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Normal Elementary Maps

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Abstract. We say that a partial elementary map f of a structure M is normal if f can be extended to an elementary map on M whose domain or range is equal to M. In this paper, we investigate properties for normal elementary maps.

We prepare some notations. We fix a complete theory T of a countable language L. Throughout this paper, we work in a big model $\mathscr{M}$ of T. We denote subsets of $\mathscr{M}$ by $A, B, \cdots$, elementary submodels of $\mathscr{M}$ by $M, N, \cdots$ and finite tuples of $\mathscr{M}$ by $\bar{a}, \bar{b}, \cdots$. And we denote types (possibly with parameters) by $p, q, \cdots$ and formulas (possibly with parameters) by $\varphi, \psi, \cdots$. The set of realizations of a formula φ in a set A is denoted by φ^A. A type of $\bar{a}$ over A is denoted by $tp(\bar{a}/A)$, and for $tp(\bar{a}/\varnothing)$ we write simply $tp(\bar{a})$. We write $RM(p)$ for the Morley rank of a type p. We denote mappings by $f, g, \cdots$ and $\sigma, \tau, \cdots$. We write $dom(f)$ and $ran(f)$ for the domain and the range of a mapping f respectively. We denote the group of automorphisms of a structure M which leave A pointwise fixed. We say a partial elementary map of M is maximal if it is maximal in the set $\{g : g$ elementary map on M and $g \supseteq f\}$.

LEMMA 1. *Let T be ω-stable and M a model of T. If f is a maximal elementary map on M then $dom(f) \prec M$ and $ran(f) \prec M$.*

PROOF. It is enough to show that $dom(f) \prec M$. Assume the contrary. Put $S = \{\varphi(x, \bar{a}) \in L(dom(f)) : M \models \exists x \varphi(x, \bar{a})$ and $\varphi(x, \bar{a})^M \subseteq M \setminus dom(f)\}$. By the Tarski-Vaught test, S is not empty. Put $S_{rank} = \{\psi(x, \bar{b}) : \psi(x, \bar{b})$ is Morley rank minimal in $S\}$. Take a formula $\psi_0(x, \bar{b}_0)$ $(\in S_{rank})$ whose Morley degree is minimal in S_{rank}. Take an element $c \in M \setminus dom(f)$ and an element $d \in M \setminus ran(f)$ such that $M \models \psi_0(c, \bar{b}_0)$ and $M \models \psi_0(d, f(\bar{b}_0))$. Since $\psi_0(x, \bar{b}_0)$ isolates a type over $dom(f)$, f can be extended to an elementary map f^* on M such that $f^*(c) = d$. This contradicts the maximality of f. So we have $dom(f) \prec M$. $\square$

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COROLLARY 2. *Let T be ω-stable and M an $\aleph_0$-saturated model to T. If an elementary map f on M is maximal then dom(f) and ran(f) are $\aleph_0$-saturated elementary submodels of M.*

PROOF. Let D be the domain of f and R the range of f. By lemma 1, D and R are elementary submodels of M. Assuume that D is not $\aleph_0$-saturated. Put $P=\{p : p\in S(A)$ for some finite subset A of D and p is not realized in $D\}$. By the assumption $P\neq\varnothing$. Let p be Morley rank minimal in P. By ω-stability of T, we may assume that p is stationary. Since M is $\aleph_0$-saturated, we can choose a tuple $\bar{a}\in M$ which realizes p.

CLAIM 3. *$\bar{a}$ and D are independent over dom(p).*

PROOF OF CLAIM. Assume that $\bar{a}$ and D are dependent over $dom(p)$. Then there is a finite tuple $\bar{c}\in D$ such that $tp(\bar{a}/dom(p)\cup\bar{c})$ forks over $dom(p)$. So we have $RM(p)<RM(\bar{a}/dom(p)\cup\bar{c})$. This contradicts the choice of p. □

By the $\aleph_0$-saturation of M, there is a tuple $\bar{b}\in M\backslash R$ which realizes $f(p)$. By a similar argument in claim 3, $\bar{b}$ *and* R are also independent over $f(dom(p))$. Let f^* ($\in Aut(\mathscr{M})$) be an extension of f. Since $f(p)$ is stationary and $\bar{b}$ and R are independent over $f(dom(p))$, we have $tp(b/R)=tp(f^*(a)/R)$. Thus we get $tp(bR)=tp(f^*(a)R)=tp(aD)$. This contradicts the maximality of f. So D is $\aleph_0$-saturated. By a similar argument, we can prove that R is $\aleph_0$-saturated. □

DEFINITION 4. Let M be a model of T of f an elementary map on M. f is a *normal* elementary map on M if f can be extended to an elementary map on M whose domain or range is equal to M.

