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A Note on Ando's Paper "Pluricanonical Systems of Algebraic Varieties of General Type of Dimension $\leqq 5$"

Shigetaka FUKUDA

Nagoya University
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§1. Introduction.

Let X be a non-singular projective variety of dimension n over an algebraically closed field k of characteristic zero.

DEFINITION. A Cartier divisor T is called *numerically trivial* if $(T, C)_X = 0$ for every curve C on X. A Cartier divisor D is called *nef* if $(D, C)_X \geqq 0$ for every curve C on X.

Ando [1] has shown the following results:
1) Assume that the canonical divisor K_X is nef and big, and that $n=4, 5$. Then there exists a positive integer $m(n)$ depending only on $\dim X$ such that the rational map $\Phi_{|mK_X|}$ associated with $|mK_X|$ is a birational map onto its image for every $m \geqq m(n)$. Here $m(n)$ is given as follows:

$$m(4)=16\,, \qquad m(5)=29\,.$$

2) Assume that $-K_X$ is nef and big, and that $n \leqq 4$. Then $\Phi_{|-mK_X|}$ is birational for $m \geqq l(n)$, where $l(n)$ is given by $l(2)=3$, $l(3)=5$, $l(4)=12$

We now improve Ando's argument by using Reider's result [8] and Matsuki's argument [5], and show the following result:

MAIN THEOREM. (1) *Assume that K_X is nef and big and that $n=4, 5$. Then $\Phi_{|mK_X|}$ is birational for $m \geqq m(n)$, where $m(n)$ is given by $m(4)=12$, $m(5)=18$.*

(2) *Assume that K_X is numerically trivial and that $n=3, 4, 5$. Let D be a nef and big divisor on X. Then $\Phi_{|mD|}$ is birational for $m \geqq k(n)$, where $k(n)$ is given by $k(3)=6$, $k(4)=10$, $k(5)=17$.*

(3) *Assume that $-K_X$ is nef and big and that $n=3, 4$. Then $\Phi_{|-mK_X|}$ is birational for $m \geqq l(n)$, where $l(n)$ is given by $l(3)=4$, $l(4)=11$.*

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In Section 4, we present a function $m(n)$ for $n \geqq 8$.

We use the notation of Ando [1].

The author would like to thank Prof. S. Mukai, who taught him Reider's result, for his advices.

§2. Key Lemma.

Improving Ando's argument [1, Theorem 5] by using Reider's result [8] and Matsuki's argument [5, Corollary 9], we obtain the following "Key Lemma".

KEY LEMMA. *Let X be a non-singular projective variety of dimension n, R a nef and big divisor and T a numerically trivial divisor. We assume:*

(1) *For each i with $1 \leqq i \leqq n-2$, there exists a natural number r_i such that* $\dim \Phi_{|r_i R|}(X) \geqq i$.

(2) *there exists an integer $r_0 \geqq 4$ such that every integer $r \geqq r_0$ satisfies* $H^0(X, rR+K_X+T) \neq 0$.

Then $\Phi_{|K_X+mR+T|}$ is birational for all $m \geqq r_0+(r_1+\cdots+r_{n-2})$.

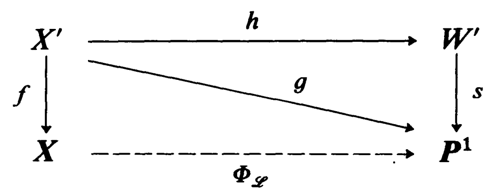
LEMMA 1 (See Tankeev [9, Lemma 2]). *Let $\mathcal{M}$ be a linear pencil free from base points, and let D be an effective divisor. Let M be a divisor with $M \in \mathcal{M}$. If, for a general member Y of $\mathcal{M}$, the rational map $\Phi := \Phi_{|M+D|}$ is birational on Y, then Φ is birational.*

PROOF. Assuming that Φ is not birational, we shall derive a contradiction. We may assume that $\mathrm{Supp}(D)$ does not include Y. Let $U = X - \mathrm{Supp}(D)$ and choose a general point $x \in U \cap Y$. There exists a point $y \in U$ such that $y \neq x$ and $\Phi(x) = \Phi(y)$. Since $|M|$ is base point free, x and y belong to the same effective divisor of $|M+D|$. So $x, y \in \mathrm{Supp}(Y+D)$. Thus $y \in Y$. Thus Φ is not birational on Y. This is a contradiction.
Q.E.D.

