TOKYO J. MATH.
VOL. 20, NO. 1, 1997

Invariant Einstein Metrics on Certain Homogeneous Spaces

Joon-Sik PARK and Yusuke SAKANE

Pusan University of Foreign Studies and Osaka University
(Communicated by T. Nagano)

0. Introduction.

Existence and non-existence of homogeneous Einstein metrics are investigated by M. Wang and W. Ziller ([4]). They proved a general existence theorem for homogeneous Einstein metrics on compact homogeneous spaces G/H with a maximal connected closed subgroup H of G and gave a criterion for existence of G-homogeneous Einstein metrics on compact homogeneous spaces G/H, when the isotropy representation decomposes into two inequivalent irreducible summands. Wang and Ziller also exhibited some compact simply connected homogeneous spaces which carry no G-homogeneous Einstein metrics.

In this paper we study existence and non-existence of homogeneous Einstein metrics on certain compact homogeneous spaces G/H whose isotropy representation decomposes into three inequivalent irreducible summands and give such examples of compact connected homogeneous spaces G/H that have G-invariant homogeneous Einstein metrics and examples of G/H which carry no G-invariant Einstein metrics.

1. Ricci tensor.

Let G be a compact connected semi-simple Lie group and H a closed subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras of G and H. Let B be the negative of the Killing form of $\mathfrak{g}$. We consider the $Ad(H)$-invariant decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ with $B(\mathfrak{h}, \mathfrak{m})=0$. Then the set of G-invariant symmetric covariant 2-tensors on G/H can be identified with the set of $Ad(H)$-invariant symmetric bilinear forms on $\mathfrak{m}$. In particular, the set of G-invariant metrics on G/H is identified with the set of $Ad(H)$-invariant inner products on $\mathfrak{m}$.

Let $\mathfrak{m}=\mathfrak{m}_1+\cdots+\mathfrak{m}_q$ be a B-orthogonal $Ad(H)$-invariant decomposition such that

Received June 14, 1995
The first author was supported by TGRC-KOSEF.

$Ad(H)|\mathfrak{m}_i$ is irreducible for $i=1, \cdots, q$ and assume that $\mathfrak{m}_i$ are mutually inequivalent irreducible $Ad(H)$-representations. Then the space of G-invariant symmetric covariant 2-tensors on G/H is given by

$$\{x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q} \mid x_1, \cdots, x_q \in \mathbf{R}\},$$

and the space of G-invariant Riemannian metrics on G/H is given by

$$\{x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q} \mid x_1 > 0, \cdots, x_q > 0\}.$$

Note that the Ricci tensor r of G-invariant Riemannian metric on G/H is a G-invariant symmetric covariant 2-tensors on G/H, and we identify r with an $Ad(H)$-invariant symmetric bilinear form on $\mathfrak{m}$. Thus r is written as

$$r = y_1 B|_{\mathfrak{m}_1} + \cdots + y_q B|_{\mathfrak{m}_q}$$

for some $y_1, \cdots, y_q$.

Now we compute components of Ricci tensor r of G-invariant Riemannian metrics on G/H explicitly. Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e., $e_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$). Next set $A_{\alpha\beta}^{\gamma} = B([e_\alpha, e_\beta], e_\gamma)$, so that $[e_\alpha, e_\beta]_{\mathfrak{m}} = \sum_\gamma A_{\alpha\beta}^{\gamma} e_\gamma$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A_{\alpha\beta}^{\gamma})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i$, $e_\beta \in \mathfrak{m}_j$, $e_\gamma \in \mathfrak{m}_k$, and $[\,,\,]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component. Then, $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the B-orthonormal bases chosen for $\mathfrak{m}_i$, $\mathfrak{m}_j$, $\mathfrak{m}_k$, and

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}. \tag{1.1}$$

We denote by $g_{x_1, \cdots, x_q}$ the G-invariant Riemannian metric on G/H corresponding to an inner product $x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}$ on $\mathfrak{m}$.

