(\eta^{x,y})-f(\eta)\}^2$,

(ii) $(\mathscr{L}_2 f,f)_{\nu_0}=-\dfrac{1}{2}\displaystyle\int_{\mathscr{M}_0}\nu_0(d\eta)\int_{R^d}du\,p(|u|)\chi(u\,|\,\eta)\{f(\tau_{-u}\eta)-f(\eta)\}^2$.

In particular, $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}$ are non-positive self-adjoint operators.

Proposition 1.2. *$(\eta_t,\hat{P})$ is an ergodic reversible Markov process.*

Proof. The reversibility of $(\eta_t,\hat{P})$ follows immediately from the self-adjointness of $\mathscr{L}$. So, it is sufficient to prove that $T_t f=f$ $({}^\forall t\geqq 0)$ implies $f\equiv$const. Suppose $T_t f=f$ for any $t\geqq 0$. Then, $f\in\mathscr{D}(\mathscr{L})$ and $\mathscr{L}f=0$. So $(\mathscr{L}_1 f,f)_{\nu_0}+(\mathscr{L}_2 f,f)_{\nu_0}=0$. From non-positivity of $\mathscr{L}_1$ and $\mathscr{L}_2$, $(\mathscr{L}_1 f,f)_{\nu_0}=(\mathscr{L}_2 f,f)_{\nu_0}=0$. Hence, $\mathscr{L}_1 f=0$ and so $S_t f=f$ for any $t\geqq 0$. From Proposition 1.1, this completes the proof of Proposition 1.2. □

Finally we construct the process $x(t)$, describing the motion of the ball colliding with infinitely many particles, as a process driven by η_t. Let $A \in \mathscr{B}(\boldsymbol{R}^d)$ and put

$$\Lambda = \{\eta \in \mathscr{M}_0 \ : \ \eta = \tau_{-u}\eta \text{ for some } u \in \boldsymbol{R}^d \backslash \{0\} \text{ with } \eta(B_r(u)) = 0\} \ ,$$
$$\Delta = \{(\eta, \eta) \ : \ \eta \in \mathscr{M}_0\} \cup (\Lambda \times \Lambda) \ ,$$
$$\Gamma_A = \{(\eta, \zeta) \in (\mathscr{M}_0 \times \mathscr{M}_0) \backslash \Delta \ : \ \zeta = \tau_{-u}\eta \text{ for some } u \in A\} \ .$$

Lemma 1.4. *Λ and Γ_A are measurable subsets of $\mathscr{M}_0$ and $\mathscr{M}_0 \times \mathscr{M}_0$, respectively.*

Proof. It is easy to see the measurability of Λ^c and consequently of Λ. To prove the measurability of Γ_A we consider

$$\tilde{\Gamma}_A = \{(\eta, \zeta) \in \mathscr{M}(\boldsymbol{R}^d) \times \mathscr{M}(\boldsymbol{R}^d) \ : \ \tilde{\tau}_{-u}\eta = \zeta \text{ for some } u \in A\} \ .$$

Note that $\mathscr{M}(\boldsymbol{R}^d)$ is endowed with the vague topology and the topological Borel field coincides with $\mathscr{B}(\mathscr{M}(\boldsymbol{R}^d))$. If A is compact, then $\tilde{\Gamma}_A$ is a closed subset of $\mathscr{M}(\boldsymbol{R}^d) \times \mathscr{M}(\boldsymbol{R}^d)$ and hence measurable. Therefore, $\Gamma_A = \tilde{\Gamma}_A \cap \{(\mathscr{M}_0 \times \mathscr{M}_0) \backslash \Delta\}$ is also measurable if A is compact. We put $\mathscr{A} = \{A \in \mathscr{B}(\boldsymbol{R}^d) \colon \Gamma_A \text{ is measurable}\}$. Then, $\mathscr{A}$ contains all compact subsets A of $\boldsymbol{R}^d$ and, using the fact that $\Gamma_A \cap \Gamma_B = \emptyset$ if $A \cap B = \emptyset$, it is easy to see that $\mathscr{A}$ is a σ-field. Therefore $\mathscr{A}$ coincides with $\mathscr{B}(\boldsymbol{R}^d)$. $\square$

