TOKYO J. MATH.
VOL. 14, NO. 1, 1991

On the $\wp$-Zero Value Function and the $\wp$-Zero Division Value Functions

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(Communicated by K. Katase)

Introduction.

Let $\mathscr{H}$ be the upper half-plane $\{\tau\in \boldsymbol{C} \mid \mathrm{Im}\,\tau>0\}$ and $\tau\in\mathscr{H}$. Let $\wp(u,\tau)$ denote the Weierstrass $\wp$-function with fundamental periods $(\tau, 1)$, (in more usual notation, it should be written $\wp(u;\tau,1)$ or $\wp\left(u,\begin{pmatrix}\tau\\1\end{pmatrix}\right)$). As is well known, $\wp(u,\tau)$ is a holomorphic function of two complex variables u,τ in a suitable region $\subset \boldsymbol{C}\times\mathscr{H}$, and the theorem of implicit function shows that, given a suitable region $D\subset\mathscr{H}$, there exists a holomorphic function $u_D(\tau)$ of $\tau\in D$ such that $\wp(u_D(\tau),\tau)=0$ on D. This $u_D(\tau)$ is not uniquely determined by D. We shall show in this paper that there exists a unique analytic function u in $\mathscr{H}$, called "*$\wp$-zero value function*", such that every $u_D(\tau)$ are its branch on D (Theorem 1). This function u is a "many-valued modular form" in a sense to be indicated below. We shall show also in this paper the existence of another function $\mathfrak{p}_N$ of the same kind for an integer N greater than 1, which will be called "*N^{th} $\wp$-zero division value function*" (Theorem 2), and which is expected to have interesting arithmetical applications.

ACKNOWLEDGEMENT. The author wishes to thank Professors T. Mitsui and S. Iyanaga for their warm guidance and encouragement. He also wishes to thank Dr. K. Okutsu for his kind advice.

NOTATIONS AND TERMINOLOGIES. In this paper, the symbol ":=" means that the expression on the right is the definition of that on the left. We put

$$\Gamma:=SL_2(\boldsymbol{Z}),\quad U:=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad T:=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad I:=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Furthermore, for $z\in\boldsymbol{C}$, $S=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$, we set

$$Sz:=\frac{az+b}{cz+d},\qquad S:z:=cz+d\,.$$

Received March 27, 1990

For an integer k, $S \in \Gamma$ and a function f defined in a neighborhood of $\tau_0 \in \mathscr{H}$, we define $f|_k S$ as the function defined in the neighborhood of $S^{-1}\tau_0$ as follows:

$$(f|_k S)(\tau) := (S : \tau)^{-k} f(S\tau) .$$

A function element is a pair (f, D) such that D is a region in $\boldsymbol{C}$ and f is a holomorphic function in D. An analytic function on $\mathscr{H}$ means a set of function elements (f, D), called branches of the analytic function, such that $D \subset \mathscr{H}$ and for any two function elements (f_1, D_1), (f_2, D_2) in the set there exists a curve γ in $\mathscr{H}$ such that (f_2, D_2) is an analytic continuation of (f_1, D_1) along γ, the union of all D's in the set coinciding with $\mathscr{H}$ except for a discrete set, and that this set is maximal in the sense that every function element satisfying the above condition belongs to the set.

§ 1. Definition of the N^{th} $\wp$-zero division value functions.

In this section, we assume that $\omega_1, \omega_2 \in \boldsymbol{C}$, ω_1/ω_2, $\tau \in \mathscr{H}$ and N is a positive integer. We define as usual, for $z \in \boldsymbol{C}$,

$$\wp\left(z, \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}\right) := \frac{1}{z^2} + \sum_{\substack{\omega \in \boldsymbol{Z}\omega_1 + \boldsymbol{Z}\omega_2 \\ \omega \neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\},$$

$$\sigma\left(z, \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}\right) := z \prod_{\substack{\omega \in \boldsymbol{Z}\omega_1 + \boldsymbol{Z}\omega_2 \\ \omega \neq 0}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

We write simply $\wp(z, \tau)$, $\sigma(z, \tau)$ instead of $\wp\left(z, \begin{pmatrix} \tau \\ 1 \end{pmatrix}\right)$, $\sigma\left(z, \begin{pmatrix} \tau \\ 1 \end{pmatrix}\right)$ respectively. We set $\wp'(z, \tau) := (\partial/\partial z)\wp(z, \tau)$. ($\wp(z, \tau)$ is the same expression that was already given in the Introduction.)

DEFINITION. We define two functions on $\boldsymbol{C} \times \mathscr{H}$ as follows:

$$\Lambda_N(z, \tau) := \sigma(Nz, \tau)^2 / \sigma(z, \tau)^{2N^2} ,$$

$$\Phi_N(z, \tau) := \wp(Nz, \tau)\Lambda_N(z, \tau) .$$

We know that $\Lambda_N(z, \tau), \Phi_N(z, \tau) \in \boldsymbol{Z}[15G_4(\tau), 35G_6(\tau)][\wp(z, \tau)]$, where

$$G_4(\tau) := \sum_{\substack{\omega \in \boldsymbol{Z}\tau + \boldsymbol{Z} \\ \omega \neq 0}} \frac{1}{\omega^4}, \qquad G_6(\tau) := \sum_{\substack{\omega \in \boldsymbol{Z}\tau + \boldsymbol{Z} \\ \omega \neq 0}} \frac{1}{\omega^6} .$$

Let $\lambda_N(X, \tau), \phi_N(X, \tau) \in \boldsymbol{Z}[15G_4(\tau), 35G_6(\tau)][X]$ such that

$$\lambda_N(\wp(z, \tau), \tau) = \Lambda_N(z, \tau) , \qquad \phi_N(\wp(z, \tau), \tau) = \Phi_N(z, \tau) .$$

λ_N, ϕ_N have the degrees N^2-1, N^2 in X, respectively. Moreover, we know that N^2-1 roots of λ_N are

$$\left\{\wp\left(\frac{1}{N}(a, b)\begin{pmatrix}\tau\\1\end{pmatrix}, \tau\right) \,\middle|\, a, b\in \boldsymbol{Z},\ 0\leqq a, b<N,\ (a, b)\neq(0, 0)\right\}$$

(cf. Cassels [1]).