LEMMA 1.1. *The components $r_1, \cdots, r_q$ of Ricci tensor r of $g_{x_1, \cdots, x_q}$ are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix} \qquad (k = 1, \cdots, q). \tag{1.2}$$

PROOF. For simplicity, let $\langle\,,\,\rangle$ be the inner product on $\mathfrak{m}$ corresponding to the metric $g_{x_1, \cdots, x_q}$ and let $\{X_j^{(k)}\}_{j=1}^{d_k}$ (resp. $\{e_j^{(k)}\}_{j=1}^{d_k}$) be a $\langle\,,\,\rangle$-orthonormal (resp. B-orthonormal) basis on $\mathfrak{m}_k$ $(k = 1, \cdots, q)$. We can put $X_j^{(k)} = (1/\sqrt{x_k}) e_j^{(k)}$. The Ricci tensor r of $g_{x_1, \cdots, x_q}$ is given by the following (cf. [1], pp. 184–185):

$$r(X, X) = -\frac{1}{2} \sum_j \langle [X, X_j]_{\mathfrak{m}}, [X, X_j]_{\mathfrak{m}} \rangle + \frac{1}{2} B(X, X) + \frac{1}{4} \sum_{j,i} \langle [X_j, X_i]_{\mathfrak{m}}, X \rangle^2$$

for $X \in \mathfrak{m}$. From this equation, we have

$$r_k = r(X_l^{(k)}, X_l^{(k)}) = \frac{1}{2x_k} - \frac{1}{2}\sum_{j,i}\frac{x_j}{x_k x_i}\sum_{s \in \mathfrak{m}_i} B([e_l^{(k)}, e_s^{(i)}]_{\mathfrak{m}_j}, [e_l^{(k)}, e_s^{(i)}]_{\mathfrak{m}_j})$$
$$+ \frac{1}{4}\sum_{j,i}\frac{x_k}{x_j x_i}\sum_{s,t} B([e_s^{(j)}, e_t^{(i)}]_{\mathfrak{m}_k}, e_l^{(k)})^2 .$$

As we have remarked above,

$$d_k r_k = \sum_{l=1}^{d_k} r(X_l^{(k)}, X_l^{(k)}) = \frac{d_k}{2x_k} - \frac{1}{2}\sum_{j,i}\frac{x_j}{x_k x_i}\left[\begin{matrix} j \\ ki \end{matrix}\right] + \frac{1}{4}\sum_{j,i}\frac{x_k}{x_j x_i}\left[\begin{matrix} k \\ ji \end{matrix}\right]. \qquad \text{q.e.d.}$$

From now on, we consider the case of $q=3$ and let K_1, K_2 be closed connected subgroups of G with $H \subsetneqq K_1 \subsetneqq K_2 \subsetneqq G$ and $\mathfrak{k}_i$ the subalgebra of $\mathfrak{g}$ corresponding to K_i. We assume that subgroups K_1 and K_2 satisfy the following conditions:

$$(1.3) \qquad \begin{cases} \mathfrak{k}_1 = \mathfrak{h} + \mathfrak{m}_1 & \text{with } [\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h} , \\ \mathfrak{k}_2 = \mathfrak{h} + \mathfrak{m}_1 + \mathfrak{m}_2 & \text{with } [\mathfrak{k}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2 \text{ and } [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}_1 , \\ \mathfrak{g} = \mathfrak{h} + \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 & \text{with } [\mathfrak{k}_2, \mathfrak{m}_3] \subset \mathfrak{m}_3 \text{ and } [\mathfrak{m}_3, \mathfrak{m}_3] \subset \mathfrak{k}_2 . \end{cases}$$

Homogeneous spaces K_2/K_1 and K_1/H need not be effective in general. So let K_1' (resp. K_2') be the quotient of K_1 (resp. K_2) acting effectively on K_1/H (resp. K_2/K_1). We also assume that K_1', K_2' are semisimple and $\alpha_i B|_{\mathfrak{k}_i'} = B_{\mathfrak{k}_i'}$ for some $\alpha_i > 0$ $(i=1, 2)$, where $B_{\mathfrak{k}_i'}$ is the negative of the Killing form of $\mathfrak{k}_i'$.