Put

$$\hat{\mathscr{F}}_t = \bigcap_{\varepsilon > 0} \{\text{the } \hat{P}\text{-completion of } \sigma(\eta_s \colon s \in [0, t+\varepsilon])\} \ ,$$
$$N((0, t] \times A) = \sum_{s \in (0,t]} \mathbf{1}_{\Gamma_A}(\eta_{s-}, \eta_s) \ .$$

Then, $N(dtdu)$ is an $\hat{\mathscr{F}}_t$-adapted σ-finite random measure. We define the process $x(t)$ by

$$x(t) = x(0) + \int_{(0,t]} \int_{\boldsymbol{R}^d} u \, N(dsdu) \ . \tag{1.12}$$

Remark 1.2. Let F be an $\boldsymbol{R}^d$-valued bounded $\mathscr{B}(\mathscr{M}_0)$-measurable function. From the reversibility of $(\eta_t, \hat{P})$ we have

$$\hat{E}\{(x(t) - x(0)) \cdot F(\eta_t)\} = -\hat{E}\{(x(t) - x(0)) \cdot F(\eta_0)\} \ .$$

Lemma 1.5. *For $t > 0$ and a bounded set $A \in \mathscr{B}(\boldsymbol{R}^d)$ we set*

$$M((0, t] \times A) = N((0, t] \times A) - \int_{(0,t]} ds \int_A du \, p(|u|) \chi(u \,|\, \eta_s) \ .$$

Then

(i) *$M((0,t]\times A)$ is a square integrable $\hat{\mathscr{F}}_t$-martingale.*

(ii) *$M((0,t]\times A)^2-\int_{(0,t]}ds\int_A du\, p(|u|)\chi(u\,|\,\eta_s)$ is an $\hat{\mathscr{F}}_t$-martingale.*

PROOF. We can prove this lemma following the proof of Lemma 2.4 in [6]. We only give the outline of the proof. For a Borel measurable subset B of $X_0\times X_0$ we put

$$\Lambda_B=\{(\eta,\zeta)\in(\mathscr{M}_0\times\mathscr{M}_0)\backslash\Delta\ :\ \zeta=\eta^{x,y}\text{ for some }(x,y)\in B\}$$

(the measurability of Λ_B can be proved as in Lemma 1.4) and define an $\hat{\mathscr{F}}_t$-adapted σ-finite random measure $N'(dtdxdy)$ by

$$N'((0,t]\times B)=\sum_{s\in(0,t]}\mathbf{1}_{\Lambda_B}(\eta_{s-},\eta_s)\,,\qquad B\in\mathscr{B}_0\otimes\mathscr{B}_0\,.$$

We also put

$$M'((0,t]\times B)=N'((0,t]\times B)-\int_{(0,t]}ds\int_B\int\eta(dx)dy\,p(|x-y|)\,.$$

Noting that for any $f\in\mathfrak{A}$

$$\begin{aligned}f(\eta_t)-f(\eta_0)=&\int_{(0,t]}\int_{R^d}\{f(\tau_{-u}\eta_{s-})-f(\eta_{s-})\}N(dsdu)\\&+\int_{(0,t]}\iint_{X_0\times X_0}\{f(\eta_{s-}^{x,y})-f(\eta_{s-})\}N'(dsdxdy)\,,\end{aligned}$$

and that $f(\eta_t)-\int\mathscr{L}f(\eta_s)ds$ is an $\hat{\mathscr{F}}_t$-martingale, we see that

$$\begin{aligned}&\int_{(0,t]}\int_{R^d}\{f(\tau_{-u}\eta_{s-})-f(\eta_{s-})\}M(dsdu)\\&\qquad+\int_{(0,t]}\int_{X_0\times X_0}\{f(\eta_{s-}^{x,y})-f(\eta_{s-})\}M'(dsdxdy)\end{aligned}$$

is an $\hat{\mathscr{F}}_t$-martingale. Also it is easily seen that for $g\in\mathfrak{A}$

$$\begin{aligned}&\int_{(0,t]}\int_{R^d}\{f(\tau_{-u}\eta_{s-})-f(\eta_{s-})\}g(\eta_{s-})M(dsdu)\\&\qquad+\int_{(0,t]}\iint_{X_0\times X_0}\{f(\eta_{s-}^{x,y})-f(\eta_{s-})\}g(\eta_{s-})M'(dsdxdy)\end{aligned}$$