The following two lemmas follow easily from the well known properties of $\wp$-function and σ-function.

LEMMA 1. *We fix $\tau\in\mathscr{H}$. Let $\Delta_\tau:=\{\mu_1\tau+\mu_2 \mid 0\leqq\mu_1, \mu_2<1\}$. Then the function $z\mapsto\Phi_N(z, \tau)$ is an elliptic function of order $2N^2$ with fundamental periods $(\tau, 1)$ and*

$$\left\{\frac{1}{N}(\alpha+a, \beta+b)\begin{pmatrix}\tau\\1\end{pmatrix}, \frac{1}{N}(\alpha'+a', \beta'+b')\begin{pmatrix}\tau\\1\end{pmatrix} \,\middle|\, \begin{matrix} a, b, a', b'\in \boldsymbol{Z} \\ 0\leqq a, b, a', b'<N\end{matrix}\right\}$$

is the set of all zeros of Φ_N in Δ_τ where $\alpha\tau+\beta$, $\alpha'\tau+\beta'$ are two zeros of $\wp(z, \tau)$ in Δ_τ $(0\leqq\alpha, \beta, \alpha', \beta'<1)$.

LEMMA 2. *We fix $\tau\in\mathscr{H}$. Let $\alpha, \beta\in\boldsymbol{R}$ such that $\wp(\alpha\tau+\beta, \tau)=0$. Then the following N^2 elements are all roots of the polynomial $\phi_N(X, \tau)$ in X:*

$$\wp\left(\frac{1}{N}(\alpha+a, \beta+b)\begin{pmatrix}\tau\\1\end{pmatrix}, \tau\right), \qquad a, b\in\boldsymbol{Z},\quad 0\leqq a, b<N .$$

Hereafter, we assume $N>1$.

Let $D(\phi_N)(\tau)$ be the discriminant of the polynomial $\phi_N(X, \tau)$ in X. Take $\tau_0\in\mathscr{H}$ and choose $\alpha, \beta\in\boldsymbol{R}$ such that $\wp(\alpha\tau_0+\beta, \tau_0)=0$. It is easy to see that $D(\phi_N)(\tau_0)=0$ is equivalent to $2\alpha, 2\beta\in\boldsymbol{Z}$. On the other hand, we have $\lambda_2(X, \tau_0)=4X^3-60G_4(\tau_0)X-140G_6(\tau_0)$, and so we find $\tau_0\in\Gamma\sqrt{-1}$ if and only if $2\alpha, 2\beta\in\boldsymbol{Z}$ since $\tau_0\in\Gamma\sqrt{-1}$ if and only if $G_6(\tau_0)=0$. Therefore $\tau_0\in\Gamma\sqrt{-1}$ is equivalent to $D(\phi_N)(\tau_0)=0$. Hence, from the implicit function theorem, there exists an analytic function on $\mathscr{H}$ such that $\phi_N(g(\tau), \tau)=0$ on D for a branch (g, D) of it. Moreover, by above arguments, we can express $\phi_N(X, \tau)$ at $\tau_0\in\Gamma\sqrt{-1}$ as

$$(\sharp)\qquad \phi_N(X, \tau_0)=\begin{cases} X\displaystyle\prod_{i=1}^{(N^2-1)/2}(X-\alpha_{\tau_0,i}^{(N)})^2 & \text{(for odd } N\text{)}\\ \displaystyle\prod_{i=1}^{N^2/2}(X-\alpha_{\tau_0,i}^{(N)})^2 & \text{(for even } N\text{)}\end{cases}$$

$$(\alpha_{\tau_0,i}^{(N)}\neq0,\quad \alpha_{\tau_0,i}^{(N)}\neq\alpha_{\tau_0,j}^{(N)}\quad \text{for } i\neq j).$$

Now, for $(a, b)\in\boldsymbol{Z}^2$ and $(a, b)\not\equiv(0, 0)$ mod N, the function

$$\wp_{N,(a,b)}(\tau):=\wp\left(\frac{1}{N}(a, b)\begin{pmatrix}\tau\\1\end{pmatrix}, \tau\right)$$

is an entire modular form of weight 2 for $\Gamma[N]$, where

$$\Gamma[N]:=\{S\in\Gamma \mid S\equiv I \bmod N \text{ or } S\equiv -I \bmod N\}.$$

$\wp_{N,(a,b)}$ is called the N^{th} $\wp$-division value. It is a value of $\wp(z,\tau)$ for z = an "N-division point of a pole of $\wp(z,\tau)$". In analogy, we shall consider "N^{th} $\wp$-zero division value function" defined as follows:

DEFINITION. We call an analytic function on $\mathscr{H}$ such that $\phi_N(g(\tau),\tau)=0$ on D for a branch (g, D) of it as N^{th} $\wp$-zero division value function, and denote it by $\mathfrak{p}_N$.