By our assumption (1.3), we have

$$(1.4) \qquad \left[\begin{matrix} 1 \\ 11 \end{matrix}\right] = \left[\begin{matrix} 2 \\ 11 \end{matrix}\right] = \left[\begin{matrix} 3 \\ 11 \end{matrix}\right] = \left[\begin{matrix} 2 \\ 13 \end{matrix}\right] = \left[\begin{matrix} 2 \\ 22 \end{matrix}\right] = \left[\begin{matrix} 2 \\ 23 \end{matrix}\right] = \left[\begin{matrix} 3 \\ 33 \end{matrix}\right] = 0 .$$

Let $\{Z_i\}$ be an orthonormal basis of $\mathfrak{h}$ with respect to $B|_{\mathfrak{h}}$ and $C = -\sum_j (adZ_j)^2$. Since each $Ad(H)|\mathfrak{m}_i$ is irreducible, there exists positive constant c_i such that $C|_{\mathfrak{m}_i} = c_i I|_{\mathfrak{m}_i}$, $(i=1, 2, 3)$. Now we have the following result by Wang and Ziller ([4], p. 181, Lemma 1.5):

$$(1.5) \qquad \sum_{j,k}\left[\begin{matrix} k \\ ij \end{matrix}\right] = (1-2c_i)d_i \qquad (i=1, 2, 3) .$$

We have

$$(1.6) \qquad \left[\begin{matrix} 3 \\ 13 \end{matrix}\right] = (1-\alpha_2)d_1 ,$$

since

$$\left[\begin{matrix} 3 \\ 13 \end{matrix}\right] = -\sum_{e_\alpha \in \mathfrak{m}_1} tr_{\mathfrak{m}_3}(pr_{\mathfrak{m}_3}(ade_\alpha))^2 = \sum_{e_\alpha \in \mathfrak{m}_1}\{-tr_{\mathfrak{g}}(ade_\alpha)^2 + tr_{\mathfrak{k}_2'}(ade_\alpha)^2\}$$
$$= \sum_{e_\alpha \in \mathfrak{m}_1}\{B(e_\alpha, e_\alpha) + B_{\mathfrak{k}_2'}(e_\alpha, e_\alpha)\} = (1-\alpha_2)d_1 .$$

Similarly we can get

(1.7) $$\begin{bmatrix} 3 \\ 23 \end{bmatrix} = (1-\alpha_2)d_2 .$$

From (1.4), (1.5), (1.6), we obtain

(1.8) $$\begin{bmatrix} 2 \\ 12 \end{bmatrix} = (\alpha_2 - 2c_1)d_1 .$$

Let c_1^* be the Casimir constant of $\mathfrak{m}_1$ with respect to $B_{\mathfrak{k}'_1}|_{\mathfrak{h}}$. From $\alpha_1 B|_{\mathfrak{k}'_1} = B_{\mathfrak{k}'_1}$, we have $c_1 = \alpha_1 c_1^*$. Since K_1/H is a symmetric space, we have $c_1^* = 1/2$ ([3], p. 568, Corollary I.1.6). Thus we have

(1.9) $$\begin{bmatrix} 2 \\ 12 \end{bmatrix} = (\alpha_2 - \alpha_1)d_1 .$$

Now we normalize G-invariant Riemannian metrics by putting $x_3 = 1$ and denote by g_{t_1,t_2} the metric defined by $t_1 B|_{\mathfrak{m}_1} + t_2 B|_{\mathfrak{m}_2} + B|_{\mathfrak{m}_3}$. Letting $r_1 = r_2$ and $r_2 = r_3$ in (1.2), we get the following:

THEOREM 1.2. *For a compact homogeneous manifold G/H which satisfies the conditions* (1.3), *a G-invariant Riemannian metric g_{t_1,t_2} on G/H is Einstein if and only if $t_1 > 0$, $t_2 > 0$ satisfy the following equations*:

(1.10) $$(1-\alpha_2)d_2 t_1 t_2^3 + 2\alpha_2 d_2 t_1 t_2 = (\alpha_2 - \alpha_1)(2d_1 + d_2)t_1^2 + (1-\alpha_2)d_2 t_1^2 t_2^2 + 2\alpha_1 d_2 t_2^2 , \quad \textit{and}$$

(1.11) $$2d_2 d_3 t_2^2 - 2(1-\alpha_2)d_1 d_2 t_1 t_2^2 = (1-\alpha_2)d_2(2d_2 + d_3)t_2^3 - 2(\alpha_2 - \alpha_1)d_1 d_3 t_1 + 2\alpha_2 d_2 d_3 t_2 ,$$

where $d_i = \dim \mathfrak{m}_i$ $(i = 1, 2, 3)$.