is an $\hat{\mathscr{F}}_t$-martingale. In particular, if $f(\eta)g(\eta)=0$ for all $n\in\mathscr{M}_0$, then

$$\int_{(0,t]}\int_{R^d}f(\tau_{-u}\eta_{s-})g(\eta_{s-})M(dsdu)+\int_{(0,t]}\iint_{X_0\times X_0}f(\eta_{s-}^{x,y})g(\eta_{s-})M'(dsdxdy)$$

is an $\hat{\mathscr{F}}_t$-martingale. Then, it is easily seen that for any bounded measurable function H on $\mathscr{M}_0\times\mathscr{M}_0$ with $H(\eta, \eta)=0$ and for any bounded measurable subsets B_1 and B_2 of X_0

$$\int_{(0,t]}\int_{R^d} H(\tau_{-u}\eta_{s-}, \eta_{s-})M(dsdu)+\int_{(0,t]}\iint_{B_1\times B_2} H(\eta_{s-}^{x,y}, \eta_{s-})M'(dsdxdy)$$

is an $\hat{\mathscr{F}}_t$-martingale, from which (i) follows. (ii) is immediate from (i) and the following identity which can be obtained by integration by parts:

$$M((0, t]\times A)^2=N((0, t]\times A)+2\int_{(0,t]} M((0, s)\times A)M(ds, A) . \qquad \square$$

From Lemma 1.5 we have

$$x(t)-x(0)=\int_{(0,t]}\int_{R^d} u\, M(dsdu)+\int_{(0,t]} ds\, G(\eta_s) , \tag{1.13}$$

where $G=(G_1, G_2, \cdots, G_d)$ is an R^d-valued function on $\mathscr{M}_0$ defined by

$$G(\eta)=\int_{R^d} du\, p(|u|)\chi(u\,|\,\eta)u . \tag{1.14}$$

REMARK 1.3. Since the distribution $p_x(dy)=p(|x-y|)dy$ is rotation invariant, the processes η_t and $x(t)-x(0)$ are also rotation invariant.

§ 2. Central limit theorem for $x(t)$.

Let $x(t)$ be the tagged particle process colliding with infinitely many particles, which is defined by (1.12) with $x(0)=0$ on the probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$. In this section we study the asymptotic behavior of $x(t)$ as $t\to\infty$.

First we prepare some lemmas. Let G be the function defined in (1.14).

LEMMA 2.1. *For any* $f\in L^2(\mathscr{M}_0, \nu_0)$ *and* $i=1, 2, \cdots, d$,

$$|(G_i, f)_{\nu_0}|^2\leqq c_1(f, -\mathscr{L}_2 f)_{\nu_0} , \tag{2.1}$$

where

$$c_1=\frac{1}{2}\int_{\mathscr{M}_0}\nu_0(d\eta)\int_{R^d} du\, p(|u|)\chi(u\,|\,\eta)u_1^2 .$$

PROOF. From the shift invariance of ν, we have

$$\int_{\mathscr{M}_0}\nu_0(d\eta)\chi(u\,|\,\eta)f(\eta)=\int_{\mathscr{M}_0}\nu_0(d\eta)\chi(-u\,|\,\eta)f(\tau_u\eta) .$$

Hence,

$$
\begin{aligned}
(G_i, f)_{\nu_0} &= \int_{R^d} du\, p(|u|)u_i \int_{\mathscr{M}_0} \nu_0(d\eta)\chi(u\,|\,\eta)f(\eta) \\
&= \int_{R^d} du\, p(|u|)u_i \int_{\mathscr{M}_0} \nu_0(d\eta)\chi(-u\,|\,\eta)f(\tau_u\eta) \\
&= -\int_{R^d} du\, p(|u|)u_i \int_{\mathscr{M}_0} \nu_0(d\eta)\chi(u\,|\,\eta)f(\tau_{-u}\eta)\,.
\end{aligned}
$$

Therefore we have

$$
(G_i, f)_{\nu_0} = -\frac{1}{2}\int_{\mathscr{M}_0} \nu_0(d\eta)\int_{R^d} du\, p(|u|)\chi(u\,|\,\eta)u_i\{f(\tau_{-u}\eta)-f(\eta)\}\,.
$$

