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On the Conformal Transformation Group of a Compact Riemannian Manifold with Constant Scalar Curvature

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§1. Introduction.

Let Conf(M, g) and Isom(M, g) be the conformal transformation group and the isometry group of a Riemannian n-manifold (M, g), respectively. It is obvious that $\mathrm{Isom}(M, g) \subset \mathrm{Conf}(M, g)$ and if g' is conformal to g then $\mathrm{Conf}(M, g') = \mathrm{Conf}(M, g)$. In late 1950's, conformal transformation groups of Einstein manifolds and Riemannian manifolds with parallel Ricci tensor were studied by Yano, Nagano, and Tanaka, and their results are stated as follows:

(1) *If* $\mathrm{Isom}_0(M, g) \neq \mathrm{Conf}_0(M, g)$ *holds for a compact connected Einstein n-manifold* (M, g), $n \geqq 3$, *then* (M, g) *is isometric to a Euclidean n-sphere, where* $\mathrm{Conf}_0(M, g)$ (*resp.* $\mathrm{Isom}_0(M, g)$) *denotes the connected component of the identity of* $\mathrm{Conf}(M, g)$ (*resp.* $\mathrm{Isom}(M, g)$) (Yano-Nagano [12]).

(2) *If* $\mathrm{Isom}(M, g) \neq \mathrm{Conf}(M, g)$ *holds for a compact connected Riemannian n-manifold* (M, g), $n \geqq 3$, *with parallel Ricci tensor, then* (M, g) *is isometric to a Euclidean n-sphere* (Tanaka [10], Nagano [7]).

(1) is also true if we replace the condition $\mathrm{Conf}_0(M, g) \neq \mathrm{Isom}_0(M, g)$ by $\mathrm{Conf}(M, g) \neq \mathrm{Isom}(M, g)$ (see (2) or [9] Proposition 6.2). On the other hand, Yamabe's theorem ([11]), which has been called the Yamabe problem later, says that every compact connected Riemannian n-manifold ($n \geqq 3$) is conformal to a Riemannian manifold with constant scalar curvature. Then it is natural to ask whether the same conclusion as (1) and (2) holds for Riemannian manifolds satisfying $\mathrm{Conf}(M, g) \neq \mathrm{Isom}(M, g)$ with constant scalar curvature. There are many results concerning this question and the above results (see [3], [8], [9] and their references). Ejiri, however in [3], gave a negative answer to this question. He proved that certain warped products of S^1 and Riemannian manifolds with positive constant scalar curvature satisfy $\mathrm{Conf}(M, g) \neq \mathrm{Isom}(M, g)$ and have constant scalar curvature. Note that the above

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warped products are rewritten in the form of what are obtained by deforming product metrics conformally with positive functions depending only on S^1. Then this result is explained as follows;

We can deform product metrics conformally with non-constant functions so that resulting metrics have constant scalar curvature. Then the natural isometric S^1-action with respect to product metrics turns out to be non-isometric but conformal with respect to resulting metrics.

Combined with an existence theorem for a solution of a certain nonlinear partial differential equation, this observation gives the following natural extension of Ejiri's result. As a consequence of our result, we get examples of compact simply connected Riemannian manifolds satisfying $\mathrm{Conf}(M, g) \neq \mathrm{Isom}(M, g)$ with constant scalar curvature (see Remark 4).

THEOREM. *Let (M_1, g_1) be a compact connected homogeneous Riemannian m_1-manifold and (M_2, g_2) a Riemannian m_2-manifold with positive constant scalar curvature ($m_1 \geqq 1, m_2 \geqq 2$). Then on the product $M = M_1 \times M_2$ there exists a Riemannian metric g with constant scalar curvature such that* $\mathrm{Conf}(M, g) \neq \mathrm{Isom}(M, g)$.

REMARK 1. It should be noticed that neither the compactness nor the connectedness is assumed for M_2 in the theorem.

REMARK 2. The metric g in the theorem is conformal to the product metric $g_1 + \varepsilon^2 g_2$ for some positive real number ε. In case $m_1 = 1$, M_1 must be S^1 and the theorem coinsides with Ejiri's.

REMARK 3. If (M_2, g_2) is a compact connected homogeneous Riemannian manifold with positive scalar curvature then the scalar curvature of g_2 is constant by homogeneity. The theorem in this case has been shown by O. Kobayashi in [4].

REMARK 4. M may be simply connected. In fact, for example we can take $M_1 = M_2 = S^m$ $(m \geqq 2)$.

REMARK 5. In contrast to the theorem, Obata ([9]) and Lelong-Ferrand ([5], [6]) proved the following theorem.

