TOKYO J. MATH.
VOL. 15, NO. 1, 1992

Multipliers of the Range of Composition Operators

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(Communicated by T. Nagano)

Abstract. The structure of the space of multipliers of the range of a composition operator C_ϕ on the Hardy space is studied. We provide necessary and/or sufficient conditions in terms of appropriate measurements of the distance of $|\phi|$ to 1 for the containment or the inclusion of the space of multipliers in standard spaces.

1. Introduction.

Let D denote the unit disc of the complex plane C, ∂D the unit circle and $d\sigma$ the normalized Lebesgue arc length measure on ∂D. For $1 \leq p \leq \infty$, the Lebesgue spaces $L^p(\partial D, d\sigma)$ are simply denoted by L^p and the Hardy spaces of analytic functions on D by H^p. Each $f \in H^p$ has, for a.e. $\zeta \in \partial D$, a radial limit

$$f(\zeta) = \lim_{r \to 1-} f(r\zeta),$$

and for $1 \leq p < \infty$ the *p-norm* of f is given by

$$\|f\|_p^p = \sup_{0<r<1} \int |f(r\zeta)|^p \, d\sigma(\zeta) = \int |f|^p .$$

The unadorned integral sign always means that the integral is over ∂D and all integrals unless otherwise indicated are with respect to the measure $d\sigma$. The abbreviation 'a.e.' always refers to $d\sigma$. We will use the same symbol to denote a holomorphic function on D in H^p and its radial limit function; the precise meaning of this statement will be clear from the context.

The letters ϕ and ψ with or without subscripts are reserved to denote nontrivial holomorphic self-maps of D. For $1 \leq p < \infty$, the composition operator $C_\phi: H^p \to H^p$ is defined by the equation

$$C_\phi(f) = f \circ \phi \qquad (f \in H^p).$$

Let T be an operator on a functional Hilbert space H. We say that $f \in H$ is a multiplier of the range of T if $f T(H) \subseteq H$. It is reasonable expect that some operator

Received April 26, 1991

properties of T should be reflected in the structure of the (Banach) space of multipliers of $T(H)$. This note provides some results of this type in the case of the composition operator C_ϕ on the Hardy space H^2. The containment, respectively, inclusion of the space of multipliers (of $C_\phi(H^2)$) in standard spaces is either related to the (appropriately taken) distance of $|\phi|$ to 1 or to an operator property of C_ϕ. The space of multipliers is contained in $BMOA$ if and only if it is contained in H^∞ if and only if ϕ is a finite Blaschke product (Proposition 12). This observation leads to the Cima-Thompson-Wogen characterization of Fredholm composition operators on the Hardy space (Proposition 18). If ϕ is locally well-behaved, in the sense of having an angular derivative at a point $\zeta \in \partial D$, then the multipliers are also on their best behavior—they are bounded on nontangential approach regions to ζ (Corollary 8). Proposition 14 notes that ϕ is not an extreme point of the unit ball of H^∞ if and only if bH^2 is contained in the space of multipliers for some non-zero $b \in H^\infty$. A related condition for C_ϕ to be Hilbert-Schmidt is given in Corollary 15. Finally Proposition 16 tells when the multipliers are a Hilbert space. In the proofs of some results de Branges spaces lurk around in the background but their explicit role is not identified.

Throughout this paper, the letter c will denote a constant, not necessarily of the same value at each of it's occurrences.

Acknowledgement. The idea of considering the multipliers of the range of operators was introduced to me by Professors Alan L. Lambert and Barnet M. Weinstock.

2. Preliminaries and point estimates.

The Hardy space H^2 is of course a Hilbert space, with the inner product

$$\langle f, g\rangle = \int f\bar{g} \qquad (f \text{ and } g \in H^2).$$

For each point in $w \in D$, the reproducing kernel

$$k_w(z) = (1-\bar{w}z)^{-1} \qquad (z \in D) \tag{1}$$

belongs to H^2, and represents the linear functional of point evaluation at w:

$$f(w) = \langle f, k_w\rangle \qquad (f \in H^2). \tag{2}$$

In particular

$$\|k_w\|_2^2 = \langle k_w, k_w\rangle = k_w(w) = (1-|w|^2)^{-1}. \tag{3}$$

From (2) and (3) we can derive a standard point estimate for functions in H^2;

$$|f(w)| \leq \|f\|_2 \|k_w\|_2 = (1-|\omega|^2)^{-1/2} \|f\|^2. \tag{4}$$

The Littlewood Subordination Principle [8] may be stated as, for $1 \leq p < \infty$,

$$\int |f \circ \phi|^p \leq \int |f|^p \qquad (f \in H^p), \tag{5}$$

provided $\phi(0)=0$. In the language of operator theory this says that $C_\phi\colon H^p \to H^p$ is bounded; and the operator norm of C_ϕ is in fact 1 when $\phi(0)=0$.