Using Schwarz's inequality, we have

$$
\begin{aligned}
(2.2)\qquad |(G_i, f)_{\nu_0}|^2 \leqq \frac{1}{4}\int_{\mathscr{M}_0} \nu_0(d\eta)\int_{R^d} du\, p(|u|)\chi(u\,|\,\eta)u_i^2 \\
\cdot\int_{\mathscr{M}_0} \nu_0(d\eta)\int_{R^d} du\, p(|u|)\chi(u\,|\,\eta)\{f(\tau_{-u}\eta)-f(\eta)\}^2\,.
\end{aligned}
$$

From the rotation invariance of ν_0, the right-hand side of (2.2) is independent of i. Thus, we have our assertion from Lemma 1.3. □

LEMMA 2.2. *There exists a positive constant c_2 such that for any $f \in \mathscr{D}(\mathscr{L}_1)$ and $i=1, 2, \cdots, d$,*

$$
|(G_i, f)_{\nu_0}|^2 \leqq c_2(f, -\mathscr{L}_1 f)_{\nu_0}\,. \tag{2.3}
$$

Proof of Lemma 2.2 is given in §3.

REMARK 2.1. Let c be a positive constant. Then, the following statements (i) and (ii) are equivalent.

(i) $|(G_i, f)_{\nu_0}|^2 \leqq c(f, -\mathscr{L}f)_{\nu_0}$ for any $f \in \mathscr{D}(\mathscr{L})$.

(ii) $\int_{[0,\infty)} dt\, (T_t G_i, G_i)_{\nu_0} \leqq c$.

LEMMA 2.3.

$$
\lim_{t\to\infty} \frac{1}{t}\hat{E}\{x_i(t)x_j(t)\} = \begin{cases} C & \text{if } i=j\,, \\ 0 & \text{if } i\neq j\,, \end{cases}
$$

where

$$
C = 2c_1 - 2\int_{[0,\infty)} dt\, (T_t G_1, G_1)_{\nu_0}\,.
$$

PROOF. From the rotation invariance of $x(t)$ we have

$$\hat{E}[x_i(t)x_j(t)]=0 \qquad \text{if} \quad i\neq j\,,$$
$$\hat{E}[x_i(t)^2]=\hat{E}[x_1(t)^2] \qquad \text{if} \quad i=1, 2, \cdots, d\,.$$

Therefore, it is enough to consider the case where $i=j=1$. Write $M(t, A)$ for $M((0, t]\times A)$. Then

$$x_1(t)=\int_{R^d} u_1\, M(t, du)+\int_{(0,t]} ds\, G_1(\eta_s)\,. \tag{2.4}$$

Using Lemma 1.5 and Proposition 1.2 we have

$$\hat{E}\left[\left\{\int_{R^d} u_1\, M(t, du)\right\}^2\right]=t\int_{\mathscr{M}_0} \nu_0(d\eta)\int_{R^d} du\; p(|u|)\chi(u\,|\,\eta)u_1^2\,, \tag{2.5}$$

$$\hat{E}\left[\left\{\int_{(0,t]} ds\, G_1(\eta_s)\right\}^2\right]=2\int_{(0,t]} ds\int_{(0,s]} dv\, (T_v G_1, G_1)_{\nu_0}\,. \tag{2.6}$$

From Remark 1.2 and Lemma 1.5 we have

$$\begin{aligned}\hat{E}[x_1(s)G_1(\eta_s)]&=-\hat{E}[x_1(s)G_1(\eta_0)]\\ &=-\hat{E}\left[\int_{R^d} u_1\, M(s, du)G_1(\eta_0)\right]-\int_{(0,s]} dv\, \hat{E}[G_1(\eta_v)G_1(\eta_0)]\\ &=-\int_{(0,s]} dv\, (T_v G_1, G_1)_{\nu_0}\,.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\hat{E}\left[\int_{R^d} u_1\, M(t, du)\int_{(0,t]} ds\, G_1(\eta_s)\right]\\ &=\int_{(0,t]} ds\, \hat{E}\left[\int_{R^d} u_1\, M(s, du)G_1(\eta_s)\right]\\ &=\int_{(0,t]} ds\, \hat{E}[x_1(s)G_1(\eta_s)]-\int_{(0,t]} ds\int_{(0,s]} dv\, \hat{E}[G_1(\eta_v)G_1(\eta_s)]\\ &=-2\int_{(0,t]} ds\int_{(0,s]} dv\, (T_v G_1, G_1)_{\nu_0}\,.\end{aligned} \tag{2.7}$$

