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Smooth Perturbations of the Self-adjoint Operator $|\Delta|^{\alpha/2}$

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1. Introduction.

In this paper we shall consider smooth perturbations of the formal self-adjoint operator $|\Delta|^{\alpha/2}$ in $L^2(\boldsymbol{R}^n)$, where $\Delta=\sum_{i=1}^{m}\partial^2/\partial x_i^{\,2}$. We shall first recall some notations in the theory of smooth perturbations.

Let H be a selfadjoint operator in a separable Hilbert space $\boldsymbol{H}$ with its resolvent denoted by $R(\zeta)=(H-\zeta)^{-1}$, $\mathrm{Im}\,\zeta\neq 0$. A densely defined closed linear operator A is said to be smooth with respect to H, H-smooth for short, if

$$\int_{-\infty}^{+\infty}\|AR(\lambda\pm i\varepsilon)u\|^2 d\lambda\leq c_1^2\|u\|^2\,,\qquad u\in\boldsymbol{H}\,,\qquad \varepsilon>0\,, \tag{1.1}$$

where c_1 is a constant independent of u and $\varepsilon>0$. Each of the following conditions (1.2) and (1.3) is equivalent to (1.1) (cf. T. Kato [2]):

$$|\mathrm{Im}(R(\zeta)A^*u,\,A^*u)|\leq c_2^2\|u\|^2\,,\qquad u\in D(A^*)\,,\quad \mathrm{Im}\,\zeta\neq 0\,; \tag{1.2}$$

$$\int_{-\infty}^{+\infty}\|Ae^{-itH}u\|^2 dt\leq c_3^2\|u\|^2\,,\qquad u\in\boldsymbol{H}\,. \tag{1.3}$$

Here $c_2>0$ and $c_3>0$ are constants independent of u and ζ. $\{e^{itH}\}_{t\in\boldsymbol{R}}$ is a unitary group generated by H, and it is understood that $\|Ae^{-itH}u\|=\infty$ if $e^{-itH}u\notin D(A)$. For more details, see T. Kato [2]. A is said to be supersmooth with respect to H, H-supersmooth for short, if

$$|(R(\zeta)A^*u,\,A^*u)|\leq c_4^2\|u\|^2\,,\qquad u\in D(A^*)\,,\quad \mathrm{Im}\,\zeta\neq 0\,, \tag{1.4}$$

where c_4 is a constant independent of u and $\mathrm{Im}\,\zeta\neq 0$. This terminology was introduced by T. Kato and K. Yajima [3], but the notion itself appeared in T. Kato [2].

T. Kato and K. Yajima proved in [2] that $A=|x|^{-\beta}|\nabla|^{1-\beta}$ with $1/2<\beta\leq 1$ is $-\Delta$-supersmooth. The $-\Delta$-smoothness was also proved by other simple methods (see M. Ben-Artzi and S. Klainerman [1], B. Simon [8]) and was extended to the Schrödinger

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operator with a potential ([1]). In the previous version of this work [12] we also showed, among other things, that the smoothness alone can be proved by a method which is rather simple. There is some overlap between the results of [1], [8] and ours, but the method is different and we shall present our result here. In what follows we first prove the smoothness somewhat generally for $A=f(x)|\nabla|^{\delta}$ and $H=|\Delta|^{\alpha/2}$ with suitable α, δ and f (Theorems 1 and 2). For another proof, see [1]. We also observe that for $f(x)=|x|^{-\beta}$ we can calculate the best constant in (1.3) (Corollary 4). For $H=-\Delta$ and $f(x)=|x|^{-1}$, this best constant is given in [8]. An immediate generalization of Theorem 1 to the case $H=P(|\nabla|)$ will be mentioned in Remark at the end of the section 3. As to the supersmoothness we shall prove an immediate generalization of Theorem 1 of T. Kato and K. Yajima [3] to the case of the $|\Delta|^{\alpha}$-supersmoothness of $A=|x|^{\gamma}|\nabla|^{\delta}$ following their method. As byproducts of our work we shall give: 1) an example of a potential $V(x)$, $x\in \boldsymbol{R}^2$, which is $-\Delta$-smooth but not $-\Delta$-supersmooth; 2) a decay estimate for the solution of the free Schrödinger operator (Proposition 5).

2. Theorems.

We begin with giving precise definitions. Let $\boldsymbol{H}=L^2(\boldsymbol{R}^m)$ with its inner product and norm denoted by $(\,,)$ and $\|\cdot\|$. We use the following notations. $\mathscr{S}$ is the space of rapidly decreasing functions and $\mathscr{S}'$ is its dual space. $\mathscr{F}$ is the Fourier transform from $\mathscr{S}'$ to $\mathscr{S}'$. We also write $\hat{u}(\xi)=(\mathscr{F}u)(\xi)$ when $\mathscr{F}u$ is a function. F is the restriction of $\mathscr{F}$ to $L^2(\boldsymbol{R}^m)$. $H^{\alpha}(\boldsymbol{R}^m)$ is the Sobolev space of order $\alpha\geq 1$. We use the notation $|x|$ to denote the operator of multiplication by $|x|$ in $\boldsymbol{H}$. $|\Delta|^{\alpha/2}$ is an operator in $\boldsymbol{H}$ which is the multiplication operator in the Fourier space and is defined as

$$|\Delta|^{\alpha/2}=|\nabla|^{\alpha}=F^{-1}|\xi|^{\alpha}F\,.$$

In particular, $D(|\Delta|^{\alpha/2})=H^{\alpha}(\boldsymbol{R}^m)$. We put

$$C_{0*}^{\infty}(\boldsymbol{R}^m)=C_0^{\infty}(\boldsymbol{R}^m\setminus\{0\})\,.$$