Let S be a subspacce of H^2. A function $f \in H^2$ is said to be a multiplier of S if $fS \subseteq H^2$, i.e., $fg \in H^2$ for every $g \in S$. The following lemma is well-known [13], Lemma 3, page 782.

LEMMA 1. *Let f be a multiplier of H^2. Then f is bounded.*

For a composition operator $C_\phi\colon H^2 \to H^2$, consider $M(\phi)$, the vector space of all multipliers of the range of C_ϕ. For $f \in M(\phi)$ define the operator map $T_f\colon H^2 \to H^2$ by

$$T_f(g) = fg \circ \phi \qquad (g \in H^2). \tag{6}$$

An application of the Closed Graph Theorem shows that T_f is bounded; so there exists a constant $c = c(f)$ such that

$$\|fg \circ \phi\|_2 \leq c\|g\|_2 \qquad (g \in H^2). \tag{7}$$

We define a norm on $M(\phi)$ by

$$\|f\|_{M(\phi)} = \|T_f\| \qquad (f \in M(\phi)),$$

where $\|T_f\|$ is the operator norm of T_f. When there is no risk of confusion, the multiplier norm of f will be written without the subscript $M(\phi)$. The following lemma shows that $M(\phi)$ with this norm is a Banach space. As usual $\mathscr{L}(H^2)$ denotes the space of bounded operators on H^2, endowed with the operator norm.

LEMMA 2. *The set $\{T_f : f \in M(\phi)\}$ is a closed subspace of $\mathscr{L}(H^2)$.*

PROOF. To prove the closedness, suppose $\{f_n\}$ is a sequence in $M(\phi)$ and that $T_{f_n} \to T$ as $n \to \infty$ for some T in $\mathscr{L}(H^2)$. (Each f_n is viewed as a function defined $d\sigma$-almost everywhere on ∂D). Write $f = T(1)$. Note that $f_n = T_{f_n}(1)$ converges to f in H^2, hence $\{f_n\}$ has a subsequence which converges to f a.e. Moreover,

$$\int |f_n|^2 |g \circ \phi|^2 \leq c \int |g|^2 \qquad (g \in H^2),$$

where $c = \sup_n \|T_{f_n}\| < \infty$. Passing into subsequential limits and applying Fatou's lemma,

$$\int |f|^2 |g \circ \phi|^2 \leq c \int |g|^2 \qquad (g \in H^2),$$

so $f \in M(\phi)$.

Fix $g \in H^\infty$. Then

$$\|T_{f_n}(g) - T_f(g)\|_2 = \|(f_n - f) g \circ \phi\|_2 \le \|g\|_\infty \|(f_n - f)\|_2 ,$$

so $T_{f_n}(g) \to T_f(g)$ as $n \to \infty$. Thus T and T_f agree on a dense subspace of H^2 (namely H^∞). Since both T and T_f are bounded, it follows that $T = T_f$. □

Since convergence in H^2 implies pointwise convergence on D, it is worthwhile to single out the following observation made during the proof as a corollary;

COROLLARY 3. *If as $n \to \infty$, $f_n \to f$ in $M(\phi)$ then $f_n \to f$ pointwise on D.*

Let $w \in D$. The Möbius map ψ_w is defined by,

$$\psi_w(z) = \frac{w - z}{1 - \bar{w}z} \qquad (z \in D) . \tag{8}$$

It is easy to verify that ψ_w is its own inverse map and that $\psi'_w = (1 - |w|^2) k_w^2$.

LEMMA 4. *Let ψ_w be a Möbius map. Then as vector spaces $M(\psi_w \circ \phi) = M(\phi)$.*

PROOF. Suppose $f \in M(\phi)$ and let $g \in H^2$. Then

$$\int |f|^2 |g \circ \psi_w \circ \phi|^2 \le \|f\|^2_{M(\phi)} \int |g \circ \psi_w|^2 = \|f\|^2_{M(\phi)} \int |g|^2 |\psi'_w| .$$

Since ψ'_w is bounded, we have that $f \in M(\psi_w \circ \phi)$. Thus $M(\phi) \subseteq M(\psi_w \circ \phi)$. Now replacing ϕ by $\psi_w \circ \phi$ we get the reverse inclusion. □

The usual pointwise estimate (4) for functions in the Hardy space can be improved for functions in $M(\phi)$ to provide a useful inequality.