(3) *Let (M, g) be a compact connected Riemannian n-manifold ($n \geqq 3$). If* $\mathrm{Isom}(M, g) \neq \mathrm{Conf}(M, g)$ *holds for any Riemannian metric g' which is conformal to g then (M, g) is conformally equivalent to a Euclidean n-sphere* ([5], [6], [9]). *Furthermore if the scalar curvature of g is constant then (M, g) is isometric to a Euclidean n-sphere* ([9]).

Thus there exists a Riemannian metric g', which is conformal to g in the theorem, such that $\mathrm{Conf}(M, g) = \mathrm{Conf}(M, g') = \mathrm{Isom}(M, g')$.

2. Preliminary.

The essential difference between the Ejiri's result and ours comes from that between an ODE and a PDE. To prove the theorem we need an existence theorem for a solution of a certain nonlinear PDE.

Let (N, h) be a compact connected Riemannian n-manifold. $C^\infty(N)$ denotes the space of smooth functions on N. The first non-zero eigenvalue of the Laplacian $\Delta_h = -\nabla^i\nabla_i$ of h is denoted by $\lambda_1(\Delta_h)$, and the volume element of h is denoted by dV_h.

LEMMA (cf. [11], [1] 16.37). *For arbitrary real numbers $k>0$, A and $q>2$ with $q<2n/(n-2)$ (if $n\geqq 3$) or $q<\infty$ (if $n=1, 2$), there exists a positive C^∞ solution u of* (2.1) *with $\int_N |u|^q dV_h = 1$:*

$$(2.1) \qquad (q-2)k\Delta_h u + Au = \mu u^{q-1}$$

where

$$(2.2) \qquad \mu = \inf I(f) = \inf \frac{(q-2)k\int_N |df|^2 dV_h + A\int_N f^2 dV_h}{\left(\int_N |f|^q dV_h\right)^{2/q}}$$

and the infimum is taken over all positive C^∞ functions. Moreover if $\lambda_1(\Delta_h) < A/k$ then the solution u of (2.1) *is a non-constant function.*

In case $n\geqq 3$, the first part of the lemma was proved by Yamabe ([11]). For the case $n=1, 2$, it can be shown similarly (or more easily) by using a variational method and a maximum principle. If we assume $u=$constant, a computation of the second variation of I at u gives $\lambda_1(\Delta_h)\geqq A/k$. Thus we obtain the second part of the lemma.

3. Proof of Theorem.

Define a Riemannian metric g_ε on $M_1\times M_2$ by $g_\varepsilon = g_1 + \varepsilon^2 g_2$ for $\varepsilon>0$. Then the scalar curvature R_{g_ε} of g_ε is constant and equal to $R_{g_1} + \varepsilon^{-2}R_{g_2}$. The scalar curvature R_g of

$$g = v^{4/(n-2)} g_\varepsilon\,, \quad v\in C^\infty(M_1\times M_2)\,, \qquad v>0$$

is given by the following formula (see for example [1] 1. 161).

$$(3.1) \qquad 4\frac{n-1}{n-2}\Delta_{g_\varepsilon} v + R_{g_\varepsilon} v = R_g v^{(n+2)/(n-2)}\,,$$

where $n = m_1 + m_2$.

First we show that for suitable $\varepsilon>0$ there exists a metric $g = u^{4/(n-2)} g_\varepsilon$ with constant scalar curvature where $u>0$ is a non-constant C^∞ function and depends only on M_1. The basic idea of the proof of this claim is the same as what Derdzinski has used in [2] to find a metric with harmonic curvature and non-parallel Ricci tensor (see also [1] 16.35).

Since g_ε is a product metric and R_{g_ε} is constant, for any $v\in C^\infty(M_1)\subset C^\infty(M_1\times M_2)$ (i.e., v depends only on M_1),

$$4\frac{n-1}{n-2}\Delta_{g_\varepsilon}v+R_{g_\varepsilon}v=4\frac{n-1}{n-2}\Delta_{g_1}v+R_{g_\varepsilon}v\,,$$

and we can consider that the right hand side is defined on M_1. Then (3.1) is reduced to the following formula defined on M_1.

$$4\frac{n-1}{n-2}\Delta_{g_1}v+R_{g_\varepsilon}v=R_g v^{(n+2)/(n-2)}\,. \tag{3.2}$$

Thus to show the claim above we have to prove that there exists a non-constant positive C^∞ solution u of the equation (3.2) with $R_g=$constant. To see this, take sufficiently small $\varepsilon>0$ so that $\lambda_1(\Delta_{g_1})<R_{g_\varepsilon}/(n-1)$, and put $A=R_{g_\varepsilon}$, $k=n-1$ and $q=2n/(n-2)$ in (2.1) (note that $2n/(n-2)<2m_1/(m_1-2)$, or ∞). Then we can apply the lemma to the equation (3.2) with $R_g\equiv\mu$, and the claim follows.