Throughout the rest of the present paper we put

$$H=|\Delta|^{\alpha/2}\,,\qquad \alpha\geq 1\,.$$

As is well-known

$$(*)\qquad (e^{-itH}u)(x)=\left(\frac{1}{2\pi}\right)^{m/2}\int \hat{u}(\xi)e^{-it|\xi|^{\alpha}+i\xi x}d\xi\,.$$

Our main results are now stated as follows.

THEOREM 1. *Let $\alpha\neq 0$, $m\geq 1$. Let f be in $L^2(\boldsymbol{R}^m)$ and g be a measurable function on $\boldsymbol{R}^+$. Assume that*

(a) $g(|\xi|)|\xi|^{(m-\alpha)/2}$ *is bounded;*

(b) *there exists a dense subset* $\mathscr{D}$ *in* $\boldsymbol{H}$ *such that* $g(|\nabla|)f(x)v \in \boldsymbol{H}$ *if* $v \in \mathscr{D}$.

Let A be the operator defined as

$$A = f(x)g(|\nabla|), \qquad D(A) = C_{0*}^{\infty}(\boldsymbol{R}_{\xi}^{m}).$$

Then A is closable and any closed extension of A is H-smooth.

THEOREM 2. *Let* $m \geq 2$, $\alpha > 1$, $\alpha - 2\beta \geq 0$ *and* $1/2 < \beta < m/2$. *Let* $f \in \mathscr{S}' \cap L_{\mathrm{loc}}^{2}$ *and assume that*

(a) $|f|^2 \in \mathscr{S}'$;

(b) $\mathscr{F}(|f|^2)(\xi)$ *is a measurable function*;

(c) $\mathscr{F}(|\nabla|^{m-2\beta}|f|^2)(\xi) = |\xi|^{m-2\beta}(\widehat{|f|^2})(\xi) \in L^{\infty}$;

(d) *there exists a dense subset* $\mathscr{D}$ *in* $\boldsymbol{H}$ *such that* $|\nabla|^{(\alpha-2\beta)/2} f v \in \boldsymbol{H}$ *if* $v \in \mathscr{D}$.

Let A be the operator defined as

$$A = f(x)|\nabla|^{(\alpha-2\beta)/2}, \qquad D(A) = C_{0*}^{\infty}(\boldsymbol{R}_{\xi}^{m}).$$

Then A is a closable operator in $\boldsymbol{H}$ *and any closed extension of A is H-smooth.*

THEOREM 3. *Let* $\alpha > 1$, $\alpha - 2\beta \geq 0$ *and* $1/2 < \beta < m/2$. *Let A be the operator defined as*

$$A = |x|^{-\beta}|\nabla|^{(\alpha-2\beta)/2}, \qquad D(A) = C_{0*}^{\infty}(\boldsymbol{R}_{\xi}^{m}).$$

Then, A is closable and any closed extension of A is H-supersmooth.

EXAMPLE. In Theorem 2 we can choose $|x|^{-\beta}$ as $f(x)$ (see E. M. Stein [9], p. 116–p. 121). And clearly $f \in \mathscr{S}$ satisfies the conditions (a)–(d).

For an H-smooth operator A, we denote by $\|A\|_H$ the smallest number $c_3 > 0$ for which (1.3) is true. We set $\|A\|_H = \infty$ if A is not H-smooth. Then we obtain the following corollary from the proof of Theorem 2.

COROLLARY 4. *Suppose that* $f(x) = |x|^{-\beta}$ *in Theorem* 2. *Then the best value of* c_3^2 *in* (1.3) *is*

$$c_3^2 = \frac{2\Gamma((m-2\beta)/2)\pi^{2\beta+1-m/2}}{\alpha\Gamma(\beta)} \int_{S^{m-1}} \frac{d\omega}{|\omega - \omega'|^{m-2\beta}}, \tag{2.1}$$

where Γ *is the* Γ*-function and* $d\omega$ *is the Lebesgue measure on* $S^{m-1} = \{x \in \boldsymbol{R}^m : |x| = 1\}$.