LEMMA 5. *Let $f \in M(\phi)$ and $w \in D$. Then*

$$|f(w)| \le \|f\| \sqrt{\frac{1 - |\phi(w)|^2}{1 - |w|^2}} .$$

PROOF. Let $g \in H^2$. Then $fg \circ \phi \in H^2$ so by (4)

$$|f(w) g(\phi(w))| \le \|fg \circ \phi\|_2 (1 - |\omega|^2)^{-1/2} \le \|f\| \|g\|_2 (1 - |w|^2)^{-1/2} .$$

Put $g = k_{\phi(w)}$ to deduce the lemma. □

Let $k_w^\phi = (1 - \bar{\phi}(w)\phi) k_w$. These are the reproducing kernels in the de Brange space $\mathscr{H}(\phi)$ [9]. In the next lemma we will evaluate the multiplier norm of k_w^ϕ, which turn out to be the same as the norm of k_w^ϕ in $\mathscr{H}(\phi)$ [9].

LEMMA 6. *Let $\omega \in D$. Then*

$$\|k_w^\phi\| = \sqrt{\frac{1-|\phi(w)|^2}{1-|w|^2}}\,.$$

PROOF. The domination of the right-hand side by the left-hand side is an immediate consequence of the pointwise estimate of Lemma 5 applied to k_w^ϕ. To establish the reverse inequality, fix $g \in H^2$ and notice that

$$\int |1-\bar{\phi}(w)\phi|^2 |\psi_w'| |g\circ\phi|^2 = \int |1-\bar{\phi}(w)\phi\circ\psi_w|^2 |g\circ\phi\circ\psi_w|^2\,, \tag{9}$$

where ψ_w is the Möbius map defined in (8). Since $\psi_{\phi(w)}\circ\psi_{\phi(w)}$ is the identity map and $\psi_{\phi(w)}\circ\phi\circ\psi_w(0)=0$, by the Littlewood Subordination Principle (5) the integral on the right-hand side of (9) is

$$\leq \int |1-\bar{\phi}(w)\psi_{\phi(w)}|^2 |g\circ\psi_{\phi(w)}|^2\,. \tag{10}$$

By a change of variable (10) is easily seen to be equal to

$$(1-|\phi(w)|^2)\int |g|^2\,.$$

Thus

$$\int |k_w^\phi|^2 |g\circ\phi|^2 \leq \left(\frac{1-|\phi(w)|^2}{1-|w|^2}\right)\int |g|^2\,,$$

which establishes the upper estimate for $\|k_w^\phi\|$ asserted in the lemma. □

Note that $|k_w| \leq (1-|\phi(w)|)^{-1}|k_w^\phi|$. Then from Lemma 5 and Lemma 6 we can easily estimate the multiplier norm of the reproducing kernels:

COROLLARY 7. *Let $w\in D$. Then*

$$\frac{1}{\sqrt{1-|\phi(w)|^2}}\frac{1}{\sqrt{1-|w|^2}} \leq \|k_w\| \leq \sqrt{\frac{1+|\phi(w)|}{1-|\phi(w)|}}\frac{1}{\sqrt{1-|w|^2}}\,.$$

We omit the proof.

We say that ϕ has a finite angular derivative at $\zeta\in\partial D$ if here exists λ with $|\lambda|=1$ such that the difference quotient $(\phi(z)-\lambda)/(z-\zeta)$ has a finite limit as z tends non-tangentially to ζ. A theorem of C. Carathéodory provides a necessary and sufficient condition for a function to have a finite angular derivative. Carathéodory [1], section 298, Theorem 2.1, gives a proof of the theorem which highlights its geometric nature; for a proof using Hilbert space techniques, see Sarason [10]. Part of Carathéodory's theorem which will be used in Corollary 8 is presented below.

THEOREM C (Carathéodory). *For $\zeta \in \partial D$, ϕ has a finite angular derivative at ζ if and only if*

$$\liminf\left\{\frac{1-|\phi(z)|^2}{1-|z|^2} : z \to \zeta \text{ unrestrictedly in } D\right\} < \infty .$$

Moreover, if ϕ has a finite angular derivative at $\zeta \in \partial D$ then the nontangential limit

$$\lim_{z \to \zeta} \frac{1-|\phi(z)|^2}{1-|z|^2}$$

exists.

