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Tracial States on the θ-deformed Plane

Ken MIYAKE

Keio University

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Abstract. Starting from a trivial pure state Ψ^0, we construct non-trivial tracial states (Ψ^i) on the θ-deformed $2m$-plane $C^{alg}(\mathbf{R}_\theta^{2m})$. Furthermore we generalize Ψ^i to another tracial state on $C^{alg}(\mathbf{R}_\theta^{2m})$. We study extreme points of the tracial state space of $C^{alg}(\mathbf{R}_\theta^{2m})$ in the case that deformation parameters are irrational numbers. Non-trivial pure states (Φ_t^k) on $C^{alg}(\mathbf{R}_\theta^{2m})$ are also given.

1. Introduction

The deformations of funtion algebras by using anti-symmetric real-valued matrix $\theta = (\theta_{ij})$ are called the θ-deformations (*cf.* [3],[4]). The C^*-algebra $C(T_\theta^m)$ (*cf.*[10]) corresponding to the algebra of continuous functions on the noncommutative torus T_θ^m is well-known as an example of θ-deformations. Besides, in[3] various examples of θ-deformations are studied in detail.

In this paper, we restrict our attention to the θ-defomed $2m$-plane $C^{alg}(\mathbf{R}_\theta^{2m})$. The algebra $C^{alg}(\mathbf{R}_\theta^{2m})$ is corresponding to the unital $*$-algebra of polynomial functions on the θ-deformed $2m$-plane $\mathbf{R}_\theta^{2m}$. The purpose of this paper is to construct non-trivial tracial states on $C^{alg}(\mathbf{R}_\theta^{2m})$ for every $m \in \mathbf{N}$. On the other hand, an algebraic probability space (*cf.* [1]) is defined to be a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital $*$-algebra and φ is a state on $\mathcal{A}$. The notion is obtained by considering a generalization of random variables and their expectation values in probability theory.

Our aim is to give non-trivial examples of algebraic probability spaces. To that end, it is crucial to give a criterion for the trivial pure state Ψ^0 on $C^{alg}(\mathbf{R}_\theta^{2m})$. Suggested by this criterion, we construct a tracial class Ψ^i, and its generalization.

We study also extreme points of the tracial state space in the case that deformation parameters θ_{ij} are irrational numbers. Further investigation for the tracial states on $C^{alg}(\mathbf{R}_\theta^{2m})$ will be given in the forthcoming paper.

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2. Preliminaries

In this paper, we use $\bar{\ }$ instead of $*$-operation in consideration of the simplification of the description. We begin by recalling the definition of the θ-deformed $2m$-plane $C^{alg}(\mathbf{R}^{2m}_\theta)$ (*cf.*[3]) which is a fundamental example of θ-deformations.

DEFINITION 1. Let $C^{alg}(\mathbf{R}^{2m}_\theta)$ be the unital $*$-algebra generated by m elements z^i $(i = 1, \dots, m)$ with the commutation relations:

$$\begin{aligned} z^i z^j &= \lambda^{ij} z^j z^i \\ \bar{z}^i \bar{z}^j &= \lambda^{ij} \bar{z}^j \bar{z}^i \\ z^i \bar{z}^j &= \lambda^{ji} \bar{z}^j z^i \qquad (1 \le i, j \le m)\,. \end{aligned} \tag{1}$$

Here λ^{ij} is defined as $\lambda^{ij} = e^{2\pi i \theta_{ij}} = \overline{\lambda^{ji}}$, where $\theta = (\theta_{ij})$ is an anti-symmetric real-valued matrix of degree m.

3. Tracial state Ψ^i

We give the notion of tracial state on $*$-algebra.

DEFINITION 2. Let $\mathcal{A}$ be a unital $*$-algebra and φ be a linear functional $\varphi : \mathcal{A} \to \mathbf{C}$. We say that φ is a state on $\mathcal{A}$ if φ satisfies the properties:
1. $\varphi(\bar{a}a) \ge 0 \quad (\forall a \in \mathcal{A})$,
2. $\varphi(\mathbf{1}_\mathcal{A}) = 1$

where $\mathbf{1}_\mathcal{A}$ is the unit element of $\mathcal{A}$. The set of states of an algebra $\mathcal{A}$ forms a convex set which is called the state space. An extreme point of state space is called a pure state. On the other hand, a non-extreme point of state space is called a mixed state.