Therefore, we conclude that

$$\lim_{t\to\infty}\frac{1}{t}\hat{E}\{x_1(t)^2\}=2c_1-2\int_{[0,\infty)} dt\, (T_t G_1, G_1)_{\nu_0}\,. \qquad \square$$

THEOREM 2.1. *The process $\varepsilon x(t/\varepsilon^2)$ converges to $\sigma B(t)$ as $\varepsilon\downarrow 0$ in the sense of law, that is, in the sense of the weak convergence of probability measures on the Skorohod space, where $B(t)$ is a d-dimensional Brownian motion and σ is a positive constant.*

PROOF. The first term in the right hand side of (1.13) is a martingale. As for the second term we can apply Theorem 1.8 of [2], by virtue of Lemma 2.1, and consequently we can treat the second term as well as the first term within the framework of the central limit theorem of martingales. Thus we can prove that $\varepsilon x(t/\varepsilon^2)$ converges in law to $DB(t)$ as $\varepsilon\downarrow 0$, where D is a symmetric $d\times d$ matrix defined by

$$(D^2)_{ij}=\lim_{t\to\infty}\frac{1}{t}\hat{E}\{x_i(t)x_j(t)\}\ .$$

Put $\sigma=\sqrt{C}$. By virtue of Lemma 2.3, $DB(t)$ and $\sigma B(t)$ have the same law. The details are much the same as the proof of Theorem 2.4 of [2].

The proof of the positivity of σ is as follows. From Lemma 2.1 and Lemma 2.2, for any $f\in\mathscr{D}(\mathscr{L})$ we have

$$|(G_1,\ f)_{\nu_0}|^2\leqq\frac{c_1c_2}{c_1+c_2}(f,\ -\mathscr{L}f)_{\nu_0}\ ,$$

and so by Remark 2.1,

$$\int_{[0,\infty)}dt(T_tG_1,\ G_1)_{\nu_0}\leqq\frac{c_1c_2}{c_1+c_2}\ .$$

Hence,

$$\sigma^2=2c_1-2\int_{[0,\infty)}dt\,(T_tG_1,\ G_1)_{\nu_0}\geqq\frac{2c_1^2}{c_1+c_2}\ . \qquad \square$$

§3. Proof of Lemma 2.2.

Let h be the positive constant in (1.3), i a positive integer $\leqq d$, m the integer such that

$$\frac{h+r}{m}\pi<h\leqq\frac{h+r}{m-1}\pi\ ,$$

and θ a rotation on $\boldsymbol{R}^d$ such that $\theta(0)=0$ and for $a_i=(0,\cdots,\underset{(i\text{-th})}{1},\cdots,0)\in\boldsymbol{R}^d$

(3.1) $$\theta^m(a_i)=-a_i\ .$$

Then, for any $k\in\boldsymbol{N}$ we have

(3.2) $$|\theta^k(x)-\theta^{k-1}(x)|<h \qquad \text{for}\quad x\in B_{h+r}(0)\ ,$$

(3.3) $$\theta^{-k}dx=dx\ .$$

From (3.1) and (3.3), for $f\in\mathfrak{A}$ we have

$$(G_i,\ f)_{\nu_0}=\frac{1}{2}\int_{R^d}du\ p(|u|)u_i(\chi(u|\cdot)-\chi(\theta^m(u)|\cdot),\ f)_{\nu_0}\ . \tag{3.4}$$