COROLLARY 8. *The function ϕ has a finite angular derivative at a point $\zeta \in \partial D$ if and only if there exists $M > 0$ and a sequence $w_n \to \zeta$ such that* $\limsup |f(w_n)| \le M\|f\|$ *for every $f \in M(\phi)$. Thus in the case ϕ has a finite angular derivative at $\zeta \in \partial D$ each multiplier is bounded on nontangential approach regions to ζ.*

PROOF. If ϕ has an angular derivative at ζ then the corresponding implication of the corollary is a trivial consequence of the estimate in Lemma 5 and Theorem C. To prove the converse, suppose there exists a constant $M > 0$ and a sequence w_n in D tending to $\zeta \in \partial D$ such that for every $f \in M(\phi)$ and for every n

$$|f(w_n)| \le M\|f\| .$$

Put $f = k^{\phi}_{w_n}$. Then applying Lemma 6 we have

$$\frac{1-|\phi(w_n)|^2}{1-|w_n|^2} \le M^2.$$

Hence by Carathéodory's theorem ϕ has a finite angular derivative at ζ. □

3. Inclusion of multipliers in standard spaces.

We say that ϕ is an inner function if it has radial limits of modulus 1 a.e.

LEMMA 9. *Suppose $M(\phi) \subseteq H^p$ for some $p > 2$. Then ϕ is an inner function.*

PROOF. Suppose ϕ is not an inner function, then there exist $E \subseteq \partial D$ with $\sigma(E) > 0$ and $0 < r < 1$ such that $|\phi| < r$. Let λ be a Lebesgue point of E and define $h \in L^2(\partial D, \sigma)$ by

$$h(\zeta) = \begin{cases} 1 & \text{if } \zeta \notin E \\ (\zeta - \lambda)^{-1/p} & \text{if } \zeta \in E \text{ and } \zeta \neq \lambda . \end{cases}$$

Then $\log|h|$ is integrable, so there exists $f \in H^2$ such that $|f| = |h|$, [6], page 53. Clearly $f \notin H^p$. However, for $g \in H^2$, $g \circ \phi$ is essentially bounded on E, thus $f \in M(\phi)$, so $M(\phi)$ is not contained in H^p. □

Let $d\sigma \circ \phi^{-1}$ denote the regular measure of the Borel sets of ∂D defined by $d\sigma \circ \phi^{-1}(E) = d\sigma(\phi^{-1}(E))$. Then we have the standard change of variable formula:

$$\int h \circ \phi d\sigma = \int h d\sigma \circ \phi^{-1}, \tag{11}$$

where h is a measurable function and $h \geq 0$ a.e. The measure $d\sigma \circ \phi^{-1}$ is absolutely continuous with respect to $d\sigma$ and in the case $\phi(0) = 0$,

$$d\sigma \circ \phi^{-1} = d\sigma. \tag{12}$$

(This well known result (12) is seen by applying the change of variable formula (11) to the integrals $\int \phi^n \bar{\phi}^m d\sigma = 0, n \neq m, \int |\phi| d\sigma = 1$ and then using the F. and M. Riesz theorem).

PROPOSITION 10. *Let $2 < p < \infty$. Then there does not exist ϕ such that $M(\phi) = H^p$.*

PROOF. Suppose there exists ϕ such that $M(\phi) = H^p$ for some $2 < p < \infty$. By Lemma 9, ϕ is an inner function and by Lemma 4 we can assume that $\phi(0) = 0$. Fix $g \in H^2$. Then $g \circ \phi H^p \subseteq H^2$, so $g \circ \phi H^p H^2 \subseteq H^1$. By the factorization theorems for functions in Hardy spaces (actually all we need is the weak factorization theorems in [3]) we have that $H^p H^2 \supseteq H^q$ where $p^{-1} + 2^{-1} = q^{-1}$ and $1 < q < \infty$. Thus $g \circ \phi H^q \subseteq H^1$. Recalling that the dual of H^q is H^r where $q^{-1} + r^{-1} = 1$, [5], Chapter IV, Theorem 4.2, pp. 242–243, we deduce that $g \circ \phi \in H^r$. Thus we have the following: if $g \in H^2$ then $g \circ \phi \in H^r$ where $p^{-1} + r^{-1} = 2^{-1}$. An application of the Closed Graph Theorem shows that

$$\|g \circ \phi\|_r \leq c \|g\|_2 \qquad (g \in H^2). \tag{13}$$

Note that $r > 2$. Applying the change of variable formula (11) to the integral on the left-hand side of (13) and using (12) we have,