We next define triples of models which are used for criteria of normality of elementary maps. The following two definitions are weaker than that of the special triple in [2]. So we call them a weakly special triple and an almost special triple.

DEFINITION 5. Let M_1, M_2 and N be models of T. The triple (M_1, M_2, N) is a *weakly special* triple if

1. $N\prec M_i$ and $N\neq M_i$ $(i=1,2)$;

2a. There is an element $a_1\in M_1\backslash N$ such that for all element $b_1\in M_2\backslash N$, $tp(a_1,N)\neq tp(b_1/N)$;

2b. There is an element $b_2\in M_2\backslash N$ such that for all element $a_2\in M_1\backslash N$, $tp(b_2/N)\neq tp(a_2/N)$.

DEFINITION 6. Let M_1, M_2 and N be models of T. The triple (M_1, M_2, N) is an *almost special* triple if

1. $N\prec M_i$ and $N\neq M_i$ $(i=1,2)$;
2. $tp(a/N)\neq tp(b/N)$ for all element $a\in M_1\backslash N$ and $b\in M_2\backslash N$.

We say that T has a weakly (almost) special triple if there are models M_1, M_2 and N of T such that (M_1, M_2, N) is a weakly (almost respectively) special triple. Clearly every almost special triple is a weakly special triple.

PROPOSITION 7. *Let T be ω-stable. If T has no almost special triple, then for any model M of T, all elementary maps on M are normal.*

PROOF. Suppose that there are a model M and elementary maps on M which are not normal. Let S be the set of all elementary maps on M which are not normal. Take an elementary map f on M which is maximal in S. By lemma 1, $dom(f)$ and $ran(f)$ are proper elementary submodels of M. Let f^* ($\in Aut(\mathcal{M})$) be an extension of f. Then $ran(f)$ is a proper elementary submodel of $f^*(M)$. Since f is maximal in S, for all element $a \in M \setminus dom(f)$ and $b \in M \setminus ran(f)$, we have $tp(a^\wedge dom(f)) \neq tp(b^\wedge ran(f))$. On the other hand, for every element $b' \in f^*(M) \setminus ran(f)$ there is an element $a' \in M \setminus dom(f)$ such that $tp(a'^\wedge dom(f)) = tp(b'^\wedge ran(f))$. Thus, for all element $b \in M \setminus ran(f)$ and $b' \in f^*(M) \setminus ran(f)$, we have $tp(b/ran(f)) \neq tp(b'/ran(f))$. So $(M, f^*(M), ran(f))$ is an almost special triple. $\square$

LEMMA 8. *If there is a weakly special triple (M_1, M_2, N) with $\|N\| = \lambda$ then for any κ with $\aleph_0 \leq \kappa \leq \lambda$, there is a weakly special triple (M_1^*, M_2^*, N^*) with $\|M_1^*\| = \|M_2^*\| = \|N^*\| = \kappa$.*

PROOF. Let (M_1, M_2, N) be a weakly special triple with $\|N\| = \lambda$. Take an element $a \in M_1 \setminus N$ such that for all element $d \in M_2 \setminus N$, $tp(a/N) \neq tp(d/N)$ and an element $b \in M_2 \setminus N$ such that for all element $c \in M_1 \setminus N$, $tp(b/N) \neq tp(c/N)$. By induction on j ($<\omega$), we construct models N^j, M_i^j ($i = 1, 2$) of cardinality κ with the following properties:

1. $N^0 \prec N$;
2. $N^0 \prec M_i^0 \prec M_i$, $a \in M_1^0$ and $b \in M_2^0$;
3. $N^j \prec N^{j+1} \prec N$, $tp(a/N^{j+1}) \neq tp(d/N^{j+1})$ for all $d \in M_2^j \setminus N^j$ and $tp(b/N^{j+1}) \neq tp(c/N^{j+1})$ for all $c \in M_1^j \setminus N^j$;
4. $M_i^j \prec M_i^{j+1} \prec M_i$ and $N^{j+1} \prec M_i^{j+1}$.