We define a set $B[u]$, $u\in \boldsymbol{R}^d$, by $B[u]=(B_r(u)\cup B_r(\theta^m(u)))\cap X_0$. Since $\chi(u|\eta)=\chi(\theta^m(u)|\eta)=1$ for any $\eta\in\mathscr{M}_0$ with $\eta(B[u])=0$, from a property of the Poisson distribution we have

$$\begin{aligned}(3.5)\qquad &(\chi(u|\cdot)-\chi(\theta^m(u)|\cdot),\ f)_{\nu_0}\\ &=\int_{\eta(B[u])=0}\nu_0(d\eta)\sum_{n=0}^{\infty}\frac{1}{n!}\int_{B[u]^n}dx_1\cdots dx_n\\ &\quad\cdot\{\chi(u|\eta\cdot x_1\cdots x_n)-\chi(\theta^m(u)|\eta\cdot x_1\cdots x_n)\}f(\eta\cdot x_1\cdots x_n)\\ &=\int_{\eta(B[u])=0}\nu_0(d\eta)\sum_{n=1}^{\infty}\frac{1}{n!}\int_{B[u]^n}dx_1\cdots dx_n\\ &\quad\cdot\{\chi(u|x_1\cdots x_n)-\chi(\theta^m(u)|x_1\cdots x_n)\}f(\eta\cdot x_1\cdots x_n)\ .\end{aligned}$$

Noting that, for any $n\in\boldsymbol{N}$ and $\eta\in\mathscr{M}_0$,

$$\begin{aligned}&\int_{B[u]^n}dx_1\cdots dx_n\,\chi(u|x_1\cdots x_n)f(\eta\cdot x_1\cdots x_n)\\ &\quad=\int_{B[u]^n}dx_1\cdots dx_n\,\chi(\theta^m(u)|x_1\cdots x_n)f(\eta\cdot\theta^m(x_1)\cdots\theta^m(x_n))\ ,\end{aligned}$$

from (3.5) we have

$$\begin{aligned}(3.6)\qquad &(\chi(u|\cdot)-\chi(\theta^m(u)|\cdot),\ f)_{\nu_0}\\ &=\int_{\eta(B[u])=0}\nu_0(d\eta)\sum_{n=1}^{\infty}\frac{1}{n!}\int_{B[u]^n}dx_1\cdots dx_n\,\chi(\theta^m(u)|x_1\cdots x_n)\\ &\quad\cdot\{f(\eta\cdot\theta^m(x_1)\cdots\theta^m(x_n))-f(\eta\cdot x_1\cdots x_n)\}\ .\end{aligned}$$

From (3.6) and

$$\begin{aligned}&\int_{B[u]^n}dx_1\cdots dx_n\,|f(\eta\cdot\theta^m(x_1)\cdots\theta^m(x_n))-f(\eta\cdot x_1\cdots x_n)|\\ &\quad\leqq n\int_{B[u]^n}dx_1\cdots dx_n\,|f(\eta\cdot x_1\cdots x_{n-1}\cdot\theta^m(x_n))-f(\eta\cdot x_1\cdots x_n)|\ ,\end{aligned}$$

we have

$$\begin{aligned}(3.7)\qquad &|(\chi(u|\cdot)-\chi(\theta^m(u)|\cdot),\ f)_{\nu_0}|\\ &\leqq\int_{\eta(B[u])=0}\nu_0(d\eta)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int_{B[u]^n}dx_1\cdots dx_n\\ &\qquad\cdot|f(\eta\cdot x_1\cdots x_{n-1}\cdot\theta^m(x_n))-f(\eta\cdot x_1\cdots x_n)|\\ &=\int_{\mathscr{M}_0}\nu_0(d\eta)\int_{B[u]}dx\,|f(\eta\cdot\theta^m(x))-f(\eta\cdot x)|\ .\end{aligned}$$

Next we define a constant M by

$$M=\inf\{|B_h(\theta(x))\cap B_h(x)\cap X_0| : x\in B_{r+h}(0)\cap X_0\}\,. \tag{3.8}$$

Since by (3.2) $M>0$, for any $x\in B_{r+h}(0)\cap X_0$ we have

$$\begin{aligned}|f(\eta\cdot\theta(x))-f(\eta\cdot x)|&\\ \leqq\frac{1}{M}\Big\{\int_{B_h(\theta(x))\cap X_0}dy\,|f(\eta\cdot y)-f(\eta\cdot\theta(x))|&\\ +\int_{B_h(x)\cap X_0}dy\,|f(\eta\cdot y)-f(\eta\cdot x)|\Big\}\,.&\end{aligned} \tag{3.9}$$