We notice that at present it is not clear that $\mathfrak{p}_N$ is uniquely determined: we shall show later that it is. Lemma 2 shows that it is appropriate to call $\mathfrak{p}_N$ as N^{th} $\wp$-zero division value function.

§2. The zeros of the Weierstrass $\wp$-function.

Since $\tau_0\in\Gamma\sqrt{-1}$ is equivalent to $D(\phi_N)(\tau_0)=0$, the set of all ramification points of $\mathfrak{p}_N$ is contained in $\Gamma\sqrt{-1}$. Moreover, noticing (#), we obtain the following lemma:

LEMMA 3. *The degree of ramification of* $\mathfrak{p}_N$ *at* $\tau_0\in\Gamma\sqrt{-1}$ *is at most* 1.

Now we consider the case $N=2$. Let $\tau_0\in\Gamma\sqrt{-1}$ and D be a neighborhood of τ_0. By the above lemma, we can develop an "algebraic element" g of $\mathfrak{p}_2$ around τ_0 in fractional power series as follows in D:

$$g(\tau)=c_0+c_1(\tau-\tau_0)^{d_1}+\cdots+c_n(\tau-\tau_0)^{d_n}+\cdots$$

$$(2d_n\in \boldsymbol{Z},\quad d_n>0,\quad d_n<d_m \text{ for } n<m,\quad c_0\neq 0).$$

Since $\phi_2(g(\tau),\tau)=0$ on D, substituting the development of g and

$$G_4(\tau)=a_0+a_1(\tau-\tau_0)+\cdots \qquad (a_0\neq 0),$$

$$G_6(\tau)=\qquad b_1(\tau-\tau_0)+\cdots \qquad (b_1\neq 0)$$

in

$$\phi_2(X,\tau)=(X^2+15G_4(\tau))^2+280G_6(\tau)X,$$

we have $d_1=1/2$. Thus we obtain the following lemma:

LEMMA 4. *For any* $\tau_0\in\Gamma\sqrt{-1}$ *and any branch* g *of* $\mathfrak{p}_N$, g *ramifies at* τ_0.

Let $z_0\in \boldsymbol{C}$, $\tau_0\in\mathscr{H}$ and $\wp(z_0,\tau_0)=0$. Since $\wp'(z,\tau)^2=\Lambda_2(z,\tau)$, $\wp'(z_0,\tau_0)=0$ is equivalent to $\tau_0\in\Gamma\sqrt{-1}$. Therefore any function element (u_D, D) such that $D\subset\mathscr{H}$ and $\wp(u_D(\tau),\tau)=0$ on D can be continued analytically along a curve $\subset\mathscr{H}-\Gamma\sqrt{-1}$ with an initial point in D. Hence there exists an analytic function u on $\mathscr{H}$ such that $\wp(u_1(\tau),\tau)=0$ on D_1 for any branch (u_1, D_1) of u. We fix such a function u.

The following proposition gives a precision of an argument found in Eichler, Zagier [2].

PROPOSITION 5. (1) *The set of all ramification points of u is $\Gamma\sqrt{-1}$. Particularly, any branch of u ramifies at $\tau_0 \in \Gamma\sqrt{-1}$.*

(2) *Let $\tau_0 \in \Gamma\sqrt{-1}$ and (u_1, D_1) be a branch of u such that $D_1 \cap \Gamma\sqrt{-1} = \emptyset$ and $\tau_0 \in \overline{D_1}$ where $\overline{D_1}$ is the closure of D_1 in $\mathcal{H}$. And let $l_1, l_2 \in \boldsymbol{R}$ such that*

$$\lim_{\substack{\tau \to \tau_0 \\ \tau \in D_1}} u_1(\tau) = \frac{l_1}{2}\tau_0 + \frac{l_2}{2}$$

(Hereafter we write simply $u_1(\tau_0)$ instead of $\lim_{\tau \to \tau_0, \tau \in D_1} u_1(\tau)$). Then $\tau_0 \in \Gamma[2]\sqrt{-1}$ if and only if l_1, l_2 are odd integers, $\tau_0 \in \Gamma[2](\sqrt{-1}+1)$ if and only if l_1 is odd and l_2 is even, and $\tau_0 \in \Gamma[2](\sqrt{-1}-1)/2$ if and only if l_1 is even and l_2 is odd. Moreover let $\tau_1 \in D_1$ sufficiently close to τ_0, and γ be the circle of center τ_0 through τ_1. Then, considering γ as a simple closed curve with the initial point τ_1, the branch $(u_1(\tau), D_1)$ is continued analytically to the function element $(-u_1(\tau)+l_1\tau+l_2, D_1)$ along γ, and τ_0 is an algebraic singularity of u with the degree of ramification 1.

PROOF. It is clear that $\Gamma\sqrt{-1}$ contains all ramification points of u. Suppose that u does not ramify at some $\tau_0 \in \Gamma\sqrt{-1}$. By assumption, there exists a branch (u_2, D_2) of u such that $\tau_0 \in D_2$. Then

$$\begin{aligned}\phi_2(X, \tau) = &\left(X - \wp\left(\frac{u_2(\tau)}{2}, \tau\right)\right)\left(X - \wp\left(\frac{u_2(\tau)+\tau}{2}, \tau\right)\right) \\ &\times \left(X - \wp\left(\frac{u_2(\tau)+1}{2}, \tau\right)\right)\left(X - \wp\left(\frac{u_2(\tau)+\tau+1}{2}, \tau\right)\right)\end{aligned}$$

on D_2 by Lemma 2. Therefore $\mathfrak{p}_2$ does not ramify at τ_0. This contradicts Lemma 4. Hence (1) holds.

