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A Class Number Problem in the Cyclotomic $\mathbf{Z}_3$-extension of $\mathbf{Q}$

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Abstract. Let Ω_n be the n-th layer of the cyclotomic $\mathbf{Z}_3$-extension of $\mathbf{Q}$ and h_n the class number of Ω_n. We claim that if ℓ is a prime number less than 10^4, then ℓ does not divide h_n for any positive integer n.

1. Introduction

Let p be a prime number. It is one of the basic cases of class number problem to ask whether a prime number ℓ divides the class numbers of the intermediate fields of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$. In the case $\ell = p$, Iwasawa [4] proved that p does not divide any of the class numbers of the n-th layers of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$. In the case $p = 2$, Fukuda and Komatsu [1] showed that ℓ does not divide any of the class numbers of the n-th layers of the cyclotomic $\mathbf{Z}_2$-extension of $\mathbf{Q}$ for $\ell < 10^7$.

In this paper, we investigate the case $p = 3$. Put $\Omega_n = \mathbf{Q}(2\cos(2\pi/3^{n+1}))$. Then Ω_n is a cyclic extension of degree 3^n over $\mathbf{Q}$ and the n-th layer of the cyclotomic $\mathbf{Z}_3$-extension of $\mathbf{Q}$. We denote the class number of Ω_n by h_n. Masley [6] showed $h_1 = h_2 = h_3 = 1$. Linden [5] showed $h_4 = 1$ if GRH (the Generalized Riemann Hypothesis) is valid.

Horie [3] proved the following theorem.

THEOREM 1 (Horie). *Let the notation be as above for $p = 3$ and ℓ a prime number. If $\ell \equiv 2, 4, 5, 7 \pmod 9$, then ℓ does not divide h_n for any positive integer n.*

In this paper, we prove the following result.

THEOREM 2. *Let $\ell \geqq 5$ be a prime number and 3^s the exact power of 3 dividing $\ell^2 - 1$. Put*

$$m_\ell = 3s + 2 + [\log_3(\ell - 1)] + \left[\log_3 \frac{\ell - 1}{2}\right] + [\log_3(2s + 1 + [\log_3(\ell - 1)])]\,,$$

where $[x]$ denotes the greatest integer not exceeding a real number x. If ℓ does not divide h_{m_ℓ}, then ℓ does not divide h_n for any positive integer n.

As a corollary to Theorem 2, we obtained the following result by numerical calculation.

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COROLLARY 1. *Let ℓ be a prime number less than* 10000. *Then ℓ does not divide h_n for any positive integer n.*

We prove Theorem 2 in Section 2. In Section 3, we show a criterion in each of four disjoint cases to determine that a prime number ℓ does not divide h_n.

2. Proof of Theorem 2

Let n be a positive integer, ℓ a prime number with $\ell \geqq 5$, χ a character mod ℓ with $\chi(-1) = -1$ and ψ_n an even character mod 3^{n+1} whose order is 3^n. Then the generalized Bernoulli number is defined by

$$B_{1,\chi\psi_n} = \frac{1}{3^{n+1}\ell} \sum_{b=1}^{3^{n+1}\ell} b\chi\psi_n(b)\,.$$

Let s be as in Theorem 2 and ζ_{ψ_n} such a primitive 3^{n+1}-th root of unity as

$$\zeta_{\psi_n}^{3^{n+1-s}} = \psi_n(1+3^{n+1-s})\,.$$

We define a rational function $f_1(T)$ in the rational function field $\mathbf{Q}_\ell(T)$ by

$$f_1(T) = \left(\sum_{\substack{b \equiv 1 \pmod{3^s} \\ 0<b<3^s\ell}} \chi(b)T^b \right) (T^{3^s\ell} - 1)^{-1}\,.$$

We put $d = s + 1 + [\log_3(\ell - 1)]$. We also put $\zeta_\ell = \cos\frac{2\pi}{\ell} + \sqrt{-1}\sin\frac{2\pi}{\ell}$ and $K_{n,\ell} = \Omega_n(\zeta_\ell)$. Let $h^-_{n,\ell}$ be the relative class number of $K_{n,\ell}$. Then we have the following result by [7] p. 387:

LEMMA 1. *Let χ, ψ_n be as above and $n \geqq 2s - 1$. If $B_{1,\chi\psi_n} \equiv 0 \pmod{\overline{\ell}}$ in $\mathbf{Z}_\ell[\zeta_{\psi_n}]$, then $f_1(\zeta_{\psi_n}) \equiv 0 \pmod{\overline{\ell}}$ in $\mathbf{Z}_\ell[\zeta_{\psi_n}]$, where $\overline{\ell}$ is the ideal of $\mathbf{Z}_\ell[\zeta_{\psi_n}]$ generated by ℓ.*

LEMMA 2. *If $d + s - 1 \leqq n$, then the prime number ℓ does not divide $h^-_{n,\ell}/h^-_{d+s-1,\ell}$.*

PROOF. Assume that $d + s - 1 \leqq n$. We put

$$g(T) = \frac{(T^{3^s\ell} - 1)f_1(T)}{T}\,.$$

Since

$$g(T) = \sum_{\substack{b \equiv 1 \pmod{3^s} \\ 0 < b \leqq 1 + 3^s(\ell - 1)}} \chi(b) T^{b-1},$$

we have $\deg g(T) \leqq 3^s(\ell - 1)$ where $\deg g(T)$ means the degree of the polynomial $g(T)$. Since

$$[\mathbf{Q}_\ell(\zeta) : \mathbf{Q}_\ell] \geqq 3^{n+1-s} \geqq 3^d > 3^s(\ell - 1) \geqq \deg g(T)$$

for a primitive 3^{n+1}-th root of unity $\zeta \in \overline{\mathbf{Q}_\ell}$, we have

$$g(\zeta) \not\equiv 0 \pmod{\overline{\ell}},$$

and hence

$$f_1(\zeta) \not\equiv 0 \pmod{\overline{\ell}}.$$

In particular, we obtain $f_1(\zeta_{\psi_n}) \not\equiv 0 \pmod{\overline{\ell}}$. By Lemma 1, we see $B_{1,\chi\psi_n} \not\equiv 0 \pmod{\overline{\ell}}$. Hence we obtain

$$\frac{h^-_{n,\ell}}{h^-_{d+s-1,\ell}} \not\equiv 0 \pmod{\overline{\ell}}$$

by the class number formula

$$h^-_{n,\ell} = Q_{n,\ell} \cdot 2 \cdot \ell \prod_{\chi} \prod_{b=1}^{3^n} \left(-\frac{1}{2} B_{1,\chi\psi_n^b} \right),$$

where $Q_{n,\ell} = 1$ or 2 and χ runs over all characters mod ℓ with $\chi(-1) = -1$. □

We denote the plus part and the minus part of the ideal class group of $K_{n,\ell}$ by $C^+(K_{n,\ell})$ and by $C^-(K_{n,\ell})$ respectively. We also denote the ℓ-rank of $C^+(K_{n,\ell})$ and $C^-(K_{n,\ell})$ by $r^+_{n,\ell}$ and by $r^-_{n,\ell}$ respectively. Then Theorem 10.11 in [7] implies

$$r^+_{n,\ell} \leqq r^-_{n,\ell}.$$

LEMMA 3. *Suppose $s + 1 \leqq n$. If ℓ divides h_n and if ℓ does not divide h_{n-1}, then $3^{n-s-1} < r^-_{n,\ell}$.*

PROOF. Let r_n be the ℓ-rank of the ideal class group of Ω_n. By Theorem 10.8 in [7], we have $r_n \geqq 3^{n-s}$ if $\ell \equiv 1 \pmod 3$ and $r_n \geqq 2 \cdot 3^{n-s}$ if $\ell \equiv 2 \pmod 3$. Since $r_n \leqq r^+_{n,\ell}$, we have $3^{n-s-1} < r^-_{n,\ell}$. □

Now we prove Theorem 2.

Since $|B_{1,\chi\psi_n^b}| \leqq 3^{n+1}\ell$, we have

$$h_{n,\ell}^- \leqq 2 \cdot 2 \cdot \ell \left(\frac{1}{2}3^{n+1}\ell\right)^{\frac{\ell-1}{2}3^n} < \ell^{3^n(n+1)\frac{\ell-1}{2}+2}.$$

Hence we obtain

$$r_{n,\ell}^- < 3^n(n+1)\frac{\ell-1}{2} + 2\,,$$

and then

$$r_{n,\ell}^- < 3^{d+s-1}(d+s)\frac{\ell-1}{2} + 2 \qquad (1)$$

by Lemma 2.