LEMMA 2 (Reider [8, Corollary 2]). *Let S be a non-singular projective surface and L a nef and big divisor on S such that $(L^2) \geqq 10$. If $\Phi_{|K_S+L|}$ is not birational, then S contains a base point free pencil E' with $(L, E')=1$ or $(L, E')=2$.*

PROOF OF KEY LEMMA. We prove this by induction on n. We put $m \geqq r_0+(r_1+\cdots+r_{n-2})$. If $n=2$, the result follows from Lemma 2.

We assume $n \geqq 3$. Let $\mathcal{L}$ be a subpencil of the complete linear system $|r_1 R|$. We consider the following commutative diagram:



where f is a succession of blowing-ups with non-singular centers such that $g:=\Phi_{\mathscr{L}}\circ f$ is a morphism, and $g=s\circ h$ is the Stein factorization.

Let R' be f^*R, S be a general fiber of h, H be a general point on $\boldsymbol{P}^1$, and $a:=\deg(s)$. Then S is a smooth $(n-1)$-fold. Note that g^*H is a disjoint union of S_i's $(1\leqq i\leqq a)$, each of which is of the same kind as S. In order to prove that the rational map $\Phi_{|K_X+mR+T|}$ is birational, it suffices to show that $\Phi_{|K_{X'}+g^*H+(m-r_1)R'+f^*T|}$ is birational.

Since

$$\begin{aligned}0 &\longrightarrow \mathcal{O}_{X'}(K_{X'}+(m-r_1)R'+f^*T)\\ &\longrightarrow \mathcal{O}_{X'}(K_{X'}+(m-r_1)R'+f^*T+g^*H)\\ &\longrightarrow \bigoplus_{j=1}^{a}\mathcal{O}_{S_j}(K_{S_j}+(m-r_1)R'|_{S_j}+f^*T|_{S_j})\longrightarrow 0\end{aligned}$$

is exact, and since $H^1(X', K_{X'}+(m-r_1)R'+f^*T)=0$ by the Kawamata-Viehweg vanishing theorem ([3], [10]), we have that

$$\begin{aligned}&H^0(X', \mathcal{O}_{X'}(K_{X'}+(m-r_1)R'+f^*T+g^*H))\\ &\qquad\longrightarrow \bigoplus_{j=1}^{a}H^0(S_j, \mathcal{O}_{S_j}(K_{S_j}+(m-r_1)R'|_{S_j}+f^*T|_{S_j}))\end{aligned}$$

is surjective. Therefore in order to prove the claim, it suffices to show that $\Phi_{|K_{S_j}+(m-r_1)R'|_{S_j}+f^*T|_{S_j}|}$ is birational. Actually by (2) we have $H^0(X', K_{X'}+(m-r_1)R'+f^*T)\neq 0$, hence we can apply Lemma 1.

Letting $r''_0:=r_0$, $r''_1:=r_2, \cdots, r''_{n-3}:=r_{n-2}$, we shall check that S_j, $R'|_{S_j}$, $f^*T|_{S_j}$, $r''_1, \cdots, r''_{n-3}$ and r''_0 satisfy the conditions (1) and (2). If this is done, then by induction, we conclude that $\Phi_{|K_{S_j}+(m-r_1)R'|_{S_j}+f^*T|_{S_j}|}$ is birational and complete the proof of the claim.

(1) Since H is general,

$$\dim \Phi_{|r''_iR'|_{S_j}|}(S_j)\geqq \dim \Phi_{|r_{i+1}R'|}(X')-1\geqq i \qquad \text{for} \quad i\geqq 1\,.$$

(2) Let $r\geqq r_0$. By assumption, $|rR'+K_{X'}+f^*T|\neq\varnothing$. Since S_j is a fiber of h, $S_j|_{S_j}$ is linearly equivalent to 0. So $K_{X'}|_{S_j}$ is linearly equivalent to K_{S_j}. Since H is general, $|rR'|_{S_j}+K_{S_j}+f^*T|_{S_j}|\neq\varnothing$. Q.E.D.

§3. Proof of Main Theorem.

PROPOSITION (Matsusaka [6], Maehara [4]). *Let R be a nef and big divisor and* $\dim X=n$. *If $h^0(mR)>m^rR^n+r$, then* $\dim\Phi_{|mR|}(X)>r$.