DEFINITION 3. A state φ is called a tracial state if φ has the property: $\varphi(xy) = \varphi(yx)$ for $\forall x, y \in \mathcal{A}$. The set of tracial states forms a convex set which is called the tracial state space.

First of all, let us recall a trivial state Ψ^0 on $C^{alg}(\mathbf{R}^{2m}_\theta)$.

DEFINITION 4. Let $n_1, n'_1, \dots, n_m, n'_m$ be in $\mathbf{Z}_{\ge 0}$, and consider the monomial $X = (z^1)^{n_1}(\bar{z}^1)^{n'_1} \cdots (z^m)^{n_m}(\bar{z}^m)^{n'_m} \in C^{alg}(\mathbf{R}^{2m}_\theta)$. We set Ψ^0 as a linear functional satisfying

$$\Psi^0(X) := \begin{cases} 1 & \text{if } n_1 = n'_1 = \cdots = n_m = n'_m = 0\,, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

for X.

We characterize this trivial functional from a little general viewpoints. We denote the unit element of $C^{alg}(\mathbf{R}^{2m}_\theta)$ by $\mathbf{1}$. First we consider a map $\phi : C^{alg}(\mathbf{R}^{2m}_\theta) \to \mathbf{C}$ which satisfying

$$\begin{array}{lll} \phi(z^i z^j) := \lambda^{ij} & \phi(z^j \bar{z}^i) := \lambda^{ij} & \phi(z^i \bar{z}^j) := \lambda^{ji} \\ \phi(\bar{z}^j \bar{z}^i) := \lambda^{ji} & \phi(z^j z^i) := 1 & \phi(\bar{z}^j z^i) := 1 \\ \phi(\bar{z}^i \bar{z}^j) := 1 & \phi(\bar{z}^i z^j) := 1 & \phi(z^i) := 0 \\ \phi(\bar{z}^i) := 0 & \phi(\mathbf{1}) := 1 & (1 \le i \le j \le m)\,. \end{array} \tag{3}$$

We put $z^1 = w_1, \bar{z}^1 = w_2, \ldots, z^m = w_{2m-1}, \bar{z}^m = w_{2m}$, and let $\iota(1), \ldots, \iota(n)$ be in $\mathbf{N}$ such that $1 \le \iota(1), \ldots, \iota(n) \le 2m$, where $\iota(1), \ldots, \iota(n)$ are allowed overlapping. Furthermore, we require that

$$\phi(w_{\iota(1)} \ldots w_{\iota(n)}) := \prod_{k<l}^{n} \phi(w_{\iota(k)} w_{\iota(l)}) \tag{4}$$

for the monomial $A = w_{\iota(1)} \cdots w_{\iota(n)}$ of degree more than 2, and linearity such that

$$\phi(\lambda X + \mu Y) = \lambda\phi(X) + \mu\phi(Y)\,, \qquad \lambda, \mu \in \mathbf{C},\ X, Y \in C^{alg}(\mathbf{R}^{2m}_\theta)\,.$$

LEMMA 5. *ϕ is well-defined uniquely by* (3) *and* (4) *as a linear functional.*

PROOF. It suffices to show following equalities based on (4).

$$\begin{aligned} &\phi(w_{\iota(1)} \cdots w_{\iota(k)}(w_{2p-1}w_{2q-1} - \lambda^{pq} w_{2q-1}w_{2p-1})w_{\iota(k+1)} \cdots w_{\iota(n)}) = 0\,, \\ &\phi(w_{\iota(1)} \cdots w_{\iota(k)}(w_{2p}w_{2q} - \lambda^{pq} w_{2q}w_{2p})w_{\iota(k+1)} \cdots w_{\iota(n)}) = 0\,, \\ &\phi(w_{\iota(1)} \cdots w_{\iota(k)}(w_{2p}w_{2q-1} - \lambda^{qp} w_{2q-1}w_{2p})w_{\iota(k+1)} \cdots w_{\iota(n)}) = 0 \\ &\qquad\qquad p, q, k \in \mathbf{N},\ 1 \le p, q \le m,\ 1 \le k \le n-1\,. \end{aligned} \tag{5}$$