Let m_ℓ be as in Theorem 2 and assume that ℓ does not divide h_{m_ℓ}. We also assume that there exists a positive integer n such that ℓ divides h_n but does not divide h_{n-1}. Then we have $m_\ell < n$. By Lemma 3 and (1), we obtain

$$3^{n-s-1} \leqq 3^{d+s-1}(d+s)\frac{\ell-1}{2}\,.$$

Hence we have

$$n - s - 1 \leqq d + s - 1 + \log_3(d+s) + \log_3\frac{\ell-1}{2}\,;$$

this implies

$$n \leqq 3s + 1 + [\log_3(\ell-1)] + \log_3(d+s) + \log_3\frac{\ell-1}{2}\,.$$

Therefore we have

$$n \leqq 3s + 2 + [\log_3(\ell-1)] + \left[\log_3\frac{\ell-1}{2}\right] + [\log_3(2s+1+[\log_3(\ell-1)])] = m_\ell\,.$$

This is a contradiction.

3. Calculation

Let $\Delta_n = \mathrm{Gal}(\Omega_n/\mathbf{Q})$ be the Galois group of Ω_n over $\mathbf{Q}$ and A_n the ℓ-part of the ideal class group of Ω_n.

For a character $\chi : \Delta_n \to \overline{\mathbf{Q}}_\ell$, we define e_χ by

$$e_\chi = \frac{1}{|\Delta_n|}\sum_{\sigma\in\Delta_n} \mathrm{Tr}(\chi^{-1}(\sigma))\sigma \in \mathbf{Z}_\ell[\Delta_n]\,,$$

where Tr is the trace map of $\mathbf{Q}_\ell(\chi(\Delta_n))/\mathbf{Q}_\ell$. We denote by $A_{n,\chi}$ the χ-part $e_\chi A_n$ of A_n. Then we have $A_n = \bigoplus_\chi A_{n,\chi}$ where χ runs over all representatives of $\mathbf{Q}_\ell$-conjugacy classes of characters of Δ_n.

In order to prove that ℓ does not divide h_n, it is sufficient to prove that ℓ does not divide the order of $A_{n,\chi}$ for each χ. If χ is not injective, then there exists a positive integer k such that $\Omega_k = \Omega_n^{\mathrm{Ker}\chi}$ and $A_{n,\chi} \cong A_{k,\chi}$. Therefore we may assume χ is injective.

Now, for $n \geqq 1$, let ζ_n denote a primitive 3^n-th root of unity in $\mathbf{C}$ and put

$$\xi_n = (\zeta_{n+1} - 1)(\zeta_{n+1}^{-1} - 1) = 2 - (\zeta_{n+1} + \zeta_{n+1}^{-1}) \in \Omega_n .$$

We fix a truncation $e_{\chi,\ell} \in \mathbf{Z}[\Delta_n]$ of e_χ satisfying

$$e_{\chi,\ell} \equiv e_\chi \pmod{\ell}$$

in order to consider an action on ξ_n. The following lemma is a special case of Lemma 1 in [2].

LEMMA 4. *If there exists a prime number p which is congruent to* 1 *modulo* $3^{n+1}\ell$ *and satisfies*

$$(\xi_n^{e_{\chi,\ell}})^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{\mathfrak{p}}$$

for some prime ideal $\mathfrak{p}$ of Ω_n lying above p, then we have $|A_{n,\chi}| = 1$; here $|A_{n,\chi}|$ denotes the order of $A_{n,\chi}$.