$$\left(\int |g|^r d\sigma \right)^{1/r} \leq c \left(\int |g|^2 d\sigma \right)^{1/2} \qquad (g \in H^2),$$

which is clearly impossible. □

The space of *BMO* functions with its applications to univalent function theory, quasiconformal mappings, partial differential equations and probability theory is one of the most well-studied spaces of functions. There is now a fairly rich literature on *BMO*, see [5], Chapter 6, and the references therein, for a good discussion of many of the now classical properties of *BMO* functions. The space of *BMOA* functions is defined by $BMOA = H^2 \cap BMO$ and for $f \in H^2$ the *BMOA* norm of f may be defined by

$$\|f\|_{BMOA}^2 = |f(0)|^2 + \sup_{w \in D} \|f \circ \psi_w - f(w)\|_2^2.$$

The estimate of $\|k_w\|_{BMOA}$ given in Lemma 11 will be used in the proof of Proposition 12. But first we recall the Littlewood-Paley identity.

Let dm denote the normalized Lebesgue area measure on D. (So $dm = r dr d\theta / \pi$.)

The Littlewood-Paley identity relates the Hardy space norm to that of a weighted Bergman space:

$$\int_{\partial D} |f|^2 d\sigma = |f(0)|^2 + 2\int_D |f'|^2(-\log|\cdot|)dm \qquad (f\in H^2).$$

LEMMA 11. *Let $w\in D$. Then*

$$\|k_w\|_{BMOA} \geq |w|(1-|w|^2)^{-1}.$$

PROOF. By the definition of the $BMOA$ norm

$$\|k_w\|^2_{BMOA} \geq \| k_w\circ\psi_w - k_w(w)\|_2^2. \tag{14}$$

By the Littlewood-Paley identity the right-hand side of inequality (14) is

$$\begin{aligned}
&=2\int |k_w'\circ\psi_w|^2|\psi_w'|^2(-\log|\cdot|)dm \\
&=2\int |k_w'|^2(-\log|\psi_w|)dm \\
&=2|w|^2(1-|w|^2)^{-2}\int |\psi_w'|^2(-\log|\psi_w|)dm \\
&=2|w|^2(1-|w|^2)^{-2}\int (-\log|\cdot|)dm = |w|^2(1-|w|^2)^{-2},
\end{aligned}$$

from which the lemma follows. □

Let ϕ be an inner function and let $\mathscr{A}$ denote the σ-algebra

$$\mathscr{A} = \{\phi^{-1}(E) : E \text{ is a Borel set of } \partial D\}. \tag{15}$$

For $f\geq 0$ a.e. the conditional expectation, $E(f|\mathscr{A})$, with respect to $\mathscr{A}$ is defined to be the unique $\mathscr{A}$ measurable function for which

$$\int_A E(f|\mathscr{A})d\sigma = \int_A f d\sigma \qquad (A\in\mathscr{A}). \tag{16}$$

When there is no confusion to the σ-algebra being referred we will simply denote $E(f|\mathscr{A})$ by $E(f)$. See [7] for a brief discussion of conditional expectation. The following change of variable formula (17) is adopted from [7], lines 1–4, page 227:

If f is a measurable function on ∂D and $g\in L^2$ then

$$\int |f|^2|g\circ\phi|^2 d\sigma = \int E(|f|^2)\circ\phi^{-1}|g|^2 d\sigma\circ\phi^{-1}. \tag{17}$$

(Even though ϕ^{-1} may not be a function a known property is that $E(|f|^2)\circ\phi^{-1}$

is well-defined). Since $L^2 = H^2 + \overline{H^2}$ from (7) and (17) we deduce the following:

Let $f \in H^2$. Then $f \in M(\phi)$ if and only if

$$\int E(|f|^2) \circ \phi^{-1} |g|^2 \, d\sigma \circ \phi^{-1} \leq c \int |g|^2 \qquad (g \in L^2).$$

Thus an equivalent condition for $f \in M(\phi)$ may be stated in the language of conditional expectation operators: *for every $f \in H^2$*,

$$f \in M(\phi) \Longleftrightarrow E(|f|^2) \circ \phi^{-1} \in L^\infty. \tag{18}$$

We will see that in order $E(|f|^2) \circ \phi^{-1} \in L^\infty$ it is not necessary that $f \in H^\infty$ (Proposition 12).

Clearly $M(\phi)$ is closed under multiplication by H^∞ functions; so it is impossible for $M(\phi) = BMOA$. Proposition 12 shows that even the inclusion $M(\phi) \subseteq BMOA$ occurs only under very special circumstances.