The following proposition is crucial in the proof. The space $\dot{H}^s(\boldsymbol{R}^m)$ and its norm are defined by

$$\dot{H}^s(\boldsymbol{R}^m) = \{u \in \boldsymbol{H} : |\xi|^{s/2}\hat{u} \in \boldsymbol{H}\}, \quad \|u\|_{\dot{H}^s} = \||\xi|^{s/2}\hat{u}\|.$$

PROPOSITION 5. *Let* $\alpha \neq 0$ *be real and let* u_0 *be in* $\dot{H}^{(m-\alpha)/2}(\boldsymbol{R}^m)$. *Then*

(2.2)
$$\int_{-\infty}^{\infty} |e^{-itH}u_0(x)|^2 dt = c_\alpha \int_0^\infty \left| \int_{S^{m-1}} \hat{u}_0(r\omega) e^{ir\omega x} d\omega \right|^2 r^{2m-\alpha-1} dr$$
$$\leq c_\alpha |S^{m-1}| \|u_0\|^2_{\dot{H}^{(m-\alpha)/2}}, \quad a.e.\ x,$$

where $c_\alpha = |\alpha|^{-1}(1/2\pi)^{m-1}$ *and* $|S^{m-1}|$ *is the surface area of* S^{m-1}.

REMARK. In (2.2) the middle member is bounded continuous function of x (see the proof of Proposition 5). The left hand side has the meaning if $e^{-itH}u_0(x)$ is chosen to be measurable in two variables x and t. Such a choice is possible and the proposition asserts that (2.2) is true for all such choices. We also remark that it suffices to prove (2.2) for one measurable choice.

In section 3 we shall first prove Proposition 5 and then Theorems 1 and 2. As to the proof of Theorem 3 we shall only give in section 4 an extension of Lemma 2.4 in T. Kato and K. Yajima [3]. Before ending this section we shall make two observations.

Firstly, the following Theorem 6 is found in C. E. Kenig, G. Ponce and L. Vega [4].

THEOREM 6 ([4], Theorem 4.1). *Let* $u_0 \in \dot{H}^{(1-\alpha)/2}(\boldsymbol{R}^m)$ *and* $R>0$. *Then we have*

(2.3)
$$\int_{|x|<R} \int_{-\infty}^{\infty} |e^{-itH}u_0(x)|^2 dt dx \leq CR \|u_0\|^2_{\dot{H}^{(1-\alpha)/2}},$$

where C *is independent of* R *and* u_0.

By Proposition 5, Theorem 6 and interpolation we have the following corollary.

COROLLARY 7. *Let* $R>0$. *Then we have*

(2.4)
$$\int_{|x|<R} \int_{-\infty}^{\infty} |e^{-itH}u_0(x)|^2 dt dx \leq CR^s \|u_0\|^2_{\dot{H}^{(s-\alpha)/2}},$$

where $s=\theta+(1-\theta)m$, $0\leq\theta\leq 1$.

Secondly, as a consequence of (2.2), we can give a concrete example of A which is H-smooth, but not H-supersmooth.

EXAMPLE. Let $m=2$ and $\alpha=2$. Let V be a negative real function in $C_0^\infty(\boldsymbol{R}^2)$ which is not identically zero. Then the operator $-\Delta+\lambda V$ has a negative eigenvalue for all $\lambda>0$. (See M. Reed and B. Simon [7] Theorem XIII.11). This means that the operator $A=(-V)^{1/2}$ is not $(-\Delta)$-supersmooth, because the H-supersmoothness of A would imply the unitary equivalence of $(-\Delta)$ and $-\Delta+\lambda V$ for small $\lambda>0$. (See T. Kato [2]).

On the other hand it follows from (2.2) that

$$\int_{-\infty}^{\infty} \|Ae^{-itH}u_0\|^2 dt \leq c\|A\|^2 \|u_0\|^2, \qquad u_0 \in L^2(\boldsymbol{R}^2).$$

Hence A is H-smooth.

3. *H*-smoothness.

PROOF OF PROPOSITION 5. We denote the middle member of (2.2) by $\phi(x; u_0)$. For $u_0 \in \dot{H}^{(m-\alpha)/2}(\boldsymbol{R}^m)$ we can easily see by Schwarz inequality that

$$\phi(x; u_0) \leq c_\alpha |\boldsymbol{S}^{m-1}| \, \|u_0\|^2_{\dot{H}^{(m-\alpha)/2}} . \tag{3.1}$$

In a similar way we can prove the continuity of ϕ in x.

For $\hat{u}_0 \in C^\infty_{0*}(\boldsymbol{R}^m)$ we substitute $(*)$ into (2.2), then by the Plancherel theorem applied to the t-variable we have (2.2) (see [4] and L. Vega [11]).