Thus, for $u\in B_h(0)$ we have

$$\begin{aligned}&\int_{B[u]}dx\,|f(\eta\cdot\theta^m(x))-f(\eta\cdot x)|\\ &\leqq\sum_{k=0}^{2m-1}\int_{B_r(u)\cap X_0}dx\,|f(\eta\cdot\theta^{k+1}(x))-f(\eta\cdot\theta^k(x))|\\ &\leqq\frac{2}{M}\sum_{k=0}^{2m-1}\int_{B_r(u)\cap X_0}dx\int_{B_h(\theta^k(x))\cap X_0}dy\,|f(\eta\cdot y)-f(\eta\cdot\theta^k(x))|\\ &=\frac{2}{M}\sum_{k=0}^{2m-1}\int_{B_r(\theta^k(u))\cap X_0}dx\int_{B_h(x)\cap X_0}dy\,|f(\eta\cdot y)-f(\eta\cdot x)|\,.\end{aligned} \tag{3.10}$$

From (3.4), (3.7) and (3.10) we have

$$\begin{aligned}|(G_i,f)_{\nu_0}|\leqq\frac{1}{M}\sum_{k=0}^{2m-1}\int_{R^d}du\,p(|u|)|u|&\\ \cdot\int_{\mathscr{M}_0}\nu_0(d\eta)\int_{B_r(\theta^k(u))\cap X_0}dx\int_{B_h(x)\cap X_0}dy\,|f(\eta\cdot y)-f(\eta\cdot x)|\,.&\end{aligned}$$

Then, from (3.3) we have

$$\begin{aligned}|(G_i,f)_{\nu_0}|\leqq\frac{2m}{M}\int_{R^d}du\,p(|u|)|u|&\\ \cdot\int_{\mathscr{M}_0}\nu_0(d\eta)\int_{B_r(u)\cap X_0}dx\int_{B_h(x)\cap X_0}dy\,|f(\eta\cdot y)-f(\eta\cdot x)|\,,&\end{aligned}$$

which, from Remark 1.1, is also dominated by

$$\begin{aligned}|(G_i,f)_{\nu_0}|\leqq\frac{2m}{M}\int_{R^d}du\,p(|u|)|u|&\\ \cdot\int_{\mathscr{M}_0}\nu_0(d\eta)\int_{B_r(u)}\eta(dx)\int_{B_h(x)\cap X_0}dy\,|f(\eta^{x,y})-f(\eta)|\,.&\end{aligned}$$

Therefore, using the Schwarz inequality, we have

$$|(G_i,\ f)_{\nu_0}|^2 \leqq \frac{c_2}{2}\int_{\mathscr{M}_0}\nu_0(d\eta)\int_{X_0}\eta(dx)\int_{X_0}dy\ p(|x-y|)\{f(\eta^{x,y})-f(\eta)\}^2\ ,$$

where

$$c_2=\frac{2}{\kappa}\left(\frac{2m}{M}\right)^2|B_r(0)|\ |B_h(0)|\left\{\int_{R^d}du\ p(|u|)|u|^2\right\}\ .$$

From Lemma 1.3, we obtain (2.3) for $f\in\mathfrak{A}$. Since $\mathfrak{A}$ is a core for $\mathscr{L}_1$, this completes the proof of Lemma 2.2.

ACKNOWLEDGEMENT. The author would like to express his thanks to Professor H. Tanaka for helping him with valuable suggestion and constant encouragement. The author also expresses his thanks to Dr. K. Kuroda and Dr. Y. Tamura for their encouragements.

References

[1] P. JAGERS, Aspects of random measure and point processes, Adv. Probab. Related Topics, **3** (1974), 179–239, Marcel Dekker, New York.

[2] C. KIPNIS and S. R. S. VARADHAN, Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, Comm. Math. Phys., **104** (1986), 1–19.

[3] C. KIPNIS, Central limit theorems for infinite series of queues and applications to simple exclusion, Ann. Probab., **14** (1986), 397–408.

[4] T. M. LIGGETT, *Interacting Particle Systems*, Springer-Verlag, 1985.

[5] T. SHIGA and Y. TAKAHASHI, Ergodic properties of the equilibrium process associated with infinitely many Markovian particles, Publ. RIMS Kyoto Univ., **9** (1974), 505–516.

[6] T. SHIGA and H. TANAKA, Central limit theorem for a system of Markov particles with mean field interactions, Z. Wahr. Verw. Geb., **69** (1985), 439–459.

Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA 223, JAPAN