Clearly we can choose N^0, M_1^0 and M_2^0 which satisfy conditions 1 and 2. Suppose that N^j, M_i^j ($j \leq k < \omega$) are defined. Since $a \in M_1 \setminus N$ is a witness of the weakly special triple (M_1, M_2, N), for each element $d \in M_2^k \setminus N^k$, there are a finite tuple $\bar{n}$ of N and a formula $\varphi(x, \bar{y})$ such that $\models \varphi(a, \bar{n}) \wedge \neg \varphi(d, \bar{n})$. Since $\|M_i^k\| = \kappa$, there is a subset D_1^k of N of cardinality κ such that for all element $d \in M_2^k \setminus N^k$, $tp(a/N^k D_1^k) \neq tp(d/N^k D_1^k)$. Similarly there is a subset D_2^k of N of cardinality κ such that for all element $c \in M_1^k \setminus N^k$, $tp(b/N^k D_2^k) \neq tp(c/N^k D_2^k)$. Thus we can choose a model N^{k+1} of cardinality κ such that $N^k \prec N^{k+1} \prec N$ and for all element $c \in M_1^k \setminus N^k$ and $d \in M_2^k \setminus N^k$, $tp(a/N^{k+1}) \neq tp(d/N^{k+1})$ and $tp(b/N^{k+1}) \neq tp(c/N^{k+1})$. And we can choose models M_i^{k+1} of cardinality κ such that $M_i^k \prec M_i^{k+1} \prec M_i$ and $N^{k+1} \prec M_i^{k+1}$. Put $M_i^* = \bigcup_{j<\omega} M_i^j$ and $N^* = \bigcup_{j<\omega} N^j$. By the construction of N^* and M_i^*, (M_1^*, M_2^*, N^*) is a weakly special triple such that $\|M_1^*\| = \|M_2^*\| = \|N^*\| = \kappa$. $\square$

LEMMA 9. *Let T be ω-stable. If there is a weakly special triple (M_1, M_2, N) then there is a weakly special triple (M_1^*, M_2^*, N^*) such that $\|M_1^*\| = \|M_2^*\| = \|N^*\| = \aleph_0$ and that $M_1^* \simeq N^* \simeq M_2^*$.*

PROOF. Let (M_1, M_2, N) be a weakly special triple. By lemma 8 we can assume that M_1, M_2 and N are countable. Put $A=M_1\backslash N$, $B=M_2\backslash N$ and $N_0=N$. Then we can construct countable models N_i $(1\leq i<\omega)$ with the following properties:
For all i.

A0. $N_i\cap AB=\varnothing$, $N_i\prec N_{i+1}\prec\mathcal{M}$ and N_iA, $N_iB\prec\mathcal{M}$;

When i is even.

E1. For every tuple $\bar{a}$, element $a\in N_iA$ and tuple $\bar{c}\in N_i$, there is an element $c\in N_{i+1}$ such that if $tp(\bar{a})=tp(\bar{c})$ then $tp(\bar{a}a)=tp(\bar{c}c)$;

E2. For every tuple $\bar{b}$, element $b\in N_iB$ and tuple $\bar{c}\in N_i$, there is an element $c\in N_{i+1}$ such that if $tp(\bar{b})=tp(\bar{c})$ then $tp(\bar{b}b)=tp(\bar{c}c)$;

When i is odd.

O1. For every tuple $\bar{c}$, element $c\in N_i$ and tuple $\bar{a}\in N_iA$, there is an element $a\in N_{i+1}A$ such that if $tp(\bar{c})=tp(\bar{a})$ then $tp(\bar{c}c)=tp(\bar{a}a)$;

O2. For every tuple $\bar{c}$, element $c\in N_i$ and tuple $\bar{b}\in N_iB$, there is an element $b\in N_{i+1}B$ such that if $tp(\bar{c})=tp(\bar{b})$ then $tp(\bar{c}c)=tp(\bar{b}b)$.

Let N_i $(i<\aleph_1)$ be such countable models. Put $N^*=\bigcup_{i<\omega}N_i$, $M_1^*=\bigcup_{i<\omega}N_iA$ and $M_2^*=\bigcup_{i<\omega}N_iB$. By the construction of N_i, (M_1^*, M_2^*, N^*) is a weakly special triple such that $\|M_1^*\|=\|M_2^*\|=\|N^*\|=\aleph_0$. By a back-and-forth argument, we have $N^*\simeq M_1^*$ by E1 and O1 and $N^*\simeq M_2^*$ by E2 and O2. □

Next theorem shows a relation between special triples and normality of elementary maps on a model.