MIYAOKA'S INEQUALITY ([7]). *Let X be a non-singular projective variety with K_X nef. Then $3c_2(X)-K_X^2$ is pseudo-effective.*

LEMMA A (Ando [1, Lemma 7']). *Assume* $\dim X=4$ *and suppose that K_X is nef and big. Then*

(1) $h^0(X, mK_X)\geqq 2$, *for* $m\geqq 3$.

(2) $\dim \Phi_{|mK_X|}(X) \geqq 2$, *for* $m \geqq 4$.

LEMMA B (Ando [1, Lemma 8′]). *Assume* $\dim X = 5$ *and suppose that* K_X *is nef and big. Then*

(1) $h^0(X, mK_X) \geqq 2$, *if* $m \geqq 3$.

(2) $\dim \Phi_{|mK_X|}(X) \geqq 2$, *if* $m \geqq 4$.

(3) $\dim \Phi_{|mK_X|}(X) \geqq 3$, *if* $m \geqq 6$.

PROOF. Let $a = K^5$, $b = (K^3, c_2(X))/12$, $c = -2\chi(\mathcal{O}_X)$. Let $P(m) := h^0(mK_X)$. Then

$$P(m) = m(m-1)(2m-1)(3m^2-3m-1)a/720 + m(m-1)(2m-1)b/12 + (2m-1)c/2 \qquad \text{for} \quad m \geqq 2\,.$$

Miyaoka's inequality implies $b \geqq a/36$. Since $P(2) \geqq 0$, it follows that $a/24 + b/2 + 3c/2 \geqq 0$. Thus

$$(*) \qquad P(m) \geqq (2m-1)\{(3m^4-6m^3+2m^2+m-10)a/720 + (m^2-m-2)b/12\} \geqq a(2m-1)(9m^4-18m^3+11m^2-2m-40)/2160\,.$$

Let $Q(m) := a(2m-1)(9m^4-18m^3+11m^2-2m-40)/2160$. Then $P(m) \geqq Q(m)$. Note that a is a positive even integer, because $7a/2 + 35\chi(\mathcal{O}_X) = P(4) - 14P(2) \in \boldsymbol{Z}$. The proof is completed in view of the following inequalities in the following cases. (Use the above proposition.)

(1) $P(m) \geqq 2$ for $m \geqq 3$, because by $(*)$ $P(m) \geqq Q(3) > 0.6a$.

(2) $P(m) > ma+1$ for $m \geqq 5$, because by $(*)$ $P(m) - (ma+1) \geqq Q(5) - (5a+1) = 10a-1$. $P(4) - (4a+1) \geqq Q(4) - 4a - 1 > 0.14a - 1$. Thus $P(4) - (4a+1) > 0$, if $a \geqq 8$.

If $a = 2$, 4 or 6, then $(K^3, c_2) \geqq 18$, because $a/24 + (K^3, c_2)/24 - 3\chi(\mathcal{O}_X) = P(2) \in \boldsymbol{Z}$. Thus $b \geqq 3/2$. $P(4) - (4a+1) > (3.9a + 35b/6) - (4a+1) \geqq (3.9a + 8.75) - (4a+1) = -0.1a + 7.75 > 0$.

(3) $P(m) > m^2a + 2$ for $m \geqq 6$, because by $(*)$ $P(m) - (m^2a+2) \geqq Q(6) - (36a+2) > 5.3a - 2$. Q.E.D.

LEMMA C. *Assume* $\dim X = 3$, *and that* K_X *is numerically trivial. Let* D *be a nef and big divisor on X. Then*

$$\dim \Phi_{|mD|}(X) \geqq 1 \qquad \text{if} \quad m \geqq 2\,.$$

PROOF. $h^0(mD) = \chi(mD) = m^3D^3/6 + m(D, c_2)/12$ for $m \geqq 1$. Miyaoka's inequality implies $(D, c_2) \geqq 0$. Thus $h^0(mD) \geqq m^3D^3/6$ for $m \geqq 1$. Q.E.D.