We next consider a general $u_0 \in \dot{H}^{(m-\alpha)/2}(\boldsymbol{R}^m)$. We can take a sequence $\{\hat{u}_n\}$ of $C^\infty_{0*}(\boldsymbol{R}^m)$ such that $\|u_n - u_0\| \to 0$ and $\|u_n - u_0\|_{\dot{H}^{(m-\alpha)/2}} \to 0$ as $n \to \infty$. Let $K \subset \boldsymbol{R}^m$ be an arbitrary compact set and let F_K be the characteristic function of K. Then we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \|F_K e^{-itH}(u_n - u_l)\|^2 dt \\
&= \int_{-\infty}^{\infty} \int_K |e^{-itH}(u_n - u_l)(x)|^2 dx dt \\
&= \int_K \int_{-\infty}^{\infty} |e^{-itH}(u_n - u_l)(x)|^2 dt dx \\
&\leq c_\alpha |\boldsymbol{S}^{m-1}| \int_K \|u_n - u_l\|^2_{\dot{H}^{(m-\alpha)/2}} dx \to 0 , \qquad n, l \to \infty .
\end{aligned} \tag{3.2}$$

Hence $F_K e^{-itH} u_n$ converges to $g(t)$ in $L^2(\boldsymbol{R}; L^2(\boldsymbol{R}^m))$. Since $F_K e^{-itH} u_n \to F_K e^{-itH} u_0$ for all $t \in \boldsymbol{R}$, we have $g(t) = F_K e^{-itH} u_0$. On the other hand, by bounded convergence theorem we see that

$$\int_K \phi(x; u_n) dx \to \int_K \phi(x; u_0) dx , \qquad n \to \infty . \tag{3.3}$$

We thus obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \|F_K e^{-itH} u_0\|^2 dt &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \|F_K e^{-itH} u_n\|^2 dt \\
&= \lim_{n\to\infty} \int_K \phi(x; u_n) dx \\
&= \int_K \phi(x; u_0) dx .
\end{aligned} \tag{3.4}$$

If $(e^{-itH} u_0)(x)$ is chosen to be (x, t)-measurable, we can use the Fubini theorem on the right side of (3.4). Then (2.2) follows from (3.4).

LEMMA 2.1. *Let A be the operator in Theorem 1. Then A is closable.*

PROOF. From the assumption on f and g we see immediately that A is well-defined on $C_{0*}^{\infty}(\boldsymbol{R}^m)$. And from the assumption (b) A^* is densely defined. Hence A is closable.

PROOF OF THEOREM 1. We shall prove that any extension of A is H-smooth. First let $u_0 \in D(A)$. By Proposition 5 applied to $g(|\nabla|)u_0$ in place of u_0, we have

$$
\begin{aligned}
(3.5)\qquad & \int_{-\infty}^{\infty} \|f(x)g(|\nabla|)e^{-itH}u_0\|^2 dt \\
&= \int_{\boldsymbol{R}^m} |f(x)|^2 dx \int_{-\infty}^{\infty} |e^{-itH}g(|\nabla|)u_0(x)|^2 dt \\
&= \int_{\boldsymbol{R}^m} |f(x)|^2 \phi(x; g(|\nabla|)u_0) dx \\
&\le C\|f\|^2 \|g(|\nabla|)u_0\|^2_{\dot{H}^{(m-\alpha)/2}} \\
&\le C_g \|f\|^2 \|\hat{u}_0\|^2 = C_g\|f\|^2\|u_0\|^2 ,
\end{aligned}
$$

which proves (1.3) for $u_0 \in D(A)$. In general, for a closed operator A, (1.3) holds for all $u \in \boldsymbol{H}$ if it holds for u belonging to a dense subset. Thus Theorem 1 is proved.

PROOF OF THEOREM 2.

Step 1: $A = f(x)|\nabla|^{(\alpha-2\beta)/2}$ is closable.

Let γ be a nonnegative number and $u \in D(A)$. Then we know by assumption (a) that $|f(x)|^2|\nabla|^{\gamma}u(x)$ is in $\mathscr{S}'$. Furthermore, $|\nabla|^{\gamma}u(x)$ is in $\mathscr{S}$. Hence we obtain

$$(3.6)\qquad \|f(x)|\nabla|^{\gamma}u\|^2 = (|f(x)|^2|\nabla|^{\gamma}u, |\nabla|^{\gamma}u) < \infty$$

which implies that $f(x)|\nabla|^{\gamma}u \in \boldsymbol{H}$. Thus, A is well-defined as an operator in $\boldsymbol{H}$. Assumption (d) implies that $\mathscr{D} \subset D(A^*)$ so that A^* is densely defined. Hence A is closable.

Step 2: $[A]$ is H-smooth.

Let $u \in D(A)$. Put $\gamma = (\alpha - 2\beta)/2$. Using the Fourier transformation, we see that

$$
\begin{aligned}
(3.7)\qquad & \|f(x)|\nabla|^{\gamma}e^{-itH}u\|^2 \\
&= (|f(x)|^2|\nabla|^{\gamma}e^{-itH}u, |\nabla|^{\gamma}e^{-itH}u) \\
&= c((\widehat{|f|^2}) * |\xi|^{\gamma}e^{-it|\xi|^{\alpha}}\hat{u}, |\xi|^{\gamma}e^{-it|\xi|^{\alpha}}\hat{u}) \\
&= c\iint_{\boldsymbol{R}^{2m}} (\widehat{|f|^2})(\xi-\eta)|\eta|^{\gamma}\hat{u}(\eta)|\xi|^{\gamma}\overline{\hat{u}(\xi)}e^{-it(|\eta|^{\alpha}-|\xi|^{\alpha})}d\xi d\eta \\
&= c\iint |\xi-\eta|^{m-2\beta}(\widehat{|f|^2})(\xi-\eta)\frac{|\eta|^{\gamma}\hat{u}(\eta)|\xi|\overline{\hat{u}(\xi)}}{|\xi-\eta|^{m-2\beta}}e^{-it(|\eta|^{\alpha}-|\xi|^{\alpha})}d\xi d\eta
\end{aligned}
$$