Note that $w_{2p-1} = z^p$, $w_{2p} = \bar{z}^p$, $w_{2q-1} = z^q$, $w_{2q} = \bar{z}^q$. We will show the first equation of (5).

$$\begin{aligned} &\phi(w_{\iota(1)} \cdots w_{\iota(k)}(w_{2p-1}w_{2q-1} - \lambda^{pq} w_{2q-1}w_{2p-1})w_{\iota(k+1)} \cdots w_{\iota(n)}) \\ &\quad = \phi(w_{\iota(1)} \cdots w_{\iota(k)} w_{2p-1}w_{2q-1} w_{\iota(k+1)} \cdots w_{\iota(n)}) \\ &\qquad - \lambda^{pq}\phi(w_{\iota(1)} \cdots w_{\iota(k)} w_{2q-1}w_{2p-1} w_{\iota(k+1)} \cdots w_{\iota(n)}) \\ &\quad = (\phi(w_{2p-1}w_{2q-1}) - \lambda^{pq}\phi(w_{2q-1}w_{2p-1})) \prod_{e=1}^{k} \phi(w_{\iota(e)}w_{2p-1})\phi(w_{\iota(e)}w_{2q-1}) \\ &\qquad \times \prod_{f=k+1}^{n} \phi(w_{2p-1}w_{\iota(f)})\phi(w_{2q-1}w_{\iota(f)}) \prod_{1 \le g < g' \le n} \phi(w_{\iota(g)}w_{\iota(g')}) \\ &\quad = 0 \end{aligned}$$

by (4).

Hence the first equation is proved. The remaining are proved similarly. Consequently we see that a map ϕ is uniquely determined as a linear functional. □

Next let us introduce some notations. Let $t, i(1), \dots, i(t)$ be in $\mathbf{N}$ such that $1 \leq i(1) \lneq \cdots \lneq i(t) \leq m,\ 1 \leq t \leq m$.

DEFINITION 6. Let $T^{i(1)\cdots i(t)}$ be the set of monomials formed by generators $\mathbf{1}, z^{i(1)}, \bar{z}^{i(1)}, \dots, z^{i(t)}, \bar{z}^{i(t)} \in C^{alg}(\mathbf{R}_\theta^{2m})$. Particularly, we denote the set $\{\mathbf{1}\}$ by T^0.

For example $\mathbf{1}, \bar{z}^1\bar{z}^3, z^1z^2 \in T^{1\cdot 2\cdot 3}$.

DEFINITION 7. We say that X is regular in $T^{i(1)\cdots i(t)}$ or simply we say that X is regular if there exists a monomial $Y \in T^{i(1)\cdots i(t)}$ such that $X = \lambda \overline{Y} Y$, $\lambda \in \mathbf{C} - \{0\}$.

EXAMPLE 8. The monomial $X = \bar{z}^2 z^1 z^2 \bar{z}^1$ is regular in $T^{1\cdot 2}$. In fact, if we set $Y = z^2\bar{z}^1 \in T^{1\cdot 2}$, then it holds $X = \lambda^{21}\overline{Y}Y$.

LEMMA 9. *The functional Ψ^0 is expressed as follows by using the above terms.*

$$\Psi^0(X) := \begin{cases} \phi(X) & \text{if } X \text{ is regular in } T^0\,, \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

for any monomial $X \in C^{alg}(\mathbf{R}_\theta^{2m})$.

Our intention is generalizing Ψ^0 in accordance with the form of (6). Let us define a functional $\Psi^{i(1)\cdots i(t)}$ on $C^{alg}(\mathbf{R}_\theta^{2m})$.

DEFINITION 10. Let $\Psi^{i(1)\cdots i(t)}$ be the linear functional defined by setting

$$\Psi^{i(1)\cdots i(t)}(X) := \begin{cases} \phi(X) & \text{if } X \text{ is regular in } T^{i(1)\cdots i(t)}\,, \\ 0 & \text{otherwise} \end{cases}$$

for any monomial $X \in C^{alg}(\mathbf{R}_\theta^{2m})$.