From (1.11), we have

(1.12) $$t_1 = \frac{d_2 t_2}{2d_1} \cdot \frac{(\alpha_2 - 1)(2d_2 + d_3)t_2^2 + 2d_3 t_2 - 2\alpha_2 d_3}{(1-\alpha_2)d_2 t_2^2 - (\alpha_2 - \alpha_1)d_3} .$$

Now we get the following corollary which gives a necessary condition for a G-invariant Riemannian metric g_{t_1,t_2} on G/H being Einstein from (1.10) and (1.12).

COROLLARY 1.3. *If there exists no positive number t_2 satisfying the following equation*

(1.13) $$(1-\alpha_2)^3 d_2^2(2d_2 + d_3)(2d_1 + 2d_2 + d_3)t_2^6 - 4(1-\alpha_2)^2 d_2^2 d_3(d_1 + 2d_2 + d_3)t_2^5 + (1-\alpha_2)d_2^2\{4d_3^2 + \alpha_2(1-\alpha_2)(16d_1 d_2 + 12d_1 d_3 + 4d_2^2 + 12d_2 d_3 + 5d_3^2) + \alpha_1(1-\alpha_2)(8d_1^2 - 4d_1 d_3 - 8d_1 d_2 - 4d_2^2 - 4d_2 d_3 - d_3^2)\}t_2^4$$

$$
\begin{aligned}
&+4(1-\alpha_2)d_2d_3\{\alpha_1(4d_1d_2+d_1d_3+2d_2^2+d_2d_3)-\alpha_2(6d_1d_2+d_1d_3+2d_2^2+3d_2d_3)\}t_2^3\\
&+4d_2d_3\{2(\alpha_2^2-\alpha_2^3)d_2(2d_1+d_2+d_3)+(\alpha_1\alpha_2^2-\alpha_1\alpha_2)(4d_1^2+2d_1d_2+2d_2^2+d_2d_3)\\
&+4\alpha_1^2(1-\alpha_2)d_1^2+(\alpha_2-\alpha_1)d_3(2d_1+d_2)\}t_2^2\\
&-8\alpha_2(\alpha_2-\alpha_1)d_2d_3^2(d_1+d_2)t_2+4(\alpha_2-\alpha_1)d_3^2\{2\alpha_1(\alpha_2-\alpha_1)d_1^2+\alpha_2^2d_2^2\}=0\,,
\end{aligned}
$$

then G/H carries no G-invariant Einstein metrics.

2. Examples.

We consider the case when $G=SO(2n+l)$, $K_1=U(n)\times SO(l)$, $K_2=SO(2n)\times SO(l)$ and $H=SO(n)\cdot U(1)\times SO(l)$ where $n\geqq 3$, $n\neq 4$ and $l\geqq 3$. Note that the imbedding of $U(n)$ into $SO(2n)$ is given by

$$A+\sqrt{-1}B\rightarrow\begin{pmatrix} A & B\\ -B & A\end{pmatrix}$$

and that the subgroup H of G is given by

$$\left\{\begin{pmatrix}\cos\theta\cdot A & \sin\theta\cdot A & 0\\ -\sin\theta\cdot A & \cos\theta\cdot A & 0\\ 0 & 0 & B\end{pmatrix}\,\middle|\, \theta\in\mathbf{R},\ A\in SO(n),\ B\in SO(l)\right\}.$$

The spaces $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3$ are given by

$$\mathfrak{m}_1=\left\{\begin{pmatrix}0 & S & 0\\ -S & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\,\middle|\, S \text{ is a real symmetric } n\times n \text{ matrix of trace } 0\right\},$$

$$\mathfrak{m}_2=\left\{\begin{pmatrix}X & Y & 0\\ Y & -X & 0\\ 0 & 0 & 0\end{pmatrix}\,\middle|\, X,\ Y\in\mathfrak{so}(n)\right\},$$

$$\mathfrak{m}_3=\left\{\begin{pmatrix}0 & Z\\ -{}^tZ & 0\end{pmatrix}\,\middle|\, Z \text{ is a real } 2n\times l \text{ matrix}\right\}.$$