$$=c\int\int \mathscr{F}(|\nabla|^{m-2\beta}|f|^2)(\xi-\eta)\frac{|\eta|^\gamma\hat{u}(\eta)|\xi|^\gamma\overline{\hat{u}(\xi)}}{|\xi-\eta|^{m-2\beta}}e^{-it(|\eta|^\alpha-|\xi|^\alpha)}d\xi d\eta.$$

We shall later prove that

$$\text{(3.8)}\qquad \lim_{\varepsilon\downarrow\infty}\int_{-\infty}^{\infty}e^{-\varepsilon|t|}dt\int\int\frac{|\xi|^\gamma\hat{u}(\xi)|\eta|^\gamma\overline{\hat{u}(\eta)}e^{it(|\xi|^\alpha-|\eta|^\alpha)}}{|\xi-\eta|^{m-2\beta}}d\xi d\eta\le C\|u\|^2\,,\qquad u\in D(A)\,,$$

where

$$C=(\pi/\alpha)\int_{S^{m-1}}\frac{d\omega'}{|\omega-\omega'|^{m-2\beta}}\,.$$

Since $\mathscr{F}(|\nabla|^{m-2\beta}|f|^2)\in L^\infty$ and $\hat{u}\in C_{0*}^\infty(\boldsymbol{R}^m)$, it follows from (3.8) that (1.3) holds for $u\in D(A)$ with c_3 independent of u. This shows that A is H-smooth because we can easily deduce (1.3) for a general $u\in \boldsymbol{H}$ from (1.3) for $u\in D(A)$ exactly in the same way as in the proof of Theorem 1.

Let P_ε be the Poisson kernel for the half plane, i.e.,

$$P_\varepsilon(x)=\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}\,,\qquad x\in\boldsymbol{R}\,,\quad \varepsilon>0\,.$$

By carrying out the integration with respect to t in (3.8), we obtain

$$\text{(3.9)}\qquad \lim_{\varepsilon\downarrow\infty}\int\int\frac{P_\varepsilon(|\xi|^\alpha-|\eta|^\alpha)|\xi|^\gamma\hat{u}(\xi)|\eta|^\gamma\overline{\hat{u}(\eta)}}{|\xi-\eta|^{m-2\beta}}d\xi d\eta$$

$$=c\int\int_{S^{m-1}\times S^{m-1}}\int_0^\infty\frac{r^\gamma\hat{u}(r\omega)r^\gamma\overline{\hat{u}(r\omega')}}{|r\omega-r\omega'|^{m-2\beta}}r^{2m-\alpha-1}dr d\omega d\omega'$$

$$=c\int\int_{S^{m-1}\times S^{m-1}}\int_0^\infty\frac{\hat{u}(r\omega)\overline{\hat{u}(r\omega')}}{|\omega-\omega'|^{m-2\beta}}r^{m-1}dr d\omega d\omega'$$

$$\le c\int_0^\infty\int\int_{S^{m-1}\times S^{m-1}}\frac{|\hat{u}(r\omega)|^2}{|\omega-\omega'|^{m-2\beta}}d\omega d\omega' r^{m-1}dr\,,$$

where $c=1/\alpha$. At the last inequality we used Schwarz inequality with respect to $d\omega d\omega'$. We integrate the right hand side first over ω'. We know that

$$\int_{S^{m-1}}\frac{d\omega'}{|\omega-\omega'|^{m-2\beta}}<\infty\,,\qquad \beta>1/2\,.$$

This proves (3.8) and finishes the proof of Theorem 2.

PROOF OF COROLLARY 4. If $\hat{u}$ is a radial function, the equality holds in the last inequality of (3.9). To prove (2.1), we use the Riesz potentials I_β. (See E. M. Stein [9],

p. 116–p. 121.) For $u, v \in \mathscr{S}$,

$$(3.10)\qquad (I_\beta u)(x)=\frac{1}{\gamma(\beta)}\int_{\boldsymbol{R}^m}\frac{u(y)}{|x-y|^{m-\beta}}dy\,,$$

$$(3.11)\qquad \int_{\boldsymbol{R}^m}(I_\beta u)(x)\overline{v(x)}dx=(2\pi)^{-\beta}\int_{\boldsymbol{R}^m}|\xi|^{-\beta}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi\,,$$

where

$$\gamma(\beta)=\pi^{m/2}2^{\beta}\frac{\Gamma(\beta/2)}{\Gamma((m-\beta)/2)}\,.$$