Owing to Lemma 4 , we may regard χ as a character of Δ_n into $\overline{\mathbf{F}}_\ell$ and define e_χ to be an element of $\mathbf{F}_\ell[\Delta_n]$ where $\overline{\mathbf{F}}_\ell$ is an algebraic closure of the finite field $\mathbf{F}_\ell = \mathbf{Z}/\ell\mathbf{Z}$. Let η_n be a primitive 3^n-th root of unity in $\overline{\mathbf{F}}_\ell$ and put $K = \mathbf{F}_\ell(\eta_n)$. Let ρ be the generator of Δ_n determined by $\zeta_{n+1} \mapsto \zeta_{n+1}^4$ and χ the character of Δ_n defined by $\chi(\rho) = \eta_n^{-1}$. Then

$$e_{\chi^j} = \frac{1}{3^n} \sum_{i=0}^{3^n-1} \mathrm{Tr}_{K/\mathbf{F}_\ell}(\eta_n^{ij})\rho^i .$$

Let p be a prime number congruent to 1 modulo $3^{n+1}\ell$ and g_p a primitive root of p. Then

$$\zeta_{n+1} \equiv g_p^{\frac{p-1}{3^{n+1}}} \pmod{\mathfrak{p}}$$

for some prime ideal $\mathfrak{p}$ of Ω_n lying above p.

Therefore, if $e_{\chi^j} = \sum_i a_{ij}\rho^i$, then we have

$$\xi_n^{e_{\chi^j}} = \prod_{i=0}^{3^n-1} (2 - \zeta_{n+1} - \zeta_{n+1}^{-1})^{a_{ij}\rho^i}$$

$$= \prod_{i=0}^{3^n-1} (2 - \zeta_{n+1}^{4^i} - \zeta_{n+1}^{-4^i})^{a_{ij}}$$

$$\equiv \prod_{i=0}^{3^n-1} (2 - g_p^{\frac{p-1}{3^{n+1}}4^i} - g_p^{-\frac{p-1}{3^{n+1}}4^i})^{a_{ij}} \pmod{\mathfrak{p}}\,.$$

The last product should be calculated modulo p. We fix positive integers z_1 and z_2 satisfying

$$z_1 \equiv g_p^{\frac{p-1}{3^{n+1}}} \pmod p$$
$$z_2 \equiv z_1^{-1} \pmod p.$$

3.1. The case $\ell \equiv 1 \pmod 3$ and $2 \leqq n \leqq s$. Since $\eta_n \in \mathbf{F}_\ell$, we have $\mathrm{Tr}_{K/\mathbf{F}_\ell}(\eta_n) = \eta_n$ and

$$e_{\chi^j} = \frac{1}{3^n} \sum_{i=0}^{3^n-1} \eta_n^{ij} \rho^i\,.$$

Let g_ℓ be a primitive root of ℓ and fix integers a_{ij} satisfying

$$a_{ij} \equiv g_\ell^{\frac{\ell-1}{3^n}ij} \pmod \ell\,.$$

There are $2 \cdot 3^{n-1}$ injective characters of Δ_n and none of them is conjugate over $\mathbf{F}_\ell$. If we put

$$X = \{j \in \mathbf{Z} \mid 1 \leqq j < 3^n, (j, 3) = 1\}\,,$$

then $\{\chi^j \mid j \in X\}$ is the set of all injective characters of Δ_n. Then Lemma 4 implies the following criterion.

CRITERION 1. Put $b = 4$. If there exists a prime number p which is congruent to 1 modulo $3^{n+1}\ell$ and satisfies

$$\left(\prod_{i=0}^{3^n-1} (2 - z_1^{b^i} - z_2^{b^i})^{a_{ij}}\right)^{\frac{p-1}{\ell}} \not\equiv 1 \pmod p \quad \text{for each } j \in X\,,$$

then ℓ does not divide h_n/h_{n-1}.

3.2. The case $\ell \equiv 1 \pmod 3$ and $s+1 \leqq n$. We have $[K : \mathbf{F}_\ell] = 3^{n-s}$. The minimal polynomial of η_n over $\mathbf{F}_\ell$ is

$$X^{3^{n-s}} - \eta_n^{3^{n-s}}\,.$$

Therefore $\mathrm{Tr}_{K/\mathbf{F}_\ell}(\eta_n^i) = 0$ if i is not divisible by 3^{n-s}. Hence we have

$$e_{\chi^j} = \frac{1}{3^n} \sum_{i=0}^{3^s-1} \mathrm{Tr}_{K/\mathbf{F}_\ell}(\eta_n^{3^{n-s}ij}) \rho^{3^{n-s}i}$$