Note that the subgroup $SO(n)\cdot U(1)$ acts on $\mathfrak{m}_2$ by

$$
\begin{aligned}
&\begin{pmatrix}\cos\theta\cdot A & \sin\theta\cdot A\\ -\sin\theta\cdot A & \cos\theta\cdot A\end{pmatrix}\begin{pmatrix}X & Y\\ Y & -X\end{pmatrix}\begin{pmatrix}\cos\theta\cdot A & \sin\theta\cdot A\\ -\sin\theta\cdot A & \cos\theta\cdot A\end{pmatrix}^{-1}\\
&=\begin{pmatrix}\cos 2\theta\cdot AXA^{-1}+\sin 2\theta\cdot AYA^{-1} & -\sin 2\theta\cdot AXA^{-1}+\cos 2\theta\cdot AYA^{-1}\\ -\sin 2\theta\cdot AXA^{-1}+\cos 2\theta\cdot AYA^{-1} & -(\cos 2\theta\cdot AXA^{-1}+\sin 2\theta\cdot AYA^{-1})\end{pmatrix}.
\end{aligned}
$$

Since $SO(n)$ ($n\geqq 3$, $n\neq 4$) is a simple Lie group, we see that $\mathfrak{m}_2$ is an irreducible $Ad(H)$-module. Note also that the subgroup H acts on $\mathfrak{m}_3$ by

$$\begin{pmatrix} \cos\theta\cdot A & \sin\theta\cdot A & 0 \\ -\sin\theta\cdot A & \cos\theta\cdot A & 0 \\ 0 & 0 & B \end{pmatrix} \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & D \\ -{}^tC & -{}^tD & 0 \end{pmatrix} \begin{pmatrix} \cos\theta\cdot A & \sin\theta\cdot A & 0 \\ -\sin\theta\cdot A & \cos\theta\cdot A & 0 \\ 0 & 0 & B \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 0 & 0 & P \\ 0 & 0 & Q \\ -{}^tP & -{}^tQ & 0 \end{pmatrix},$$

where $P=\cos\theta\cdot ACB^{-1}+\sin\theta\cdot ADB^{-1}$, $Q=-\sin\theta\cdot ACB^{-1}+\cos\theta\cdot ADB^{-1}$. Thus we see that $\mathfrak{m}_3$ is an irreducible $Ad(H)$-module. Moreover, $\mathfrak{m}_i$ $(i=1,2,3)$ are mutually inequivalent $Ad(H)$-representations and thus the homogeneous manifolds $G/H=SO(2n+l)/(SO(n)\cdot U(1)\times SO(l))$, $n\geqq 3$ $(n\neq 4)$, $l\geqq 3$ satisfy our assumption (1.3). We also have

$$\begin{cases} d_1=(n^2+n-2)/2\,, & d_2=(n^2-n)\,, \quad d_3=2nl\,, \\ \alpha_1=n/(2n+l-2)\,, & \alpha_2=(2n-2)/(2n+l-2)\,. \end{cases} \tag{2.1}$$

Thus the equation (1.13) is given by

$$\begin{aligned}
&\frac{2l^2(-1+n)^2n^3}{(-2+l+2n)^3}(l(-1+l+n)(-2-n+2ln+3n^2)t_2^6\\
&+2l(-2+l+2n)(2+3n-4ln-5n^2)t_2^5\\
&+(-12+24l+28n+24ln-52l^2n+8l^3n-11n^2-110ln^2\\
&+50l^2n^2-14n^3+62ln^3+9n^4)t_2^4\\
&+4(-2+l+2n)(-12+4l+10n+12ln+8n^2-11ln^2-6n^3)t_2^3\\
&+4(-56+32l-8l^2+80n-4l^2n+24n^2-60ln^2+4l^2n^2-73n^3+28ln^3+25n^4)t_2^2\\
&+16(2-l-2n)(-2+n)(-1+n)(2+3n)t_2+4(-2+n)(-8+4n-14n^2+9n^3)=0\,.
\end{aligned}$$

Now we consider the case of $l=3$. In this case, the above equation is given by

$$\begin{aligned}
&4(-2+n)(-8+4n-14n^2+9n^3)+16(-1-2n)(-2+n)(-1+n)(2+3n)t_2\\
&+4(-32+44n-120n^2+11n^3+25n^4)t_2^2+4(1+2n)(46n-25n^2-6n^3)t_2^3\\
&+(60-152n+109n^2+172n^3+9n^4)t_2^4+6(1+2n)(2-9n-5n^2)t_2^5\\
&+3(2+n)(-2+5n+3n^2)t_2^6=0\,.
\end{aligned}$$