LEMMA D. *Assume* $\dim X = 4$, *and that* K_X *is numerically trivial. Let* D *be a nef and big divisor on X. Then*

$$\dim \Phi_{|mD|}(X) \geqq 2 \qquad \text{if} \quad m \geqq 3\,.$$

PROOF. Let $a = (D^2, c_2)$, $b = \chi(\mathcal{O}_X)$. Let $P(m) := h^0(mD)$. Then $P(m) = m^4D^4/24 + m^2a/24 + b$. Miyaoka's inequality implies $a \geqq 0$. Since $P(1) \geqq 0$, $D^4/24 + a/24 + b \geqq 0$. Thus

$P(m) \geqq (m^4-1)D^4/24+(m^2-1)a/24 \geqq (m^4-1)D^4/24$. The proof is completed in view of the following inequalities.

$P(m)-(mD^4+1) \geqq (4^4-1)D^4/24-(4D^4+1) > 6.6D^4-1 > 0$ for $m \geqq 4$. If $D^4 \geqq 4$, $P(3)-(3D^4+1) \geqq 4/3-1 > 0$.

If $D^4=1$, 2, or 3, then $a \geqq 21$, because $D^4/24+a/24+b=P(1) \in \boldsymbol{Z}$. Thus $P(3) \geqq (3^4-1)D^4/24+(3^2-1)a/24 \geqq 80/24+21/3 > 10 \geqq 3D^4+1$. Q.E.D.

LEMMA E. *Assume* $\dim X=5$, *and that* K_X *is numerically trivial. Let* D *be a nef and big divisor on* X. *Then*

(1) $\dim \Phi_{|mD|}(X) \geqq 1$ *if* $m \geqq 3$.

(2) $\dim \Phi_{|mD|}(X) \geqq 2$ *if* $m \geqq 4$.

(3) $\dim \Phi_{|mD|}(X) \geqq 3$ *if* $m \geqq 6$.

PROOF. Let $a=D^5$, $b=(D^3, c_2)$, $c=(D, 3c_2^2-c_4)$. Let $P(m):=h^0(mD)$. Then, $P(m)=am^5/120+bm^3/72+cm/720$. Miyaoka's inequality implies $b \geqq 0$. Since $P(1) \geqq 0$, it follows that $a/120+b/72+c/720 \geqq 0$. Thus

$$(*) \qquad P(m) \geqq am(m^4-1)/120+bm(m^2-1)72 \geqq am(m^4-1)/120\,.$$

Let $Q(m):=am(m^4-1)/120$. Then $P(m) \geqq Q(m)$. The proof is completed in view of the following inequalities in the following cases.

(1) $P(m) \geqq 2$ for $m \geqq 3$, because $P(m) \geqq Q(3)=2a \geqq 2$ by $(*)$.

(2) $P(m) > am+1$ for $m \geqq 4$, because $P(m)-(am+1) \geqq Q(4)-(4a+1)=4.5a-1>0$ by $(*)$.

(3) $P(m) > am^2+2$ for $m \geqq 6$, because $P(m)-(am^2+2) \geqq Q(6)-(36a+2)=28.75a-2>0$ by $(*)$. Q.E.D.

LEMMA F (Ando [1, Proof of Theorem 9]). *Assume* $\dim X=3$ *and suppose that* $-K_X$ *is nef and big. Then*

$$h^0(X, -mK_X) \geqq 2\,, \qquad \textit{for} \quad m \geqq 1\,.$$

LEMMA G (Ando [1, Proof of Theorem 9]). *Assume* $\dim X=4$ *and suppose that* $-K_X$ *is nef and big. Then*

(1) $h^0(X, -mK_X) \geqq 1$, *for* $m \geqq 3$.

(2) $\dim \Phi_{|-4K_X|}(X) \geqq 2$.

PROOF OF MAIN THEOREM. (1) Assume that K_X is nef and big. We apply Key Lemma, where $R=K_X$ and $T=0$.

When $\dim X=4$, by Lemma A we put $r_0=4$, $r_1=3$, $r_2=4$.

When $\dim X=5$, by Lemma B we put $r_0=4$, $r_1=3$, $r_2=4$, $r_3=6$.

(2) Assume that K_X is numerically trivial. Let D be a nef and big divisor on X. We apply Key Lemma, where $R=D$ and $T=-K_X$.

When $\dim X=3$, by Lemma C we put $r_0=4$, $r_1=2$.

When $\dim X=4$, by Lemma D we put $r_0=4$, $r_1=3$, $r_2=3$.

When $\dim X=5$, by Lemma E we put $r_0=4$, $r_1=3$, $r_2=4$, $r_3=6$.