$$= \frac{1}{3^s} \sum_{i=0}^{3^s-1} \eta_s^{ij} \rho^{3^{n-s} i} .$$

Since there are $2 \cdot 3^{s-1}$ non-conjugate primitive 3^n-th roots of unity in $\overline{\mathbf{F}}_\ell$, there are the same number of $\mathbf{F}_\ell$-conjugacy classes of injective characters of Δ_n. In this case, we put

$$X = \{ j \in \mathbf{Z} \mid 1 \leqq j < 3^s, (j, 3) = 1 \} .$$

Then $\{\chi^j \mid j \in X\}$ is a set of representatives of $\mathbf{F}_\ell$-conjugacy classes of injective characters of Δ_n.

Let g_ℓ be a primitive root of ℓ and fix integers a_{ij} satisfying

$$a_{ij} \equiv g_\ell^{\frac{\ell-1}{3^s} ij} \pmod{\ell} .$$

CRITERION 2. Put $b = 4^{3^{n-s}}$. If there exists a prime number p which is congruent to 1 modulo $3^{n+1}\ell$ and satisfies

$$\left(\prod_{i=0}^{3^s-1} (2 - z_1^{b^i} - z_2^{b^i})^{a_{ij}} \right)^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{p} \quad \text{for each } j \in X ,$$

then ℓ does not divide h_n / h_{n-1}.

3.3. The case $\ell \equiv -1 \pmod 3$ and $2 \leqq n \leqq s$. We have $[K : \mathbf{F}_\ell] = 2$. Since there are 3^{n-1} non-conjugate primitive 3^n-th roots of unity in $\overline{\mathbf{F}}_\ell$, there are the same number of $\mathbf{F}_\ell$-conjugacy classes of injective characters of Δ_n. In this case, we put

$$X = \left\{ j \in \mathbf{Z} \mid 1 \leqq j \leqq \frac{3^n - 1}{2}, (j, 3) = 1 \right\} .$$

Then $\{\chi^j \mid j \in X\}$ is a set of representatives of $\mathbf{F}_\ell$-conjugacy classes of injective characters of Δ_n.

In this case, we have

$$\begin{aligned} e_{\chi^j} &= \frac{1}{3^n} \sum_{i=0}^{3^n-1} \mathrm{Tr}_{K/\mathbf{F}_\ell}(\eta_n^{ij}) \rho^i \\ &= \frac{1}{3^n} \sum_{i=0}^{3^n-1} \mathrm{Tr}_{\mathbf{F}_\ell(\eta_s)/\mathbf{F}_\ell}(\eta_s^{3^{s-n} ij}) \rho^i . \end{aligned}$$

Fix integers a_{ij} satisfying

$$a_{ij} \equiv t_{3^{s-n} ij} \pmod{\ell} ,$$

where t_i is the element of $\mathbf{F}_\ell$ defined by (2) in 3.4.

CRITERION 3. Put $b = 4$. If there exists a prime number p which is congruent to 1 modulo $3^{n+1}\ell$ and satisfies

$$\left(\prod_{i=0}^{3^n-1}(2 - z_1^{b^i} - z_2^{b^i})^{a_{ij}}\right)^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{p} \quad \text{for each } j \in X,$$

then ℓ does not divide h_n/h_{n-1}.

3.4. The case $\ell \equiv -1 \pmod 3$ and $s + 1 \leqq n$. We have $[K : \mathbf{F}_\ell] = 2 \cdot 3^{n-s}$. Let

$$X^2 - aX + 1$$

be the minimal polynomial of η_s over $\mathbf{F}_\ell$. Then the minimal polynomial of η_n over $\mathbf{F}_\ell$ is

$$X^{2\cdot 3^{n-s}} - aX^{3^{n-s}} + 1 .$$

therefore $\mathrm{Tr}_{K/\mathbf{F}_\ell}(\eta_n^i) = 0$ if i is not divisible by 3^{n-s}. Hence we have

$$\begin{aligned} e_{\chi^j} &= \frac{1}{3^n}\sum_{i=0}^{3^s-1}\mathrm{Tr}_{K/\mathbf{F}_\ell}(\eta_n^{3^{n-s}ij})\rho^{3^{n-s}i} \\ &= \frac{1}{3^s}\sum_{i=0}^{3^s-1}\mathrm{Tr}_{\mathbf{F}_\ell(\eta_s)/\mathbf{F}_\ell}(\eta_s^{ij})\rho^{3^{n-s}i} . \end{aligned}$$