Using (3.10) and (3.11), we can easily calculate the best constant c_3^2. In fact, putting $\gamma=(\alpha-2\beta)/2$, we have

$$\begin{aligned}
&\||x|^{-\beta}|\nabla|^{\gamma}e^{-itH}u\|^2\\
&=\int_{\boldsymbol{R}^m}|x|^{-2\beta}(|\nabla|^{\gamma}e^{-itH}u(x))\overline{(|\nabla|^{\gamma}e^{-itH}u(x))}dx\\
&=(2\pi)^{2\beta}\int(2\pi|x|)^{-2\beta}(|\nabla|^{\gamma}e^{-itH}u(x))\overline{(|\nabla|^{\gamma}e^{-itH}u(x))}dx\\
&=(2\pi)^{2\beta}\int I_{2\beta}(|\cdot|^{\gamma}e^{-it|\cdot|^{\alpha}}\hat{u})(\xi)|^{m}|^{\gamma}\overline{e^{-it|\xi|^{\alpha}}\hat{u}(\xi)}d\xi\\
&=\frac{(2\pi)^{2\beta}}{\gamma(2\beta)}\iint_{\boldsymbol{R}^{2m}}\frac{|\eta|^{\gamma}\hat{u}(\eta)|\xi|^{\gamma}\overline{\hat{u}(\xi)}}{|\xi-\eta|^{m-2\beta}}e^{it(|\xi|^{\alpha}-|\eta|^{\alpha})}d\xi d\eta\,.
\end{aligned}$$

After multiplying $e^{-\varepsilon|t|}$, we integrate both sides over $t\in\boldsymbol{R}$. From the above equality and (3.8) we obtain (2.1).

REMARK. In the proof of Theorem 1 we used the fact that $|\Delta|^{\alpha/2}$ is a radial function in the Fourier space, and in the proof of Theorem 2 the condition that $|\xi|^{m-2\beta}(\widehat{|f|^2})\in L^\infty$. Keeping this in mind, we can extend Theorem 1 as follows; Let P and Q be real-valued functions in $C^1(0,\infty)$ and $C(0,\infty)$, respectively. We assume that $P'(r)>0$ and $P'(0)=0$ and that

$$\left|\frac{Q^2(r)}{P'(r)}r^{m-1}\right|\le C\,.$$

Let f be in $L^2(\boldsymbol{R}^m)$ such that $A=f(x)Q(|\nabla|)$ with $D(A)=C_{0*}^{\infty}(\boldsymbol{R}_\xi^m)$ is a densely defined closable linear operator. Then any closed extension of A is $P(|\nabla|)$-smooth, i.e.,

$$\int_{-\infty}^{\infty}\|f(x)Q(|\nabla|)\exp(-itP(|\nabla|))u\|^2dt\le C\|u\|^2\,,\qquad u\in\boldsymbol{H}\,.$$

The relation corresponding to (2.2) takes the following form;

$$\int_{-\infty}^{\infty} |\exp(-itP(|\nabla|))u(x)|^2 dt$$

$$=C_m \int_0^{\infty} \left| \int_{S^{m-1}} \hat{u}(r\omega)e^{ir\omega x} d\omega \right|^2 \frac{r^{2m-2}}{P'(r)} dr , \qquad u \in C_{0*}^{\infty}(\boldsymbol{R}^m) ,$$

and can be found in [4].

4. *H*-supersmoothness.

The weighted Sobolev space $H_s^r(\boldsymbol{R}^m)$ and its norm are defined by

$$H_s^r(\boldsymbol{R}^m)=\{u\in\mathscr{S}' : \|(1+x^2)^{s/2}(1-\Delta)^{r/2}u\|<\infty\} \qquad \text{for} \quad s, r\in\boldsymbol{R} ,$$

$$\|u\|_{H_s^r}=\|(1+x^2)^{s/2}(1-\Delta)^{r/2}u\| .$$

The proof of Theorem 3 is based on the following lemma (the case $0\leq s\leq 1$ was used in [3]).

LEMMA 4.1. *For* $0\leq s<m/2$,

$$|x|^{-s}\in B(\boldsymbol{H}, H_s^{-s}(\boldsymbol{R}^m))\cap B(H_{-s}^s(\boldsymbol{R}^m), \boldsymbol{H}) . \tag{4.1}$$

PROOF. It is sufficient to prove that $|x|^{-s}$ is $B(H_{-s}^s(\boldsymbol{R}^m), \boldsymbol{H})$, since $B(H_s^{-s}(\boldsymbol{R}^m), \boldsymbol{H})$ is the dual of $B(\boldsymbol{H}, H_{-s}^s(\boldsymbol{R}^m))$. For $0\leq s<m/2$, we use that

$$\||x|^{-s}u\|\leq C\|u\|_{H^s} , \qquad u\in H^s(\boldsymbol{R}^m) \tag{4.2}$$

(cf. P. I. Lizorkin [5], V. G. Maz'ya and T. O. Shaposhnikova [6] and H. Triebel [10]). Let $u\in H_{-s}^s(\boldsymbol{R}^m)$ and $R>0$. We take a function $\chi_1\in C_0^{\infty}(\boldsymbol{R}^m)$ such that $0\leq\chi_1\leq 1$, $\chi_1=1$, if $|x|<R$, and $\chi_1=0$, if $|x|>2R$. Put $\chi_2=1-\chi_1$. By (4.2), we have