PROPOSITION 12. *The following are equivalent.*

1. $M(\phi) \subseteq BMOA$.
2. *ϕ is a finite Blaschke product.*
3. *ϕ is an inner function and for every $f \in H^2$*

$$E(|f|^2 | \mathscr{A}) \circ \phi^{-1} \in L^\infty \Longrightarrow f \in H^\infty,$$

where $\mathscr{A}$ is the σ-algebra defined in (15).

4. $M(\phi) \subseteq H^\infty$.

PROOF. We will prove the implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.

Proof of $1 \Rightarrow 2$: Suppose $M(\phi) \subseteq BMOA$. By Lemma 9, ϕ is an inner function. By a theorem of Frostman [6], page 176, there exists a Möbius map ψ_w such that $b = \psi_w \circ \phi$ is a Blaschke product. Then by Lemma 4, $M(b) \subseteq BMOA$. Consider the inclusion map i: $M(b) \rightarrow BMOA$. Since convergence in $M(b)$ implies pointwise convergence (Corollary 3), an application of the Closed Graph Theorem shows that i: $M(b) \rightarrow BMOA$ is bounded, i.e.,

$$\|f\|_{BMOA} \leq c \|f\|_{M(b)} \qquad (f \in M(b)) \tag{19}$$

for some constant c.

Let w be a zero of b. Apply (19) to the kernel $k_w^b = k_w$. By Lemma 6 and Lemma 11

$$\frac{|w|}{\sqrt{1 - |w|^2}} \leq c.$$

It follows that the number of zeros of b must be finite. Hence b is continuous across ∂D. However $\phi_w \circ b = \phi$, so ϕ is an inner function which is continuous across ∂D. Thus ϕ is a finite Blaschke product.

Proof of 2⇒3: Suppose ϕ is a finite Blaschke product. Let the number of zeros of ϕ, counting multiplicity, be n. Then $\phi: \partial D \to \partial D$ is an *n to* 1, *onto* function. Let $f \geq 0$ a.e. Define $\tilde{E}(f)$ by

$$\tilde{E}(f)(\xi) = \frac{1}{n} \sum_{\phi(\zeta)=\xi} f(\zeta) \qquad (\text{a.e. } \xi \in \partial D)\,.$$

Let $g \in L^2$. Then by a change of variable

$$\int f|g \circ \phi|^2 d\sigma = \int \tilde{E}(f)|g|^2 d\sigma \circ \phi^{-1}\,.$$

Comparing this equation with (17), we have

$$\tilde{E}(f) = E(f) \circ \phi^{-1} \qquad \text{a.e.}$$

Now clearly if $f \in H^2$ and $E(|f|^2) \circ \phi^{-1}$ is essentially bounded then $f \in H^\infty$.

Proof of 3⇒4: Follows from (18). □

Proposition 12 shows that it is hard for $M(\phi) = H^\infty$; however $C_\phi(H^2) \cap M(\phi) \subseteq H^\infty$ for functions ϕ which need not be even inner (Proposition 13 and [2], page 219).

PROPOSITION 13. *Suppose $C_\phi: H^2 \to H^2$ has closed range, $f \in H^2$ and $f \circ \phi \in M(\phi)$. Then f is bounded.*

PROOF. Note that C_ϕ is 1–1, therefore, $C_\phi^*: H^2 \to H^2$ has dense range. Suppose $C_\phi: H^2 \to H^2$ has closed range. Then C_ϕ^* has closed range, so it is *onto*, thus $C_\phi^* C_\phi: H^2 \to H^2$ is invertible. Let $f \circ \phi \in M(\phi)$, fix $g \in H^\infty$ and $h \in H^2$. Then

$$|\langle C_\phi(f) C_\phi(g), C_\phi(C_\phi^* C_\phi)^{-1}(h)\rangle| \leq c\|g\|_2 \|h\|_2\,,$$

so

$$|\langle fg, h\rangle| \leq c\|g\|_2 \|h\|_2\,.$$

Whence

$$\|fg\|_2 \leq c\|g\|_2 \|h\|_2 \qquad (g \in H^\infty, h \in H^2)\,.$$

Now after a standard application of Fatou' Lemma, we get f to be bounded by Lemma 1. □

If $f \in H^2$ and $f \circ \phi \in H^\infty$ then of course $f \circ \phi \in M(\phi)$. Now from Proposition 13 we can note that if C_ϕ has closed range then functions in H^2 which are bounded on the range of $\phi: D \to D$ are also in H^∞. Converse is false; for example, if ϕ is a conformal map from D onto the region Ω obtained from D by deleting an internally tangent disc then every $f \in H^2$ bounded on Ω is of course bounded on D, but C_ϕ does not have closed range [2], page 219.