Next, we shall prove (2). Let $\tau \in \mathcal{H}$, $A \in \Gamma$. If we choose $\alpha, \beta, \alpha', \beta' \in \boldsymbol{R}$ satisfying $\wp(\alpha\tau+\beta, \tau) = 0$ and

$$(\alpha', \beta') \equiv (\alpha, \beta)A^{-1} \quad \text{or} \quad -(\alpha, \beta)A^{-1} \quad \bmod \boldsymbol{Z},$$

then

$$\wp(\alpha' A\tau + \beta', A\tau) = (A : \tau)^2 \wp(\alpha\tau+\beta, \tau),$$

therefore $\wp(\alpha' A\tau+\beta', A\tau) = 0$. Consequently, noticing that the constant term of $\lambda_2(X, \tau)$ as a polynomial in X is $-140G_6(\tau)$, $G_6(\sqrt{-1}) = 0$ and that $\wp$ is an elliptic function of order 2, we get $\wp((1/2)\sqrt{-1}+1/2, \sqrt{-1}) = 0$. Moreover $\sqrt{-1}+1 = U\sqrt{-1}$ and $(1/2, 1/2)U^{-1} = (1/2, 0)$, therefore $\wp((1/2)(\sqrt{-1}+1), \sqrt{-1}+1) = 0$. Similarly, since $(\sqrt{-1}-1)/2 = TU\sqrt{-1}$ and $(1/2, 1/2)(TU)^{-1} = (0, 1/2)$, we obtain $\wp(1/2, (\sqrt{-1}-1)/2) = 0$. Moreover

$$\left(\frac{1}{2}, \frac{1}{2}\right)S^{-1} \equiv \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \bmod \boldsymbol{Z},$$

$$\left(\frac{1}{2}, 0\right)S^{-1} \equiv \left(\frac{1}{2}, 0\right) \qquad \bmod \boldsymbol{Z},$$

$$\left(0, \frac{1}{2}\right)S^{-1} \equiv \left(0, \frac{1}{2}\right) \qquad \bmod \boldsymbol{Z}$$

for $S \in \Gamma[2]$. Hence the first part of (2) is proved.

Let N be odd. In virtue of (#), there exist $a, b \in \boldsymbol{Z}$ such that

$$\wp\left(\frac{1}{N}(u_1(\tau_0) + a\tau_0 + b), \tau_0\right) = 0 .$$

Since $u_1(\tau_0) = (l_1/2)\tau_0 + l_2/2$ and

$$\frac{1}{N}(u_1(\tau_0) + a\tau_0 + b) \equiv u_1(\tau_0) \text{ or } -u_1(\tau_0) \qquad \bmod \boldsymbol{Z}\tau_0 + \boldsymbol{Z},$$

we obtain

$$(a, b) \equiv \left(\frac{l_1}{2}(N-1), \frac{l_2}{2}(N-1)\right) \qquad \bmod N .$$

Hence

$$g(\tau) := \wp\left(\frac{1}{N}\left(u_1(\tau) + \frac{l_1}{2}(N-1)\tau + \frac{l_2}{2}(N-1)\right), \tau\right)$$

does not ramify at τ_0 for $g(\tau_0)$ is a simple root of $\phi_N(X, \tau_0)$. Now let (u_2, D_1) be a function element such that (u_1, D_1) is continued analytically to (u_2, D_1) along γ as above, and let $\alpha_1(\tau), \beta_1(\tau), \alpha_2(\tau), \beta_2(\tau)$ be real valued functions defined in D_1 such that $u_1(\tau) = \alpha_1(\tau)\tau + \beta_1(\tau)$, $u_2(\tau) = \alpha_2(\tau)\tau + \beta_2(\tau)$ on D_1. Since g does not ramify at τ_0, we obtain

$$\begin{aligned}
&\left(\alpha_1(\tau) + \frac{l_1}{2}(N-1), \beta_1(\tau) + \frac{l_2}{2}(N-1)\right) \\
&\equiv \left(\alpha_2(\tau) + \frac{l_1}{2}(N-1), \beta_2(\tau) + \frac{l_2}{2}(N-1)\right) \\
&\text{or } -\left(\alpha_2(\tau) + \frac{l_1}{2}(N-1), \beta_2(\tau) + \frac{l_2}{2}(N-1)\right) \qquad \bmod N
\end{aligned}$$

for any $\tau \in D_1$. Assume that there exists $\tau_2 \in D_1$ such that the set of all odd numbers N satisfying

$$\left(\alpha_1(\tau_2)+\frac{l_1}{2}(N-1),\ \beta_1(\tau_2)+\frac{l_2}{2}(N-1)\right)$$
$$\equiv-\left(\alpha_2(\tau_2)+\frac{l_1}{2}(N-1),\ \beta_2(\tau_2)+\frac{l_2}{2}(N-1)\right) \qquad \mathrm{mod}\ N$$

is finite. Then, for this τ_2, the set of all odd numbers satisfying

$$(\alpha_1(\tau_2),\ \beta_1(\tau_2))\equiv(\alpha_2(\tau_2),\ \beta_2(\tau_2)) \qquad \mathrm{mod}\ N$$

is infinite. Therefore $(\alpha_1(\tau_2), \beta_1(\tau_2))=(\alpha_2(\tau_2), \beta_2(\tau_2))$, and hence $u_1(\tau_2)=u_2(\tau_2)$. Moreover $\wp'(u_1(\tau_2), \tau_2)\neq 0$ because $u_1(\tau_2)$ is not a 2-division point of τ_2 for $\tau_2\notin\Gamma\sqrt{-1}$. Consequently, from uniqueness part of the implicit function theorem, $u_1(\tau)=u_2(\tau)$ on D_1. This contradicts the fact that u_1 ramifies at τ_0. Thus, for any $\tau\in D_1$ the set of all odd numbers N satisfying