(3) Assume that $-K_X$ is nef and big. We apply Key Lemma, where $R=-K_X$ and $T=0$.

When $\dim X=3$, by Lemma F we put $r_0=4$, $r_1=1$.

When $\dim X=4$, by Lemma G we put $r_0=4$, $r_1=4$, $r_2=4$. Q.E.D.

REMARK. THEOREM (Matsuki [5]): *Assume* $\dim X=3$ *and that* K_X *is nef and big. Then* $\Phi_{|mK_X|}$ *is birational for* $m\geqq 7$.

PROOF. Let $P(m):=h^0(X, mK_X)=(2m-1)\{m(m-1)K_X^3/12-\chi(\mathcal{O}_X)\}$ for $m\geqq 2$. By Miyaoka's inequality, $\chi(\mathcal{O}_X)=-(K_X, c_2)/24\leqq -K_X^3/72<0$. Thus $P(m)>(2m-1)m\cdot(m-1)K_X^3/12>0$ for $m\geqq 2$. $P(2)>K_X^3/2$. K_X^3 is a positive even integer, because $-K_X^3/2+2\chi(\mathcal{O}_x)=\chi(\mathcal{O}_X(K_X))+\chi(\mathcal{O}_X(-K_X))\in \boldsymbol{Z}$. Thus $P(2)\geqq 2$. We apply Key Lemma, where $R=K_X$ and $T=0$. We put $r_0=4$, $r_1=2$. Q.E.D.

§4. Appendix (based on Ando's idea).

LEMMA 3 (See Tankeev [9, Lemma 2]). *Let* $\mathcal{M}$ *be a linear pencil free from base points, and let* D *be an effective divisor. Let* M *be a member of* $\mathcal{M}$ *and let* Φ *be the rational map* $\Phi_{|M+D|}$. *Let* p *be a natural number. If, for a general member* Y *of* $\mathcal{M}$, $\dim \Phi|_Y(Y)\geqq p$, *then* $\dim \Phi(X)\geqq p+1$.

PROOF. Assuming that $\dim \Phi(X)=p$, we shall derive a contradiction. We may assume that Supp(D) does not include any irreducible component of Y. Let E be an irreducible component of Y such that $\dim \Phi|_E(E)=p$. Let $U=X-\mathrm{Supp}(D)$ and choose a general point $x\in U\cap E$. Then $\dim(\Phi|_U)^{-1}(\Phi(x))=n-p$ and $\dim(\Phi|_{U\cap Y})^{-1}(\Phi(x))=n-1-p$. Thus there exists a point $y\in U-Y$ such that $\Phi(x)=\Phi(y)$. Since $|M|$ is base point free, x and y belong to the same effective divisor of $|M+D|$. So $x, y\in \mathrm{Supp}(Y+D)$. Thus $y\in Y$. This is a contradiction. Q.E.D.

LEMMA 4 (based on Ando's idea). *Let* p *be a natural number. Let* X *be a non-singular projective variety of dimension* $n\geqq p$, R *a nef and big divisor and* T *a numerically trivial divisor. We assume*:

(1) *For each* i *with* $1\leqq i\leqq p-1$, *there exists a natural number* r_i *such that* $\dim \Phi_{|r_iR|}(X)\geqq i$.

(2) *There exists an integer* r'_0 *such that evey integer* $r\geqq r'_0$ *satisfies* $H^0(X, rR+K_X+T)\neq 0$.

(3) *There exists an integer* l *such that every integer* $r\geqq l$ *satisfies* $H^0(X, rR)\neq 0$.

Then $\dim \Phi_{|K_X+mR+T|}(X)\geqq p$ *for* $m\geqq n-p+1+r'_0+l+(r_1+\cdots+r_{p-1})$.

PROOF. We prove this by induction on p. We put $m\geqq n-p+1+r'_0+l+(r_1+\cdots+r_{p-1})$.

Case $p=1$. We define a polynomial $P(x)$ by $P(r)=\chi(K_X+rR+T)$. For $r\in \boldsymbol{N}$,

$P(r)=h^0(X, K_X+rR+T)$. $P(r)\geqq 1$ for $r\in[r'_0, +\infty)\cap \boldsymbol{N}$. $P(x)$ is a polynomial of degree n. Thus there exists an integer $m_0\in[r'_0, r'_0+n]$ such that $P(m_0)\geqq 2$. So $h^0(K_X+mR+T)\geqq 2$ for $m\geqq r'_0+l+n$ $(\geqq m_0+l)$.