We need to calculate

$$t_i = \mathrm{Tr}_{\mathbf{F}_\ell(\eta_s)/\mathbf{F}_\ell}(\eta_s^i) . \tag{2}$$

We start from $t_1 = \eta_s + \eta_s^{-1}$ and proceed to

$$\begin{aligned} t_3 &= \eta_s^3 + \eta_s^{3\ell} = (\eta_s + \eta_s^\ell)^3 - 3\eta_s^{(\ell+1)}(\eta_s + \eta_s^\ell) = t_1^3 - 3t_1 \\ t_{3^2} &= \eta_s^{3^2} + \eta_s^{3^2\ell} = (\eta_s^3 + \eta_s^{3\ell})^3 - 3\eta_s^{3(\ell+1)}(\eta_s^3 + \eta_s^{3\ell}) = t_3^3 - 3t_3 \\ &\vdots \\ t_{3^{s-1}} &= \eta_s^{3^{s-1}} + \eta_s^{3^{s-1}\ell} = t_{3^{s-2}}^3 - 3t_{3^{s-2}} = -1 , \end{aligned}$$

noting $\eta_s^{\ell+1} = 1$. Reversing this procedure, we obtain t_1 recursively.

LEMMA 5. *Let $b_1 = -1 \in \mathbf{F}_\ell$. If $s \geqq 2$, we choose $b_i \in \mathbf{F}_\ell$ $(2 \leqq i \leqq s)$ by*

$$b_{i+1}^3 - 3b_{i+1} = b_i .$$

Then we have $t_1 = b_s$.

REMARK. For each step, we have three roots. Hence we have just 3^{s-1} t_1 which correspond to 3^{s-1} non-conjugate primitive 3^s-th roots of unity in $\overline{\mathbf{F}}_\ell$. We fix arbitrary one.

We obtain t_i $(2 \leqq i \leqq 3^s - 1)$ from $t_0 = 2$ and t_1 using the following recurrence formula.

LEMMA 6. *There holds* $t_{i+2} = t_{i+1}t_1 - t_i$.

PROOF. We have

$$\begin{aligned} t_1 t_{i+1} &= (\eta_s + \eta_s^{\ell})(\eta_s^{i+1} + \eta_s^{(i+1)\ell}) \\ &= \eta_s^{i+2} + \eta_s^{(i+2)\ell} + \eta_s^{i+\ell+1} + \eta_s^{i\ell+\ell+1} \\ &= (\eta_s^{i+2} + \eta_s^{(i+2)\ell}) + \eta_s^{\ell+1}(\eta_s^{i} + \eta_s^{i\ell}) \\ &= t_{i+2} + t_i \,. \end{aligned}$$

□

Since there are 3^{s-1} non-conjugate primitive 3^n-th roots of unity in $\overline{\mathbf{F}}_\ell$, there are the same number of $\mathbf{F}_\ell$-conjugacy classes of injective characters of Δ_n. In this case, we put

$$X = \{ j \in \mathbf{Z} \mid 1 \leqq j \leqq \frac{3^s - 1}{2}, (j, 3) = 1 \} \,.$$

Then $\{\chi^j \mid j \in X\}$ is a set of representatives of $\mathbf{F}_\ell$-conjugacy classes of injective characters of Δ_n. We fix integers a_{ij} satisfying

$$a_{ij} \equiv t_{ij} \pmod{\ell} \,.$$

Note that ij in the left hand side is a subscript with two indices and that in the right is the product of i and j.

CRITERION 4. Put $b = 4^{3^{n-s}}$. If there exists a prime number p which is congruent to 1 modulo $3^{n+1}\ell$ and satisfies

$$\left(\prod_{i=0}^{3^s-1} (2 - z_1^{b^i} - z_2^{b^i})^{a_{ij}} \right)^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{p} \quad \text{for each } \ j \in X \,,$$

then ℓ does not divide h_n / h_{n-1}.

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