$$\left(\alpha_1(\tau)+\frac{l_1}{2}(N-1),\ \beta_1(\tau)+\frac{l_2}{2}(N-1)\right)$$
$$\equiv-\left(\alpha_2(\tau)+\frac{l_1}{2}(N-1),\ \beta_2(\tau)+\frac{l_2}{2}(N-1)\right) \qquad \mathrm{mod}\ N$$

is infinite. By a similar argument, we obtain

$$(\alpha_2(\tau),\ \beta_2(\tau))=(-\alpha_1(\tau)+l_1,\ -\beta_1(\tau)+l_2)$$

on D_1. Hence $u_2(\tau)=-u_1(\tau)+l_1\tau+l_2$ on D_1. ■

§ 3. The main theorem on the zeros of $\wp$-function.

Our main theorem on the zeros of $\wp$-function states as follows:

THEOREM 1. *Let D_1, D_2 be two regions in $\mathscr{H}$ and (u_1, D_1), (u_2, D_2) be function elements such that $\wp(u_1(\tau), \tau)=0$ on D_1 and $\wp(u_2(\tau), \tau)=0$ on D_2. Then (u_1, D_1) can be continued analytically to (u_2, D_2) in $\mathscr{H}$. And, for any $S\in\Gamma$, $(u_1|_{-1}S, S^{-1}D_1)$ is another function element which can be continued analytically to (u_1, D_1) in $\mathscr{H}$.*

Notice that since $\wp((u_1|_{-1}S)(\tau), \tau)=(S:\tau)^{-2}\wp(u(S\tau), S\tau)$ for all $\tau\in S^{-1}D_1$, $(u_1|_{-1}S, S^{-1}D_1)$ is a function element such that $\wp((u_1|_{-1}S)(\tau), \tau)=0$ on $S^{-1}D_1$, and hence the latter part of Theorem 1 follows from the first part.

In order to prove Theorem 1, we show the following 4 lemmas.

LEMMA 6. *Let D_1, D_2 be two regions in $\mathscr{H}$ and (u_1, D_1), (u_2, D_2) be function elements such that $\wp(u_1(\tau), \tau)=0$ on D_1 and $\wp(u_2(\tau), \tau)=0$ on D_2. Then the following propositions hold:*

(1) *If $D_1\cap D_2\neq\varnothing$, then there exist uniquely $\varepsilon\in\{\pm1\}$, m, $n\in\boldsymbol{Z}$ such that*

$u_1(\tau)=\varepsilon u_2(\tau)+m\tau+n$ for all $\tau\in D_1\cap D_2$.

(2) *Let γ be a curve with an initial point in D_1 and a terminal point in D_2, and $\gamma\subset\mathscr{H}-\Gamma\sqrt{-1}$. Then there exist uniquely $\varepsilon\in\{\pm 1\}$, m, $n\in\boldsymbol{Z}$ such that $(u_1(\tau), D_1)$ is continued analytically to $(\varepsilon u_2(\tau)+m\tau+n, D_2)$ along γ.*

PROOF. (1) Since $D_1\cap\Gamma\sqrt{-1}=\varnothing$ and $D_2\cap\Gamma\sqrt{-1}=\varnothing$ by Proposition 5 (1), $\wp'(u_1(\tau),\tau)\neq 0$ for all $\tau\in D_1$ and $\wp'(u_2(\tau),\tau)\neq 0$ for all $\tau\in D_2$. Therefore we can easily see (1) from the properties of $\wp$-function and uniqueness part of the implicit function theorem.

(2) Let (u_3, D_3) be a function element such that D_3 contains the terminal point of γ and (u_1, D_1) is continued analytically to (u_3, D_3) along γ. Since $\gamma\subset\mathscr{H}-\Gamma\sqrt{-1}$, $\wp(u_3(\tau),\tau)=0$ on D_3 by the theorem of invariance of analytic relations. Therefore from (1), we have (2). ■

We put $\mathscr{H}_1:=\{\tau\in\mathscr{H}\mid \operatorname{Im}\tau>1\}$ and $\overline{\mathscr{H}_1}:=\{\tau\in\mathscr{H}\mid \operatorname{Im}\tau\geqq 1\}$.
The following lemma is due to Professor D. Zagier [4].

LEMMA 7. *There exists a unique function u_0 satisfying the following conditions:*

$$u_0 \text{ is continuous on } \overline{\mathscr{H}_1} \text{ and holomorphic on } \mathscr{H}_1\,, \tag{7.1}$$

$$\wp(u_0(\tau),\tau)=0 \qquad \text{for all} \quad \tau\in\overline{\mathscr{H}_1}\,, \tag{7.2}$$

$$u_0(\tau+1)=u_0(\tau) \qquad \text{for all} \quad \tau\in\overline{\mathscr{H}_1}\,, \tag{7.3}$$

$$u_0(\sqrt{-1})=\frac{1}{2}\sqrt{-1}+\frac{1}{2}\,. \tag{7.4}$$

PROOF. Let

$$\Delta(\tau):=\exp(2\pi\sqrt{-1}\tau)\prod_{n=1}^{\infty}(1-\exp(2n\pi\sqrt{-1}\tau))^{24}\,,$$

$$E_6(\tau):=\frac{945}{2\pi^6}G_6(\tau)\,,$$

and put

$$u_0(\tau):=\frac{1}{2}+\left(\frac{\log(5+2\sqrt{6})}{2\pi}-144\pi\sqrt{6}\int_{\tau}^{i\infty}(t-\tau)\frac{\Delta(t)}{E_6(t)^{3/2}}dt\right)\sqrt{-1} \qquad (\tau\in\mathscr{H})\,,$$

where the integral is to be taken over the vertical line $t=\tau+\sqrt{-1}\boldsymbol{R}_+$ in $\mathscr{H}$ ($\boldsymbol{R}_+:=\{\beta\in\boldsymbol{R}\mid\beta>0\}$). The following theorem is given by Eichler, Zagier [2].