$$\begin{aligned}
\||x|^{-s}u\| &\leq C(\||x|^{-s}\chi_1 u\|+\||x|^{-s}\chi_2 u\|)\\
&\leq C(1+4R^2)^{s/2}\||x|^{-s}(1+x^2)^{-s/2}\chi_1 u\|+C\|(1+x^2)^{-s/2}u\|\\
&\leq C\|(1+x^2)^{-s/2}u\|_{H^s}+C\|(1+x^2)^{-s/2}u\|\\
&\leq C\|(1-\Delta)^{s/2}(1+x^2)^{-s/2}u\|+C\|(1-\Delta)^{s/2}(1+x^2)^{-s/2}u\|\\
&\leq C\|(1+x^2)^{-s/2}(1-\Delta)^{s/2}u\|\\
&=C\|u\|_{H_{-s}^s} .
\end{aligned}$$

The rest of the proof refers to the proof of Theorem 1 of [3]. We shall make a sketch of proof.

For $s>1/2$ and $r\in\boldsymbol{R}$, a $B(H_s^r(\boldsymbol{R}^m), \Sigma)$-valued function $\gamma(k)$ is defined by

$$\gamma(k)g(\omega)=k^{(m-1)/2}\hat{g}(k\omega)\,, \qquad g\in H^r_s(\boldsymbol{R}^m)\,, \quad k\in\boldsymbol{R}^+\,.$$

Let $M=|x|^{-s}$, $1/2<s<m/2$. Then a $B(H, \Sigma)$-valued function $\Psi(k)$ defined by

$$\Psi(k)=\gamma(k)M\,, \qquad k\in\boldsymbol{R}^+ \tag{4.3}$$

is locally Hölder continuous. In particular one has the following inequalities:

$$\|\Psi(k)g\|_\Sigma\le\begin{cases} Ck^{s-1/2}\|g\|\,, & 1/2<s<3/2 \\ Ck|\log k|\,\|g\|\,, & s=3/2 \\ Ck\|g\|\,, & s>3/2\,, \end{cases} \tag{4.4}$$

for sufficiently small $k\in\boldsymbol{R}^+$.

Let $E(\lambda)$ be a spectral decomposition of H. Put $M=|x|^{-\beta}$. For $\hat{u}$, $\hat{v}\in C^\infty_{0*}(\boldsymbol{R}^m)$ and $\operatorname{Im}\zeta\neq 0$ we have

$$(R(\zeta)A^*u, A^*v)=\int_{\boldsymbol{R}^m}\frac{|\xi|^{\alpha-2\beta}(FMu)(\xi)\overline{(FMv)(\xi)}}{|\xi|^\alpha-\zeta}d\xi\,. \tag{4.5}$$

We take δ and K such that $0<\delta\ll 1<K$. Let $2\delta<\operatorname{Re}\zeta<K$. Splitting the integral of (4.5) into three parts $|\xi|^\alpha\le\delta$, $\delta<|\xi|^\alpha\le 2K$ and $|\xi|^\alpha>2K$, we have that

$$\begin{aligned}(R(\zeta)A^*u, A^*v)&=(AR(\zeta)E((2K,\infty))A^*u, v)\\ &\quad+\left(\int_0^{\delta^{1/\alpha}}+\int_{\delta^{1/\alpha}}^{(2K)^{1/\alpha}}\right)\frac{k^{\alpha-2\beta}(\Psi(k)u, \Psi(k)v)}{k^\alpha-\zeta}dk\,.\end{aligned}$$

Under the assumptions of Theorem 3 we can see that $A(1+|\Delta|^{\alpha/2})^{-1/2}$, $(1+|\Delta|^{\alpha/2})^{-1/2}$ and $A(1+|\Delta|^{\alpha/2})^{-1}A^*$ are in $B(H)$. The first term of the right hand side is

$$\begin{aligned}&[AR(\zeta)E((2K,\infty))A^*]\\ &=[A(1+|\Delta|^{\alpha/2})^{-1/2}]R(\zeta)(1+|\Delta|^{\alpha/2})E((2K,\infty))[(1+|\Delta|^{\alpha/2})^{-1/2}A^*]\,.\end{aligned}$$

Therefore the first term of the right hand side is in $B(H)$, and that is analytic in $\delta<\operatorname{Re}\zeta<K$. We shall estimate the remainder terms. Let $\zeta=\lambda\pm i\varepsilon$. Since $\Psi(k)^*\Psi(k)\in B(H)$, it is sufficient to prove that the following limits exist with respect to the operator norm of $B(H)$ when ε tends to 0:

$$\lim_{\varepsilon\downarrow 0}\left(\int_0^{\delta^{1/\alpha}}+\int_{\delta^{1/\alpha}}^{(2K)^{1/\alpha}}\right)\frac{k^{\alpha-2\beta}\Psi(k)^*\Psi(k)}{k^\alpha-\lambda\mp i\varepsilon}dk\,. \tag{4.6}$$