Let $f(t_2)$ be the left hand side of the above equation and put $f_2(t_2)=df^2/dt_2^2$ and $f_4(t_2)=df^4/dt_2^4$. Then

$$\begin{aligned}
f_2(t_2)=&\,8(-32+44n-120n^2+11n^3+25n^4)+24n(1+2n)(46-25n-6n^2)t_2\\
&+12(60-152n+109n^2+172n^3+9n^4)t_2^2+120(-1-2n)(2+n)(-1+5n)t_2^3\\
&+90(2+n)^2(-1+3n)t_2^4\,,
\end{aligned}$$

$$f_4(t_2)=24(60-152n+109n^2+172n^3+9n^4)+720(-1-2n)(2+n)(-1+5n)t_2$$
$$+1080(2+n)^2(-1+3n)t_2^2\,.$$

Now the discriminant D of $f_4(t_2)$ is given by

$$D=103680(2+n)^2(65-362n+510n^2+145n^3-7n^4-27n^5)\,.$$

For $n\geqq 4$ we see that $D<0$ and thus $f_4(t_2)>0$ for all t_2. This implies that $f_2(t_2)$ is a convex function. Consider tangent lines z, u of $f_2(t_2)$ at $t_2=4/3$ and $t_2=7/6$. Since $z=(df_2/dt_2)(4/3)(t_2-4/3)+f_2(4/3)$, we have

$$z=\frac{4096}{9}-\frac{1696n}{3}+96n^2-\frac{232n^3}{9}+8n^4$$
$$+\left(-\frac{640}{3}-\frac{400n}{3}-\frac{712n^2}{3}+320n^3\right)\left(t_2-\frac{4}{3}\right).$$

Note that

$$-\frac{640}{3}-\frac{400n}{3}-\frac{712n^2}{3}+320n^3>0 \qquad \text{if} \quad n\geqq 2\,.$$

Since $u=(df_2/dt_2)(7/6)(t_2-7/6)+f_2(7/6)$, we have

$$u=\frac{2629}{6}-\frac{4154n}{9}+\frac{10631n^2}{72}-\frac{5473n^3}{73}+11n^4$$
$$+\left(\frac{1120}{3}-\frac{3086n}{3}-\frac{965n^2}{3}+287n^3-36n^4\right)\left(t_2-\frac{7}{6}\right).$$

Note that

$$\frac{1120}{3}-\frac{3086n}{3}-\frac{965n^2}{3}+287n^3-36n^4<0 \qquad \text{if} \quad n\geqq 6\,.$$

Solving $z=u$, we get the solution

$$t_2^0=\frac{-(53060-81080n-7955n^2-2995n^3-3240n^4)}{24(-1760+2686n+253n^2+99n^3+108n^4)}$$

and at the point t_2^0 we have

$$z^0=u^0=(-2489920+6727832n-4728660n^2+897654n^3-379595n^4-106665n^5$$
$$+29008n^6-14616n^7+2592n^8)/3(-1760+2686n+253n^2+99n^3+108n^4)\,.$$

We see that $z^0=u^0>0$ for $n\geqq 6$ and thus $f_2(t_2)>0$, because $f_2(t_2)$ is a convex function. This implies that $f(t_2)$ is a convex function for $n\geqq 6$.

Now consider tangent lines p, q of $f(t_2)$ at $t_2=6/5$ and $t_2=4/3$. Since $p=$

$(df/dt_2)(6/5)(t_2-6/5)+f(6/5)$, we have

$$p=-\frac{1229312}{15625}+\frac{556544n}{15625}+\frac{241808n^2}{15625}-\frac{98096n^3}{15625}+\frac{324n^4}{625}$$
$$+\left(-\frac{35072}{3125}+\frac{268544n}{3125}-\frac{209152n^2}{3125}+\frac{47104n^3}{3125}-\frac{144n^4}{125}\right)\left(t_2-\frac{6}{5}\right).$$

Note that

$$-\frac{35072}{3125}+\frac{268544n}{3125}-\frac{209152n^2}{3125}+\frac{47104n^3}{3125}-\frac{144n^4}{125}<0 \quad \text{if} \quad n\geqq 2.$$