"*The zeros of $\wp(z,\tau)$ ($\tau\in\mathscr{H}$, $z\in\boldsymbol{C}$) are given by $z=\pm u_0(\tau)+m\tau+n$ ($m, n\in\boldsymbol{Z}$)*"

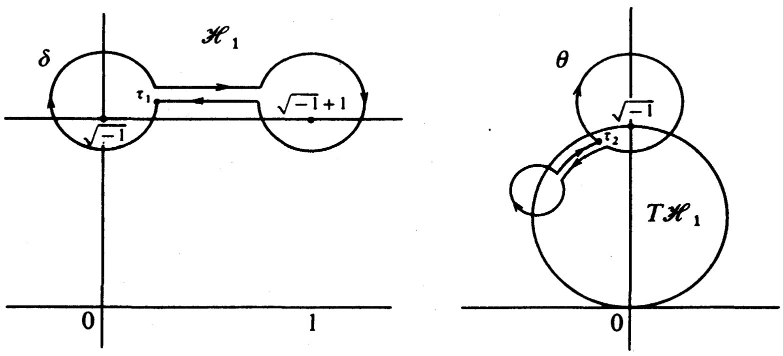
Thus it is clear that u_0 satisfies the condition (7.2) and it is easy to see that u_0 satisfies the condition (7.1), (7.3), therefore we have only to show $u_0(\sqrt{-1})=(1/2)\sqrt{-1}+1/2$.

Let $z(s):=u_0(\sqrt{-1}s)$ and

$$\beta(s):=\frac{\log(5+2\sqrt{6})}{2\pi}+144\pi\sqrt{6}\int_s^{\infty}(t-s)\frac{\Delta(\sqrt{-1}t)}{E_6(\sqrt{-1}t)^{3/2}}dt\,.$$

Then $z(s)=(1/2)+\beta(s)\sqrt{-1}$ on $\{s\in\boldsymbol{R}\mid s>0\}$, and $\beta(s)$ is a real, positive, monotone decreasing and continuous function on $\{s\in\boldsymbol{R}\mid s\geqq 1\}$. Here, notice that "$\wp(z_0,\tau)=0$ implies $\tau\in\Gamma\sqrt{-1}$" if and only if $z_0\in(\tau/2)\boldsymbol{Z}+(1/2)\boldsymbol{Z}$. As $\wp(z(1),\sqrt{-1})=0$, there is a positive integer N_0 such that $\beta(1)=N_0/2$. Assume $N_0>1$. Since $\beta(s)/s$ is continuous on $\{s\in\boldsymbol{R}\mid s\geqq 1\}$, $\lim_{s\to\infty}\beta(s)/s=0$ and $\beta(1)/1=N_0/2$, there exists $s_0>1$ such that $\beta(s_0)/s_0=1/2$. For this s_0, we have $\wp(s_0\sqrt{-1}/2+1/2,s_0\sqrt{-1})=0$. Therefore $s_0\sqrt{-1}\in\Gamma\sqrt{-1}$. This contradicts $s_0>1$. Hence $N_0=1$ and we obtain $u_0(\sqrt{-1})=z(1)=(1/2)\sqrt{-1}+1/2$. ■

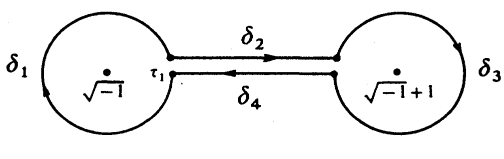
We fix now τ_1 sufficiently close to $\sqrt{-1}$ such that $\operatorname{Im}\tau_1>1$ and $\operatorname{Re}\tau_1>0$, and put $\tau_2=T\tau_1$. Let δ, θ be closed curves with initial points τ_1, τ_2 respectively as shown in the figures:



Let u_0 be the function of Lemma 7.

Lemma 8. *$u_0(\tau)$ is continued analytically to $u_0(\tau)+1$ along δ.*

Proof. We split δ into 4 curves δ_1, δ_2, δ_3, δ_4 as shown in the figure:



$(\delta=\delta_1+\delta_2+\delta_3+\delta_4)$

First we consider the curve δ_1. Since $u_0(\sqrt{-1})=(1/2)\sqrt{-1}+1/2$, $u_0(\tau)$ is continued analytically to $-u_0(\tau)+\tau+1$ along δ_1 by Proposition 5 (2). Next since $u_0(\tau)$ is holomorphic on $\mathscr{H}_1$, $-u_0(\tau)+\tau+1$ is continued analytically to $-u_0(\tau)+\tau+1$ along δ_2. Furthermore since $u_0(\tau+1)=u_0(\tau)$ for all $\tau\in\mathscr{H}_1$, $u_0(\sqrt{-1}+1)=(1/2)(\sqrt{-1}+1)+0/2$, and hence, again by Proposition 5 (2), $-u_0(\tau)+\tau+1$ is continued analytically to $u_0(\tau)+1$

along δ_3. Finally, by the same reason as for δ_2, $u_0(\tau)+1$ is continued analytically to $u_0(\tau)+1$ along δ_4. This completes the proof of the lemma. ■