In the following we denote $\Psi^{i(1)\cdots i(t)}$ simply by Ψ^i.

REMARK 11. In definition 10, if $T^{i(1)\cdots i(t)} = T^0$, then $\Psi^{i(1)\cdots i(t)} = \Psi^0$.

Now we will prove that Ψ^i is a tracial state on $C^{alg}(\mathbf{R}_\theta^{2m})$. The following Lemma is fundamental.

LEMMA 12. *We put $z^1 = w_1, \bar{z}^1 = w_2, \dots, z^m = w_{2m-1}, \bar{z}^m = w_{2m}$. Suppose that $X = w_{j(1)} \cdots w_{j(k)} \in T^{i(1)\cdots i(t)}$ for $j(1), \dots, j(k) \in \{1, \dots, 2m\}$, $k \in \mathbf{N}$. Then we have*

$$\Psi^i(\overline{X}X) = 1\,. \tag{7}$$

PROOF. If $X = \mathbf{1}$, then (7) is obvious. Therefore, we assume that $X \neq \mathbf{1}$. Let p, q be in $\mathbf{N}$ such that $1 \leq q \lneq p \leq k$. Then we get

$$\Psi^i(\overline{X}X) = \Psi^i(\bar{w}_{j(k)} \cdots \bar{w}_{j(1)} w_{j(1)} \cdots w_{j(k)})$$

$$
\begin{aligned}
&= \phi(\bar{w}_{j(k)} \cdots \bar{w}_{j(1)} w_{j(1)} \cdots w_{j(k)}) \\
&= \prod_{p \neq q} \underbrace{\phi(\bar{w}_{j(p)} \bar{w}_{j(q)}) \phi(w_{j(q)} w_{j(p)})}_{1} \underbrace{\phi(\bar{w}_{j(p)} w_{j(q)}) \phi(\bar{w}_{j(q)} w_{j(p)})}_{1} \\
&= 1 .
\end{aligned}
$$

□

LEMMA 13. *If X and Y are regular, then XY, $\overline{X}Y$ are regular.*

Note that the converse of Lemma 13 is not true in general.

PROPOSITION 14. *If X and Y are regular, then we have*

$$\Psi^i(XY) = \Psi^i(X)\Psi^i(Y) . \tag{8}$$

PROOF. If X or Y is a scalar multiplication of $\mathbf{1}$, then (8) is obvious. Therefore, we assume $X, Y \neq \lambda\mathbf{1}, \lambda \in \mathbf{C} - \{0\}$. Let z^j be one of elements which forms X. Then $\bar{z}^j$ is also one of elements which forms X. Similarly, let $z^k, \bar{z}^k$ be elements which form Y. Then we obtain the following equality.

$$\phi(z^j z^k)\phi(z^j \bar{z}^k)\phi(\bar{z}^j z^k)\phi(\bar{z}^j \bar{z}^k) = 1 .$$

Since the elements z^j and z^k are arbitrary, the result is given. □

In relation to Proposition 14, we have the following Lemma.

LEMMA 15. *If X is regular and Y is not regular, then*

$$\Psi^i(XY) = \Psi^i(\overline{X}Y) = 0 . \tag{9}$$

PROOF. If X is regular and Y is not regular, then XY, $\overline{X}Y$ are not regular. Namely, (9) is proved by Definition 10. □

Ψ^i has the following property.

PROPOSITION 16. $\Psi^i(xy) = \Psi^i(yx)$ *for* $\forall x, y \in C^{alg}(\mathbf{R}^{2m}_\theta)$.

PROOF. By Definition of Ψ^i, it suffices to consider the case that xy is regualr. We put $z^j = e_1, \bar{z}^j = e_2, z^k = e_3, \bar{z}^k = e_4,\ j, k = 1, \ldots, m$. Suppose that $k_1, k_2, k_3, k_4 \in \{1, \ldots, 4\}$, however $i \neq j \Rightarrow k_i \neq k_j, i, j = 1, \ldots, 4$. Then it is easy to see that

$$e_{k_1} e_{k_2} e_{k_3} e_{k_4} = e_{k_2} e_{k_3} e_{k_4} e_{k_1} = e_{k_3} e_{k_4} e_{k_1} e_{k_2} = e_{k_4} e_{k_1} e_{k_2} e_{k_3} .$$

Hence if $X = xy$ is regular, then it holds $xy = yx$. □

By the commutation relations of $C^{alg}(\mathbf{R}^{2m}_\theta)$, it holds the following property concerned with Ψ^i.