Now, assume that $\mathrm{Conf}(M_1\times M_2, g)=\mathrm{Isom}(M_1\times M_2, g)$. For any $\varphi\in\mathrm{Isom}(M_1, g_1)$, we can define $\tilde{\varphi}\in\mathrm{Isom}(M_1\times M_2, g_\varepsilon)$ by $\tilde{\varphi}(p, x)=(\varphi(p), x)$ where $p\in M_1$, $x\in M_2$. Hence $\mathrm{Isom}(M_1\times M_2, g)=\mathrm{Conf}(M_1\times M_2, g)=\mathrm{Conf}(M_1\times M_2, g_\varepsilon)\supset\mathrm{Isom}(M_1, g_1)$. Therefore the action of $\mathrm{Isom}(M_1, g_1)$ leaves g invariant. Let us denote g at $(p, x)\in M_1\times M_2$ by $g(p, x)$ and put $f\equiv u^{4/(n-2)}$. If $\varphi(p)=q$ for $\varphi\in\mathrm{Isom}(M_1, g_1)$ and $p, q\in M_1$ then

$$\begin{aligned}(\tilde{\varphi}^*g)(p, x)&=\tilde{\varphi}^*\{f(q)g_\varepsilon(q, x)\}=f(q)\{\varphi^*(g_1(q))+\varepsilon^2 g_2(x)\}\\&=f(q)\{g_1(p)+\varepsilon^2 g_2(x)\}=f(q)g_\varepsilon(p, x)\\&=g(p, x)=f(p)g_\varepsilon(p, x)\end{aligned}$$

holds for any $x\in M_2$, where $\tilde{\varphi}^*g$ (resp. φ^*g_1) is the pull-back of g (resp. g_1) by $\tilde{\varphi}$ (resp. φ). Thus $f(p)=f(q)$. Since $\mathrm{Isom}(M_1, g_1)$ acts transitively on M_1 and f depends only on M_1, f must be a constant function. That is, u is a constant function, a contradiction. Thus $\mathrm{Conf}(M_1\times M_2, g)\neq\mathrm{Isom}(M_1\times M_2, g)$. This completes the proof.

REMARK. Let Y be a gradient vector field of u on M_1 with respect to g_1 and take $p\in M_1$ such that $Y\neq 0$ at p. Since $\mathrm{Isom}(M_1, g_1)$ acts transitively on M_1, there exists a Killing vector field X of (M_1, g_1) such that $X=Y$ at p. This vector field X can be lifted to a Killing vector field of $(M_1\times M_2, g_\varepsilon)$ and it is easy to see that X is not a Killing vector field of $(M_1\times M_2, g)$. That is, $(M_1\times M_2, g)$ admits a non-isometric conformal vector field.

References

[1] A. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.

[2] A. Derdzinski, An easy construction of new compact Riemannian manifold with harmonic curvature (Preliminary report), Preprint SFB 40-MPI no. 21, 1983, Bonn.

[3] N. Ejiri, A negative answer to a conjecture of conformal transformations of Riemannian manifolds, J. Math. Soc. Japan, **33** (1981), 261–266.

[4] O. Kobayashi, On the Yamabe problem (in Japanese), Seminar on Mathematical Sciences, **16**, Keio University, 1990.

[5] J. Lelong-Ferrand, Transformations conformes et quasi-conformes des variétés riemanniennes compactes (démonstration de la conjecture de A. Lichnerowicz), Acad. Roy. Belg. Cl. Sci. Mem. Coll. 8°(2), **39**, no. 5, 1971.

[6] J. Lelong-Ferrand, Geometrical interpretation of scalar curvature and regularity of conformal homeomorphism, *Differential Geometry and Relativity*, ed. by M. Cahen and M. Flato, Reidel Publ., 1976.

[7] T. Nagano, The conformal transformations on a space with parallel Ricci tensor, J. Math. Soc. Japan, **11** (1959), 10–14.

[8] M. Obata, Conformal transformations of Riemannian manifolds, J. Diff. Geom., **4** (1970), 311–333.

[9] M. Obata, The conjectures on conformal transformations of Riemannian manifolds, J. Diff. Geom., **6** (1971), 247–258.

[10] N. Tanaka, Conformal connections and conformal transformations, Trans. Amer. Math. Soc., **92** (1959), 168–190.

[11] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J., **12** (1960), 21–37.

[12] K. Yano and T. Nagano, Einstein spaces admitting a one-parameter group of conformal transformations, Ann. of Math., **69** (1959), 451–461.

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