LEMMA 9. *There exists a closed curve χ in $\mathscr{H}$ with the initial point τ_1 such that $u_0(\tau)$ is continued analytically to $u_0(\tau)+\tau$ along χ.*

PROOF. Let $u_4(\tau):=(u_0|_{-1}T)(\tau)$. Then $(u_4, T\mathscr{H}_1)$ is the function element satisfying $\wp(u_4(\tau), \tau)=0$ on $T\mathscr{H}_1$. Since $u_4(\sqrt{-1})=(1/2)\sqrt{-1}-1/2$ and $u_4((\sqrt{-1}-1)/2)=(0/2)((\sqrt{-1}-1)/2)-1/2$, we see, in a similar way as in the proof of Lemma 8, that $u_4(\tau)$ is continued analytically to $u_4(\tau)+\tau$ along θ. Now, let θ_1 be a curve with the initial point τ_1 and the terminal point τ_2, and $\theta_1\subset\mathscr{H}-\Gamma\sqrt{-1}$. Then, by Lemma 6 (2), there exist uniquely $\varepsilon\in\{\pm 1\}$, $m, n\in \boldsymbol{Z}$ such that $u_0(\tau)$ is continued analytically to $\varepsilon u_4(\tau)+m\tau+n$ along θ_1. Thus, putting $\chi:=\theta_1+\theta+(-\theta_1)$, $u_0(\tau)$ is continued analytically to $u_0(\tau)+\tau$ along χ. This χ satisfies the conditions of the lemma. ■

PROOF OF THEOREM 1. Let γ_1 be a curve with an initial point in D_1 and the terminal point τ_1, and let γ_2 be a curve with an initial point in D_2 and the terminal point τ_1. Let χ be a curve satisfying the conditions of Lemma 9. By Lemma 6 (2), there exist uniquely $\varepsilon_1, \varepsilon_2\in\{\pm 1\}$, $m_1, m_2, n_1, n_2\in\boldsymbol{Z}$ such that $u_1(\tau)$ is continued analytically to $\varepsilon_1 u_0(\tau)+m_1\tau+n_1$ along γ_1 and $u_2(\tau)$ is continued analytically to $\varepsilon_2 u_0(\tau)+m_2\tau+n_2$ along γ_2. Let δ_1 be the closed curve in the proof of Lemma 8. Replacing γ_1 by $\gamma_1+\delta_1$ if necessary, we may assume $\varepsilon_1=1$ by Proposition 5 (2), and we may assume $\varepsilon_2=1$ by the same reason. Thus, putting $\gamma:=\gamma_1+(n_2-n_1)\delta+(m_2-m_1)\chi+(-\gamma_2)$, $u_1(\tau)$ is continued analytically to $u_2(\tau)$ along γ. ■

COROLLARY 10. *There exists a unique analytic function u on $\mathscr{H}$ such that $\wp(u_1(\tau), \tau)=0$ on D_1 for a branch (u_1, D_1) of u.*

This analytic function u will be called "*the $\wp$-zero value function*", it is many-valued with countably infinitely many values. The latter part of Theorem 1 shows its "*modular invariance*". In this sense it is called a "*many-valued modular form*".

§4. The main theorem on the $\wp$-zero division value functions.

Let (u_1, D_1) be a branch of our $\wp$-zero value function u. For a positive integer N and integers a, b, we put

$$g_{N,(a,b)}(\tau):=\wp\left(\frac{1}{N}\left(u_1(\tau)+(a, b)\begin{pmatrix}\tau\\1\end{pmatrix}\right), \tau\right).$$

Then, for any (a, b), $(a', b')\in\boldsymbol{Z}^2$, $(u_1(\tau), D_1)$ can be continued analytically to $\left(u_1(\tau)+(a'-a, b'-b)\begin{pmatrix}\tau\\1\end{pmatrix}, D_1\right)$ in $\mathscr{H}$ by Theorem 1, therefore the function element $(g_{N,(a,b)}, D_1)$

can be also continued analytically to $(g_{N,(a',b')}, D_1)$ in $\mathscr{H}$. And we have $\phi_N(g_{N,(a,b)}(\tau), \tau)=0$ for all $\tau \in D_1$ by Lemma 2. Moreover, for $S \in \Gamma$, since

$$(\#\#) \qquad (g_{N,(a,b)}|_2 S)(\tau)=(S:\tau)^{-2}\wp\left(\frac{1}{N}\left(u_1(S\tau)+(a,b)\begin{pmatrix} S\tau \\ 1 \end{pmatrix}\right), S\tau\right)$$
$$=\wp\left(\frac{1}{N}\left((u_1|_{-1}S)(\tau)+(a,b)S\begin{pmatrix} \tau \\ 1 \end{pmatrix}\right), \tau\right)$$

for all $\tau \in S^{-1}D_1$ and $\wp((u_1|_{-1}S)(\tau), \tau)=0$ on $S^{-1}D_1$, the function element $(g_{N,(a,b)}|_2 S, S^{-1}D_1)$ can be continued analytically to $(g_{N,(a,b)S}, D_1)$ in $\mathscr{H}$.

Therefore we obtain the following theorem.

THEOREM 2. *Let N be an integer greater than 1. Then an N^{th} $\wp$-zero division value function $\mathfrak{p}_N$ is uniquely determined and it is an N^2-valued analytic function of $\mathscr{H}$. Moreover, for a branch (g, D) of $\mathfrak{p}_N$ and $S \in \Gamma$, another function element $(g|_2 S, S^{-1}D)$ is also a branch of $\mathfrak{p}_N$.*

This theorem shows that $\mathfrak{p}_N$ is another "*many-valued modular form*" like u.