Case $p\geqq 2$. Let $\mathscr{L}$ be a subpencil of the complete linear system $|r_1R|$. We consider the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\quad h \quad} & W' \\ {\scriptstyle f}\downarrow & \overset{g}{\searrow} & \downarrow{\scriptstyle s} \\ X & \dashrightarrow & \boldsymbol{P}^1 \\ & \Phi_{\mathscr{L}} & \end{array}$$

where f is a succession of blowing-ups with non-singular centers such that $g:=\Phi_{\mathscr{L}}\circ f$ is a morphism, and $g=s\circ h$ is the Stein factorization.

Let R' be f^*R, S be a general fiber of h, H be a general point on $\boldsymbol{P}^1$, and $a:=\deg(s)$. Then S is a smooth $(n-1)$-fold. Note that g^*H is a disjoint union of S_i's $(1\leqq i\leqq a)$, each of which is of the same kind as S. In order to prove that $\dim\Phi_{|K_X+mR+T|}(X)\geqq p$, it suffices to show that $\dim\Phi_{|K_{X'}+g^*H+(m-r_1)R'+f^*T|}(X')\geqq p$.

Since

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{X'}(K_{X'}+(m-r_1)R'+f^*T) \\ &\longrightarrow \mathcal{O}_{X'}(K_{X'}+(m-r_1)R'+f^*T+g^*H) \\ &\longrightarrow \textstyle\bigoplus_{j=1}^{a}\mathcal{O}_{S_j}(K_{S_j}+(m-r_1)R'|_{S_j}+f^*T|_{S_j})\longrightarrow 0 \end{aligned}$$

is exact, and since $H^1(X', K_{X'}+(m-r_1)R'+f^*T)=0$ by the Kawamata-Viehweg vanishing theorem ([3], [10]), we have that

$$\begin{aligned} &H^0(X', \mathcal{O}_{X'}(K_{X'}+(m-r_1)R'+f^*T+g^*H)) \\ &\qquad\longrightarrow \textstyle\bigoplus_{j=1}^{a}H^0(S_j, \mathcal{O}_{S_j}(K_{S_j}+(m-r_1)R'|_{S_j}+f^*T|_{S_j})) \end{aligned}$$

is surjective. Therefore in order to prove the claim, it suffices to show that $\dim\Phi_{|K_{S_j}+(m-r_1)R'|_{S_j}+f^*T|_{S_j}|}(S_j)\geqq p-1$. Actually by (2) we have $H^0(X', K_{X'}+(m-r_1)R'+f^*T)\neq 0$, hence we can apply Lemma 3.

Letting $r''_0:=r'_0$, $l'':=l$, $r''_1:=r_2, \cdots, r''_{p-2}:=r_{p-1}$, we shall check that S_j, $R'|_{S_j}$, $f^*T|_{S_j}$, $r''_1, \cdots, r''_{p-2}$, r''_0 and l'' satisfy the condition (1), (2) and (3). If this is done, then by induction, we conclude that $\dim\Phi_{|K_{S_j}+(m-r_1)R'|_{S_j}+f^*T|_{S_j}|}(S_j)\geqq p-1$ and complete the proof of the claim.

(1) Since H is general,

$$\dim\Phi_{|r''_iR'|_{S_j}|}(S_j)\geqq\dim\Phi_{|r_{i+1}R'|}(X')-1\geqq i \qquad \text{for} \quad i\geqq 1.$$

(2) Let $r\geqq r'_0$. By assumption, $|rR'+K_{X'}+f^*T|\neq\varnothing$. Since S_j is a fiber of h, $S_j|_{S_j}$ is linearly equivalent to 0. So $K_{X'}|_{S_j}$ is linearly equivalent to K_{S_j}. Since H is

general, $|rR'|_{S_j}+K_{S_j}+f^*T|_{S_j}|\neq\varnothing$.

(3) Let $r\geqq l$. By assumption, $|rR'|\neq\varnothing$. Since H is general, $|rR'|_{S_j}|\neq\varnothing$. Q.E.D.