Since $q=(df/dt_2)(4/3)(t_2-4/3)+f(4/3)$, we have

$$q=-\frac{18496}{243}+\frac{10304n}{243}+\frac{1856n^2}{243}-\frac{128n^3}{27}+\frac{4n^4}{9}$$
$$+\left(\frac{448}{9}+\frac{128n}{9}-\frac{464n^2}{9}+\frac{80n^3}{9}\right)\left(t_2-\frac{4}{3}\right).$$

Note that

$$\frac{448}{9}+\frac{128n}{9}-\frac{464n^2}{9}+\frac{80n^3}{9}>0 \quad \text{if} \quad n\geqq 5.$$

Solving $p=q$, we get the solution

$$t_2^1=\frac{-(3667652-4316212n+921287n^2-368937n^3+69120n^4)}{27(-107228+126056n-27023n^2+10871n^3-2025n^4)},$$

and at the point t_2^1 we have

$$p^1=q^1=4(-57487616+97531392n-41806576n^2+198704n^3$$
$$+2786616n^4-1640696n^5+631881n^6-105813n^7+6075n^8)/$$
$$27(107228-126056n+27023n^2-10871n^3+2025n^4).$$

Now we see that $p^1=q^1>0$ for $n\geqq 8$ and thus $f(t_2)>0$, because $f(t_2)>0$ is a convex function.

For $n=3$, we have

$$f(t_2)=484-2464t_2+5368t_2^2-6972t_2^3+5958t_2^4-2940t_2^5+600t_2^6,$$

$$\frac{df}{dt_2}=-2464+10736t_2-20916t_2^2+23832t_2^3-14700t_2^4+3600t_2^5,$$

$$\frac{d^2f}{dt_2^2}=10736-41832t_2+71496t_2^2-58800t_2^3+18000t_2^4,$$

$$\frac{d^3f}{dt_2^3} = -41832 + 142992t_2 - 176400t_2^2 + 72000t_2^3$$

$$\frac{d^4f}{dt_2^4} = 142992 - 352800t_2 + 216000t_2^2 .$$

Now it is easy to see that $(d^4f/dt_2^4)(t_2) > 0$ for $t_2 \geqq 1$ and $(d^3f/dt_2^3)(2) > 0$. Thus we have $(d^3f/dt_2^3)(t_2) > 0$ for $t_2 \geqq 2$. We also have $(d^2f/dt_2^2)(2) > 0$ and thus $(d^2f/dt_2^2)(t_2) > 0$ for $t_2 \geqq 2$, and $(df/dt_2)(2) > 0$ and thus $(df/dt_2)(t_2) > 0$ for $t_2 \geqq 2$. And we have $f(2) > 0$ and thus $f(t_2) > 0$ for $t_2 \geqq 2$. Now we consider the Taylor series expansion of $f(t_2)$ at $t_2 = 63/100$. Then we have

$$f(t_2) = \frac{16053450317}{5000000000} - \frac{7302547}{6250000}\left(t_2 - \frac{63}{100}\right) + \frac{44568589}{100000}\left(t_2 - \frac{63}{100}\right)^2$$
$$- \frac{78267}{125}\left(t_2 - \frac{63}{100}\right)^3 + \frac{2691}{10}\left(t_2 - \frac{63}{100}\right)^4 - 672\left(t_2 - \frac{63}{100}\right)^5 + 600\left(t_2 - \frac{63}{100}\right)^6 .$$

Put $t = t_2 - 63/100$ and

$$g = \frac{16053450317}{5000000000} - \frac{7302547t}{6250000} + \frac{44568589t^2}{100000} - \frac{78267t^3}{125} + \frac{2691t^4}{10} - 672t^5 + 600t^6 .$$

For $t < 2 - 63/100$, we have $16053450317/5000000000 - 7302547t/6250000 > 0$. Now we claim that

$$w = \frac{44568589}{100000} - \frac{78267t}{125} + \frac{2691t^2}{10} - 672t^3 + 600t^4 > 0 \qquad \text{for} \quad t < 2 .$$

We consider the Taylor series expansion of w at $t = 909/1000$. Then we have

$$w = \frac{18962806943}{5000000000} - \frac{100083}{1250000}\left(t - \frac{909}{1000}\right) + \frac{3527919}{2500}\left(t - \frac{909}{1000}\right)^2$$
$$+ \frac{7548}{5}\left(t - \frac{909}{1000}\right)^3 + 600\left(t - \frac{909}{1000}\right)^4 .$$

Thus we see that

$$\frac{18962806943}{5000000000} - \frac{100083}{1250000}\left(t - \frac{909}{1000}\right) > 0 \qquad \text{for} \quad -\frac{63}{100} < t < 2$$

and

$$\frac{3527919}{2500}\left(t - \frac{909}{1000}\right)^2 + \frac{7548}{5}\left(t - \frac{909}{1000}\right)^3 + 6000\left(t - \frac{909}{1000}\right)^4 > 0 .$$

Therefore we see that $f(t_2)>0$ for all $t_2>0$.