PROPOSITION 14. *Let $b\neq 0$ be in H^∞. Then $bH^2\subseteq M(\phi)$ if and only if* $|b(z)|\leq c\sqrt{1-|\phi(z)|^2}$.

REMARK. Recall that ϕ is an extreme point of the unit ball of H^∞ if and only if $\log(1-|\phi|^2)$ is not integrable [6], Chapter 9, page 138. Thus before proceeding with the proof of the proposition we may note an equivalent form of it.

There exists $b\neq 0$ in the unit ball of H^∞ such that $bH^2\subseteq M(\phi)$ if and only if ϕ is not an extreme point of the unit ball of H^∞.

PROOF. Now to prove Proposition 14, suppose $|b(z)|\leq c\sqrt{1-|\phi(z)|^2}$ and let $g\in H^2$. Then by (4) $bg\circ\phi$ is bounded, so $bH^2\subset M(\phi)$.

Conversely suppose for some non-zero $b\in H^\infty$, $bH^2\subset M(\phi)$. Applying the Closed Graph Theorem to the map (from H^2 to $M(\phi)$)

$$f\to bf \qquad (f\in H^2),$$

we have

$$\|bf\|\leq c\|f\|_2 \qquad (f\in H^2).$$

Then by Lemma 5

$$|b(w)f(w)|\leq c\|f\|_2\sqrt{\frac{1-|\phi(w)|^2}{1-|w|^2}} \qquad (w\in D).$$

Put $f=k_w$ to deduce the desired inequality. □

COROLLARY 15. *Suppose $H^\infty\subseteq bH^2\subseteq M(\phi)$. Then C_ϕ is Hilbert-Schmidt.*

PROOF. Since $H^\infty\subseteq bH^2$, $b^{-1}\in H^2$. Hence $(1-|\phi|^2)^{-1}$ is integrable, so the corollary follows from [11], Theorem 3.1. □

PROPOSITION 16. *Suppose $M(\phi)$ is a Hilbert space. Then there exists b in the unit ball of H^∞ such that $M(\phi)=bH^2$, where $|b|^2+|\phi|^2=1$ and if $f\in M(\phi)$ then* $\|f\|=\|fb^{-1}\|_2$.

PROOF. Suppose $M(\phi)$ is a Hilbert space. Clearly multiplication by z acts as an isometry in $M(\phi)$, so by de Branges extension of Beurling's theorem [6], $M(\phi)=bH^2$ and if $f\in M(\phi)$ then $\|f\|=\|fb^{-1}\|_2$. By applying Lemma 5 to bk_w we get the inequality of Proposition 14 with $c=1$, i.e.,

$$|b(z)|\leq\sqrt{1-|\phi(z)|^2} \qquad (z\in D). \tag{20}$$

Let $f\in M(\phi)$. Then

$$\|f\|^2=\sup_{\|g\|_2=1}\int|f|^2|g\circ\phi|^2\leq\int|f|^2(1-|\phi|^2)^{-1}\leq\int|f|^2|b|^{-2}=\|f\|^2.$$

Thus equality holds throughout and in view of (20), $|b|^2 = 1 - |\phi|^2$. $\square$

REMARK. In particular suppose $M(\phi) = H^2$. Then $bH^2 = H^2$, so $b^{-1}H^2 = H^2$. Therefore by Lemma 1, $b^{-1} \in H^\infty$, i.e., $|b| > c > 0$ for some c. Thus we may note that $M(\phi) = H^2$ if and only if $|\phi| < r < 1$ for some $r > 0$.

4. Fredholm composition operators.

An operator T on a Hilbert space H is called Fredholm if the range of T is closed and the dimension of the kernel and the co-kernel of T are finite. In case of the composition operator C_ϕ on H^2 the kernel is trivial, so C_ϕ is Fredholm if and only if the range is closed and has finite co-dimension.

In a 1976 paper Cima, Thompson and Wogen [2] investigated among other things, Fredholm composition operators on H^2. They proved that C_ϕ is Fredholm if and only if ϕ is a conformal automorphism of the disc. We can now prove the same result by taking a quite different viewpoint from theirs; namely, we will consider the multipliers of the range of C_ϕ. If C_ϕ is Fredholm, then it's range is "very large" therefore we would expect it to be hard for a function $f \in H^2$ to be in $M(\phi)$. Lemma 17 states this fact in more precise language.