By the change of variable $k^\alpha=\mu$, (4.6) is equal to

$$\frac{1}{\alpha}\lim_{\varepsilon\downarrow 0}\left(\int_0^\delta+\int_\delta^{2K}\right)\frac{\mu^{(1-2\beta)/\alpha}\Psi(\mu^{1/\alpha})^*\Psi(\mu^{1/\alpha})}{\mu-\lambda\mp i\varepsilon}d\mu\,.$$

Since $\mu^{(1-2\beta)/\alpha}\Psi(\mu^{1/\alpha})^*\Psi(\mu^{1/\alpha})$ is locally Hölder continuous on $\boldsymbol{R}^+$, by Privalov's theorem, the second term converges in $B(H)$ and the limit operator is locally Hölder continuous. To estimate the first term, we handle the following three cases separately; $1/2<\beta<3/2$, $\beta=3/2$ and $\beta>3/2$. We use (4.4).

Case 1. $1/2<\beta<3/2$.

$$\|\mu^{(1-2\beta)/\alpha}\Psi(\mu^{1/\alpha})^*\Psi(\mu^{1/\alpha})\|\leq C\mu^{(1-2\beta)/\alpha}\mu^{2/\alpha(\beta-1/2)}=C\,.$$

Case 2. $\beta=3/2$. We take $0<s<1$. Then we have

$$\begin{aligned}\|\mu^{-2/\alpha}\Psi(\mu^{1/\alpha})^*\Psi(\mu^{-1/\alpha})\| &\leq C\mu^{-2/\alpha}\mu^{2/\alpha}|\log\mu^{1/\alpha}|^2\\ &=C\mu^{-s}(\mu^s|\log\mu^{1/\alpha}|^2)\\ &\leq C\mu^{-s}\,.\end{aligned}$$

Case 3. $\beta>3/2$.

$$\begin{aligned}\|\mu^{(1-2\beta)/\alpha}\Psi(\mu^{1/\alpha})^*\Psi(\mu^{1/\alpha})\| &\leq C\mu^{(1-2\beta)/\alpha}\mu^{2/\alpha}\\ &=C\mu^{(3-2\beta)/\alpha}\,.\end{aligned}$$

Since $\alpha\geq 1$ and $\alpha-2\beta\geq 0$, we know that $(3-2\beta)/\alpha>-1$.

Consequently for $\delta<\lambda<K$, the limit of the first term exists in all cases as $\varepsilon\downarrow 0$.

When $\lambda=0$, as is known, $\mathscr{F}\in B(H_s^r(\boldsymbol{R}^m);\,H_r^s(\boldsymbol{R}^m))$. Therefore

$$\begin{aligned}|(AR(\pm i\varepsilon)A^*u,\,u)| &\leq\int_{\boldsymbol{R}^m}|\xi|^{-2\beta}|(FMu)(\xi)|^2d\xi\\ &=\|MFMu\|^2\\ &\leq C\|u\|^2\,.\end{aligned}\tag{4.7}$$

When $\lambda<0$, it follows from (4.7) that

$$|(AR(\lambda\pm i\varepsilon)A^*u,\,u)|\leq C\|u\|^2\,.\tag{4.8}$$

In particular we obtain for $\zeta=\pm 1+i\varepsilon$

$$\sup_{\varepsilon\neq 0}\|[AR(\pm 1+i\varepsilon)A^*]\|<\infty\,.\tag{4.9}$$

The uniform boundedness (1.4) follows by scaling argument. (cf. T. Kato and K. Yajima [2].) We define $S(\rho)$ by

$$S(\rho)f(x)=\rho^{m/2}f(\rho x)\,,\qquad \rho>0\,.\tag{4.10}$$

$S(\rho)$ is unitary on H, $S(\rho)D(A^*)=D(A^*)$ and

$$[S(\rho)AS(\rho)^{-1}]=\rho^{m/2}[A]\,,\tag{4.11}$$

$$S(\rho)(|\Delta|^{\alpha/2}-\zeta)^{-1}S(\rho)^{-1}=\rho^{\alpha}(|\Delta|^{\alpha/2}-\rho^{\alpha}\zeta)^{-1}\,.\tag{4.12}$$

Combining (4.8)–(4.12), we have that

$$\sup_{\lambda\neq 0,\varepsilon\neq 0}\|[AR(\lambda+i\varepsilon)A^*]\| \tag{4.13}$$

$$=\sup\|[S(|\lambda|^{-1/\alpha})AR(\lambda+i\varepsilon)A^*S(|\lambda|^{1/\alpha})]\|$$

$$=\sup_{\varepsilon\neq 0}\|[AR(\pm 1+i\varepsilon)A^*]\|<\infty\ .$$

By (4.7) and (4.13) we obtain that $[A]$ is H-supersmooth.

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