COROLLARY 11. *Let N be an integer greater than 1. Then $\phi_N(X, \tau)$ is an irreducible polynomial in $\boldsymbol{Z}[15G_4(\tau), 35G_6(\tau)][X]$.*

PROOF. Since any root of $\phi_N(X, \tau)$ is expressed by a branch of $\mathfrak{p}_N$, this follows from Theorem 2. ■

COROLLARY 12. *Let p be a prime number, and $(g_1, D), \cdots, (g_{p^2}, D)$ be p^2 branches of $\mathfrak{p}_p$. Let $\alpha_1, \cdots, \alpha_{p^2} \in \boldsymbol{C}$. Then*

$$\alpha_1 g_1 + \cdots + \alpha_{p^2} g_{p^2} = 0 \qquad \textit{on } D$$

if and only if $\alpha_1 = \cdots = \alpha_{p^2}$. Therefore p^2-1 distinct branches of $\mathfrak{p}_p$ with the same region are linearly independent over $\boldsymbol{C}$.

PROOF. Since the second term of the polynomial $\phi_N(X, \tau)$ in X vanishes (cf. Cassels [1]), we get $\alpha_1 g_1 + \cdots + \alpha_{p^2} g_{p^2} = 0$ if $\alpha_1 = \cdots = \alpha_{p^2}$.

As shown in the first part of this section,

$$\{g_{(a,b)} \mid 0 \leqq a, b < p, a, b \in \boldsymbol{Z}\} \qquad (g_{(a,b)} := g_{p,(a,b)})$$

is the set of all branches of $\mathfrak{p}_p$ on D_1. Therefore it suffices to show that $\alpha_1 g_{(0,0)} + \alpha_2 g_{(0,1)} + \cdots + \alpha_{p^2} g_{(p-1,p-1)} = 0$ on D_1 implies $\alpha_1 = \cdots = \alpha_{p^2}$. We set $\boldsymbol{F} := \boldsymbol{Z}/p\boldsymbol{Z}$ and $\hat{G} := SL_2(\boldsymbol{F})$. $\hat{G}$ acts on $\boldsymbol{F}^2 - \{(\bar{0}, \bar{0})\}$ by $(\bar{S}, (\bar{a}, \bar{b})) \mapsto (\bar{a}, \bar{b})\bar{S}$ where $\bar{a} := a \bmod p$, and this action is transitive. We know that $\hat{G}$ consists of $p(p^2-1)$ elements. Let G be a subset of Γ such that

$$\hat{G} = \left\{\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\right\}$$

and G consists of $p(p^2-1)$ elements. Assume that $(p^2-1)\alpha_j \neq \sum_{i=1, i\neq j}^{p^2} \alpha_i$ for some j $(1 \leqq j \leqq p^2)$. By Theorem 2, we may assume $j=1$, i.e. $(p^2-1)\alpha_1 \neq \sum_{i=2}^{p^2} \alpha_i$. For any $S \in G$, we have that

$$\alpha_1 g_{(0,0)}|_2 S + \alpha_2 g_{(0,1)}|_2 S + \cdots + \alpha_{p^2} g_{(p-1,p-1)}|_2 S = 0$$

on $S^{-1}D_1$. By Theorem 1, there exists a curve γ_S such that $(u_1|_{-1}S, S^{-1}D_1)$ is continued analytically to (u_1, D_1) along γ_S. Then, by (##), for all integers a, b, $(g_{(a,b)}|_2 S, S^{-1}D_1)$ is continued analytically to $(g_{(a,b)S}, D_1)$ along γ_S, and hence

$$\alpha_1 g_{(0,0)S} + \alpha_2 g_{(0,1)S} + \cdots + \alpha_{p^2} g_{(p-1,p-1)S} = 0$$

on D_1. Therefore we obtain

(###) $$\alpha_1 \sum_{S \in G} g_{(0,0)S} + \alpha_2 \sum_{S \in G} g_{(0,1)S} + \cdots + \alpha_{p^2} \sum_{S \in G} g_{(p-1,p-1)S} = 0$$

on D_1. We can easily see that the stabilizer of $(\bar{0}, \bar{1})$ consists of p elements and for $(a, b), (a', b') \in \boldsymbol{Z}^2$ if $(a, b) \equiv (a', b') \bmod p$, then $g_{(a,b)} = g_{(a',b')}$ on D_1. Thus, by $g_{(0,0)} + g_{(0,1)} + \cdots + g_{(p-1,p-1)} = 0$ on D_1,

$$\begin{aligned}\sum_{S \in G} g_{(a,b)S} &= p(g_{(0,1)} + \cdots + g_{(p-1,p-1)}) \\ &= (-p) g_{(0,0)}\end{aligned}$$

on D_1 for $(a, b) \not\equiv (0, 0) \bmod p$. Hence, by (###),

$$0 = \left(\alpha_1 p(p^2-1) - \sum_{i=2}^{p^2} \alpha_i p\right) g_{(0,0)}$$

on D_1 noticing that $\sum_{S \in G} g_{(0,0)S} = p(p^2-1) g_{(0,0)}$ on D_1. Thus $g_{(0,0)} = 0$ on D_1. This contradicts the fact that $(g_{(0,0)}, D_1)$ is a branch of the p^2-valued analytic function $\mathfrak{p}_p$. Therefore $(p^2-1)\alpha_j = \sum_{i=1, i\neq j}^{p^2} \alpha_i$ for any j $(1 \leqq j \leqq p^2)$. This leads to $\alpha_1 = \cdots = \alpha_{p^2}$. ■

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