LEMMA 17. *The multipliers of closed subspaces of H^2 of finite co-dimension are bounded.*

PROOF. Let $M \subseteq H^2$ be any closed subspace of finite co-dimension. We prove the lemma by induction on n, the co-dimension of M. When $n = 0$, we are looking at the multipliers of H^2, so they are bounded (Lemma 1). Suppose the conclusion of the Lemma holds for all closed subspaces of co-dimension n, and now let M be a closed subspace of H^2 of co-dimension $n+1$. If M is closed under multiplication by ζ, then by a theorem of Beurling [6], Chapter 7, pp 99–100, $M = \phi H^2$ for some inner function ϕ. Then the multipliers of M are also multipliers of H^2, hence are bounded (Lemma 1). So now suppose there exists $f \in M$ such that $\zeta f \notin M$. Then

$$M' = \{M + a\zeta f : a \in C\}$$

is a closed subspace of co-dimension n, hence by the induction hypothesis, multipliers of M' are bounded. However, a multiplier of M is also a multiplier of M', so the result follows. $\square$

We are ready to classify Fredholm composition operators on H^2.

PROPOSITION 18 (Cima, Thompson and Wogen). *The composition operator $C_\phi : H^2 \to H^2$ is Fredholm if and only if ϕ is a conformal automorphism of the disc.*

PROOF. If ϕ is a conformal automorphism of the disc, then as noted by Schwartz [12], $C_\phi : H^2 \to H^2$ is invertible, so is trivially Fredholm. To prove the converse, suppose

C_ϕ is Fredholm. Then the range of C_ϕ is a closed subspace of finite co-dimension, so by Lemma 17, $M(\phi)=H^\infty$. Thus by Proposition 12, ϕ is a finite Blaschke product. Let us assume for the moment that $\phi(0)=0$. Then $d\sigma\circ\phi^{-1}=d\sigma$ (12) and therefore $C_\phi^* C_\phi=I$ where I is the identity operator on H^2. However, C_ϕ is invertible in the Calkin algebra [4], Chapter 5, 5.13 Definition and 5.17 Theorem, pp 127–129, so

$$C_\phi C_\phi^* = I + K \tag{21}$$

for some compact operator K: $H^2 \to H^2$.

Let $w \in D$ and k_w be the reproducing kernel at w. Then $\|k_w\|_2^{-1} k_w$ are of unit norm and tend to zero weakly in the Hilbert space H^2 as $|w|\to 1$. Whence $\|k_w\|_2^{-1} K(k_w)\to 0$ as $|w|\to 1$ in the H^2 norm, [4], Chapter 5, 5.6 Proposition, page 123. Then from (21)

$$\|k_w\|_2^{-2}\langle C_\phi C_\phi^*(k_w), k_w\rangle - \|k_w\|_2^{-2}\langle k_w, k_w\rangle \to 0$$

as $|w|\to 1$. Thus,

$$\|k_w\|_2^{-2}\langle C_\phi^*(k_w), C_\phi^*(k_w)\rangle - \|k_w\|_2^{-2} k_w(w) \to 0 \tag{22}$$

as $|w|\to 1$. Using the well known fact $C_\phi^*(k_w)=k_{\phi(w)}$, (3) and (22), we have that

$$\frac{1-|w|^2}{1-|\phi(w)|^2}\to 1 \qquad \text{as} \quad |w|\to 1\,.$$

By Schwarz-Pick lemma

$$|\phi'(w)|(1-|w|^2)\le 1-|\phi(w)|^2\,,$$

hence $|\phi'(\zeta)|\le 1$ for all $\zeta\in\partial D$ (ϕ' exists on ∂D because ϕ is a finite Blaschke product). Recall the usual formula for the number of zeros of ϕ in D:

$$\frac{1}{2\pi i}\int_{\partial D}\frac{\phi'(\zeta)}{\phi(\zeta)}d\zeta\,.$$

Since $|\phi'|\le 1$ on ∂D and $|\phi|=1$ on ∂D, we deduce that ϕ has at most one zero inside D; but $\phi(0)$ was assumed to be 0, so ϕ has exactly one zero in D. Hence $\phi(z)=\lambda z$ ($z\in D$) for some $\lambda\in\partial D$. To handle the general case, write $w=\phi(0)$ and as usual consider

$$C_{\psi_w\circ\phi}=C_\phi C_{\psi_w}\,.$$

Since C_{ψ_w} is an invertible operator on H^2, $C_{\psi_w\circ\phi}$ is also Fredholm with $\psi_w\circ\phi(0)=0$, from which the desired result follows. □

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