Lemma 5 (Ando [1, Proposition 2′]). *If K_X is nef and big and $n\geqq 6$, then $|mK_X|\neq\varnothing$ for any $m\geqq 2[n/2]-2$.*

THEOREM. *Assume that K_X is nef and big and $n\geqq 6$. Then $\Phi_{|mK_X|}$ is birational for $m\geqq m(n)$, where $m(n)$ is given by*

$$m(6)=204\,,\qquad m(7)=444\,,$$
$$m(n)=2^{n-2}\cdot(n+4[n/2]-5)-2[n/2]+1\qquad \textit{for}\quad n\geqq 8\,.$$

PROOF. We apply Lemma 4 and Key Lemma, where $R=K_X$ and $T=0$. By Lemma 5, we put $r'_0=2[n/2]-3$ and $l=2[n/2]-2$. By Lemma 4, we put

$$r_1=n+1+r'_0+l\,,\quad r_2=n+r'_0+l+r_1\,,\quad \cdots\cdots,\quad r_{n-1}=n-(n-3)+r'_0+l+r_1+\cdots+r_{n-2}\,,\quad r_n=n-(n-2)+r'_0+l+r_1+\cdots+r_{n-1}\,.$$

So $r_p=2^{p-1}\cdot(n+r'_0+l)+1$ for $p\geqq 1$. Thus

$$1+r'_0+(r_1+r_2+\cdots+r_{n-2})=1+r_{n-1}-l-3=2^{n-2}\cdot(n+r'_0+l)-l-1$$
$$=2^{n-2}\cdot(n+4[n/2]-5)-2[n/2]+1\,.$$

When $n=6,7$, we put $r_0=r'_0+1=4$. When $n\geqq 8$, we put $r_0=r'_0$. Then we put $m(n)=1+r_0+(r_1+r_2+\cdots+r_{n-2})$. Q.E.D.

ACKNOWLEDGEMENT. Lemma 4 is based on Prof. Ando's letter to the author which tells him that the method of the proof of Key Lemma is usefull to get good r_i ($i\geqq 1$). The author would like to express his gratitude to Prof. Ando for his advice.

§5. Further appendix.

LEMMA 6. *If $-K_X$ is nef and big, then $|m(-K_X)|\neq\varnothing$ for any $m\geqq 2[n/2]$.*

PROOF. We define a polynomial $P(x)$ by $P(m)=\chi(m(-K_X))$. Let r be the number of integral roots of $P(x)$ in $x>-1/2$ counting the multiplicity precisely and $\alpha=\max\{x\in \boldsymbol{Z}\,|\,x>-1/2,\ P(x)=0\}$. By the Serre duality, $P(x)=(-1)^nP(-x-1)$. Thus $r\leqq[n/2]$. Since $H^i(X,m(-K_X))=0$ for $i\geqq 1$ and $m\geqq 0$, we have $P(m)\geqq 0$ for integers $m\geqq 0$. By the same argument as in (I) in the proof of Proposition 2 of Ando's paper [1], we conclude that $\alpha\leqq 2r-1$. Q.E.D.

THEOREM. *Assume that $-K_X$ is nef and big and $n\geqq 5$. Then $\Phi_{|-mK_X|}$ is birational for $m\geqq l(n)$, where $l(n)$ is given by*

$$l(n)=2^{n-2}\cdot(n+4[n/2]-1)-2[n/2]-1\,.$$

PROOF. We apply Lemma 4 and Key Lemma, where $R=-K_x$ and $T=0$. By

Lemma 6, we put $r'_0=2[n/2]+1$ and $l=2[n/2]$. By Lemma 4, we put $r_p=n-p+r'_0+l+r_1+\cdots+r_{p-1}$ for $p\geqq 1$. So $r_p=2^{p-1}\cdot(n+r'_0+l-2)+1$ for $p\geqq 1$. We put $r_0=r'_0$. Thus

$$-1+r_0+(r_1+r_2+\cdots+r_{n-2})=-1+r_{n-1}-l-1=2^{n-2}\cdot(n+4[n/2]-1)-2[n/2]-1\ .$$

Q.E.D.

REMARK. Oguiso proved the following result in his preprint "On polarized Calabi-Yau 3-folds":

When X is Calabi-Yau 3-fold, for any ample divisor L on X, $\Phi_{|mL|}$ is birational if $m\geqq 5$.

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Present Address:
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, NAGOYA UNIVERSITY
FURO-CHO, CHIKUSA-KU NAGOYA 464–01, JAPAN