THEOREM 10. *Let T be ω-stable. The following are equivalent.*

1. *T has no weakly special triple.*
2. *T has no almost special triple.*
3. *For any model M of T, all elementary maps on M are normal.*

PROOF. $1)\Rightarrow 2)$ is clear. By proposition 7, we have $2)\Rightarrow 3)$. We prove $3)\Rightarrow 1)$. Suppose that there is a weakly special triple (M_1, M_2, N). By lemma 9, we may assume that $\|M_1\|=\|M_2\|=\|N\|=\aleph_0$ and $M_1\simeq N\simeq M_2$. Let $g: M_1\to M_2$ be the isomorphism. Put $f=g^{-1}|N$.

CLAIM 11. *f is not normal on M_1.*

PROOF OF CLAIM. Since (M_1, M_2, N) is a weakly special triple, we can choose an element $a\in M_1\backslash N$ such that for all $d\in M_2\backslash N$, $tp(a/N)\neq tp(d/N)$ and an element $b\in M_2\backslash N$ such that for all $c\in M_1\backslash N$, $tp(b/N)\neq tp(c/N)$. Assume that f is normal on M_1.
Case 1) Assume that f can be extended to an elementary map h_1 on M_1 whose domain is M_1. Then we have $tp(aN)=tp(h_1(aN))=tp(h_1(a)g^{-1}(N))=tp(g\circ h_1(a)N)$. But $g\circ h_1(a)\in M_2\backslash N$. This contradicts the choice of a.
Case 2) Assume that f can be extended to an elementary map h_2 on M_1 whose range is M_1. By a similar argument in case 1, this contradicts the choice of b. □

This completes the proof of the theorem. □

We next concentrate on elementary maps on $\aleph_0$-saturated models. Corollary 12 shows a relation between triples of $\aleph_0$-saturated models and normality of elementary maps on $\aleph_0$-saturated models.

COROLLARY 12. *Let T be ω-stable. The following are equivalent.*

1. *There is no weakly special triple of $\aleph_0$-saturated models.*
2. *There is no almost special triple of $\aleph_0$-saturated models.*
3. *For any $\aleph_0$-saturated model M of T, all elementary maps on M are normal.*

PROOF. By corollary 2 and a similar argument in the proof of theorem 10, we obtain the proof. □

In the proof of 3) $\Rightarrow$ 1) in theorem 10, we constructed a non-normal elementary map f on a countable model M. But f may have a property that $|M \backslash dom(f)| = |M \backslash ran(f)| = |dom(f)|$ because there is a theory which has no almost special triple (M_1, M_2, N) such that $|M_1 \backslash N| = |M_2 \backslash N| < \|N\|$.

EXAMPLE. Let G be a proper elementary extension of $(Z_2^\omega, +)$ and H a proper elementary extension of $(Z_3^\omega, +)$. Put $N = Z_2^\omega \oplus Z_3^\omega$, $M_1 = G \oplus Z_3^\omega$ and $M_2 = Z_2^\omega \oplus H$. Then, it can be seen that (M_1, M_2, N) is an almost special triple of models of $Th(N)$.

We next think about a non-normal elementary map f on a model M such that $|M \backslash dom(f)| = |M \backslash ran(f)| < |dom(f)|$. We construct a model M^* and an elementary map f on M^* such that $|M^* \backslash dom(f)| = |M^* \backslash ran(f)| \le \aleph_0$, $|dom(f)| = \aleph_1$ and f is not normal.

THEOREM 13. *Let T be ω-stable. If T has an almost special triple (M_1, M_2, N) of countable models with the following properties*:

1. *N is $\aleph_0$-saturated*;
2. *$M_1 \simeq M_2$*;
3. *There is a finite tuple $\bar{c} \in N$ such that $AB \downarrow_{\bar{c}} N$ where $A = M_1 \backslash N$ and $B = M_2 \backslash N$.*

Then there are a model M^* of T and an elementary map f on M^* such that f is not normal, $|dom(f)| = |ran(f)| = \aleph_1$ and $|M^* \backslash dom(f)|$, $|M^* \backslash ran(f)| \le \aleph_0$.

PROOF. Let (M_1, M_2, N) be an almost special triple of countable models which satisfies condition 1 and 2. Since T is ω-stable, we can construct $\{N_\alpha; \alpha < \aleph_1\}$ with following properties:

i. $N_0 = N$;
ii. $N_{\alpha+1}$ $(\supset N_\alpha)$ is countably saturated and $N_{\alpha+1} \downarrow_{N_\alpha} AB$;
iii. $N_\delta = \bigcup_{\alpha<\delta} N_\alpha$ (δ is limit).