For $n=5$ we have

$$f(t_2)=9444-35904t_2+56752t_2^2-50380t_2^3+29150t_2^4-11088t_2^5+2058t_2^6\,,$$

for $n=6$ we have

$$f(t_2)=16(1456-5200t_2+7672t_2^2-6240t_2^3+3243t_2^4-1131t_2^5+204t_2^6)$$

and for $n=7$ we have

$$f(t_2)=48420-165600t_2+232776t_2^2-177660t_2^3+84942t_2^4-27540t_2^5+4860t_2^6\,.$$

Now by the same way as in the case of $n=3$, we see that, for $n=5$, 6 and 7, $f(t_2)>0$ for all $t_2>0$.

Thus we get

THEOREM 2.1. *If $n\geqq 3$, $n\neq 4$ and $l=3$, $SO(2n+l)/(SO(n)\cdot U(1)\times SO(l))$ carries no homogeneous Einstein metrics.*

Now we consider the case of $n=3$. In this case, t_1 is given by

$$t_1=\frac{3(2-t_2)t_2(2t_2+lt_2-4)}{5(t_2-1)(t_2+1)}$$

and the equation (1.13) is equivalent to

$$\begin{aligned}&242-176(4+l)t_2+4(227+124l+8l^2)t_2^2-2(4+l)(72+59l)t_2^3\\&+3(54+130l+49l^2+4l^3)t_2^4-2l(4+l)(17+6l)t_2^5+l(2+l)(11+3l)t_2^6=0\,.\end{aligned}$$

Let $g(t_2)$ be the left hand side of the above equation. For $t_2=2-1/l$, we see that

$$g(2-1/l)=\frac{22+51l-489l^2+1079l^3-1066l^4+486l^5-72l^6}{l^5}$$

and thus $g(2-1/l)<0$ for $l\geqq 4$. For $t_2=3/2$, we see that

$$g(3/2)=\frac{6728-2226l-711l^2+243l^3}{32}$$

and thus $g(3/2)>0$ for $l\geqq 4$. For $t_2=2$, we have $g(2)=450$. Hence, for $l\geqq 4$ we have 2 solutions t_2^1, t_2^2 of $g=0$ such that $3/2<t_2^1<2-1/l$ and $2-1/l<t_2^2<2$. Note that $(2+l)t_2>9$ for $l\geqq 4$ and $t_2>3/2$. Thus we see that

$$t_1=\frac{3(2-t_2)t_2(2t_2+lt_2-4)}{5(t_2-1)(t_2+1)}>0$$

for $3/2<t_2<2$.

Therefore we get

THEOREM 2.2. *If $n=3$ and $l \geqq 4$, $SO(2n+l)/(SO(n)\cdot U(1)\times SO(l))$ carries homogeneous Einstein metrics.*

References

[1] A. L. BESSE, *Einstein Manifolds*, Springer Verlag (1987).

[2] S. HELGASON, *Differential Geometry and Symmetric Spaces*, Academic Press (1978).

[3] M. WANG and W. ZILLER, On normal homogeneous Einstein manifolds, Ann. Sci. Ecole Norm. Sup. (4) **18** (1985), 563–633.

[4] M. WANG and W. ZILLER, Existence and non-existence of homogeneous Einstein metrics, Invent. Math. **84** (1986), 177–194.

Present Addresses:

JOON-SIK PARK
DEPARTMENT OF MATHEMATICS, PUSAN UNIVERSITY OF FOREIGN STUDIES,
UAM-DONG, NAM-KU, PUSAN, 608–738 KOREA.

YUSUKE SAKANE
DEPARTMENT OF MATHEMATICS, OSAKA UNIVERSITY,
TOYONAKA, 560 JAPAN.