It is clear that N_δ is also $\aleph_0$-saturated when δ is limit.

CLAIM 14. *$tp(N_\alpha/\bar{c}AB) = tp(N/\bar{c}AB)$ for all $\alpha < \aleph_1$.*

PROOF OF CLAIM. By the ω-stability of T, we can assume that $tp(AB/N)$ is the unique non-forking extension of $tp(AB/\bar{c})$. We fix $\alpha < \aleph_1$. Since N and N_α are countably saturated, we have $N_\alpha \simeq_{\bar{c}} N$. So there is an automorphism f ($\in Aut_{\bar{c}}(\mathcal{M})$) with $f: N \to N_\alpha$. By the construction of N_α, $tp(AB/N_\alpha)$ is the unique non-forking extension of $tp(AB/\bar{c})$. Since $AB \downarrow_{\bar{c}} N$, we have $f(AB) \downarrow_{\bar{c}} N_\alpha$. Then $tp(f(AB)/N_\alpha)$ is also the unique non-forking extension of $tp(f(AB)/\bar{c})$ $(=tp(AB/\bar{c}))$. So we have $tp(f(AB)/N_\alpha) = tp(AB/N_\alpha)$. Thus we have $tp(ABN/\bar{c}) = tp(f(AB)N_\alpha/\bar{c}) = tp(ABN_\alpha/\bar{c})$. □

Put $N_{\aleph_1} = \bigcup_{\alpha < \aleph_1} N_\alpha$.

CLAIM 15. $N_{\aleph_1}A \simeq N_{\aleph_1}B$.

PROOF OF CLAIM. By claim 14, we have $N_\alpha A \simeq N_\alpha B$ for all $\alpha < \aleph_1$. Let $g_\alpha: N_\alpha A \to N_\alpha B$ be an isomorphism for each $\alpha < \aleph_1$. By the elementary chain principle, for all $\alpha < \aleph_1$, $N_\alpha A$ and $N_\alpha B$ are elementary submodels of $N_{\aleph_1}A$ and $N_{\aleph_1}B$ respectively. Let $\varphi(\bar{a})$ be an $L(N_{\aleph_1}A)$-sentence. For every $L(N_{\aleph_1}A)$-sentence ψ, there is β $(<\aleph_1)$ such that $\psi \in L(N_\beta A)$. So we have, for some $\gamma < \aleph_1$, $N_{\aleph_1}A \models \varphi(\bar{a})$ if and only if $N_\gamma A \models \varphi(\bar{a})$. Thus we have $N_{\aleph_1}A \models \varphi(\bar{a})$ if and only if $N_{\aleph_1}B \models \varphi(g_\gamma(\bar{a}))$. □

Let $\sigma: N_{\aleph_1}A \to N_{\aleph_1}B$ be an isomorphism and $\tau: N_{\aleph_1} \to \sigma^{-1}(N_{\aleph_1})$ an elementary map on $N_{\aleph_1}A$.

CLAIM 16. *The model $N_{\aleph_1}A$ and the elementary map τ on $N_{\aleph_1}A$ are what we look for.*

PROOF OF CLAIM. By the construction of $N_{\aleph_1}A$ and τ, we have $|dom(\tau)| = |N_{\aleph_1}| = \aleph_1$ and $|N_{\aleph_1}A \setminus dom(\tau)| = |A| \leq \aleph_0$. Assume that τ is normal. Then, for example, τ can be extended to an elementary map ρ on $N_{\aleph_1}A$ whose domain is $N_{\aleph_1}A$. Then we have $tp(aN_{\aleph_1}) = tp(\rho(aN_{\aleph_1})) = tp(\rho(a)\sigma^{-1}(N_{\aleph_1})) = tp(\sigma \circ \rho(a)N_{\aleph_1})$ for all $a \in A$. Since $\sigma \circ \rho(a) \in B$, this is a contradiction. When τ can be extended to an elementary map ρ' whose range is $N_{\aleph_1}A$, we can prove similarly. □

This completes the proof of theorem 13. □

QUESTION. Is there another condition for a theory to have an elementary map f on a model M such that $|M \setminus dom(f)| = |M \setminus ran(f)| < |dom(f)|$ and f is not normal?

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