LEMMA 17. *Let x be regular and let y be a monomial in $C^{alg}(\mathbf{R}^{2m}_\theta)$. Then it holds $\Psi^i(xy) = \Psi^i(yx)$.*

DEFINITION 18. Let X, Y be in $T^{i(1)\cdots i(t)}$. We denote by $X \sim Y$ if $\overline{X}Y$ is regular.

The relation $\sim$ is an equivalence relation. We denote the equivalence class of A by $[A]$ for $A \in T^{i(1)\cdots i(t)}$. Let $\deg(A)$ denote the degree of a monomial A, and let $[A]_{\min}$ be the subset of $[A]$ such that

$$[A]_{\min} = \{x \in [A] \mid \deg(x) \leq \deg(y),\ \forall y \in [A]\}.$$

LEMMA 19. *Let X, Y be in $T^{i(1)\cdots i(t)}$. If $X \nsim Y$, then*

$$\Psi^i(\overline{X}Y) = 0.$$

PROOF. If $X \nsim Y$, then $\overline{X}Y$ is not regular. i.e. $\Psi^i(\overline{X}Y) = 0$. □

PROPOSITION 20. *Ψ^i is a positive functional.*

PROOF. By Lemma 19 and Definition of Ψ^i, it suffices to prove $\Psi^i(\overline{X}X) \geq 0$ for $X = \sum_{t=1}^k r_t x_t,\ x_p \sim x_q,\ p, q = 1, \ldots, k,\ r_1, \ldots, r_k \in \mathbf{C}$ in order for Ψ^i to be a positive functional. We can denote X by $\sum_{t=1}^k r'_t u y_t$, where $u \in [x_t]_{\min}$, $y_1, \ldots, y_k$ are regular, and $r'_1, \ldots, r'_k \in \mathbf{C}$. Then it follows from Proposition 14 and Lemma 17 that

$$\begin{aligned}
\Psi^i(\overline{X}X) &= \Psi^i\bigg(\bigg(\sum_{t=1}^k \bar{r}'_t \bar{y}_t \bar{u}\bigg)\bigg(\sum_{t=1}^k r'_t u y_t\bigg)\bigg) \\
&= \Psi^i(\bar{u}u)\bigg(\sum_{t=1}^k \bar{r}'_t \Psi^i(\bar{y}_t)\bigg)\bigg(\sum_{t=1}^k r'_t \Psi^i(y_t)\bigg) \\
&= \Psi^i(\bar{u}u)\overline{\bigg(\sum_{t=1}^k r'_t \Psi^i(y_t)\bigg)}\bigg(\sum_{t=1}^k r'_t \Psi^i(y_t)\bigg) \geq 0.
\end{aligned}$$

Thus it is proved that Ψ^i is a state on $C^{alg}(\mathbf{R}_\theta^{2m})$. □

It follows from Proposition 16 and Proposition 20 that the following Theorem.

THEOREM 21. *Ψ^i is a tracial state on $C^{alg}(\mathbf{R}_\theta^{2m})$.*

4. Generalization of Ψ^i

Tracial state Ψ^i is generalized naturally. Let us define a linear functional $\Psi_{x_1\cdots x_t}^{i(1)\cdots i(t)}$ on $C^{alg}(\mathbf{R}_\theta^{2m})$. Suppose that $x_1, \ldots, x_t \gneq 0,\ n_1, n'_1, \ldots, n_m, n'_m \in \mathbf{Z}_{\geq 0}$.

DEFINITION 22. Let $\Psi_{x_1\cdots x_t}^{i(1)\cdots i(t)}$ be the linear functional defined by setting

$$\Psi_{x_1\cdots x_t}^{i(1)\cdots i(t)}(X) := \begin{cases} x_1^{n_{i(1)}} \cdots \mathrm{x}_t^{n_{i(t)}} \phi(X) & \text{if } X \text{ is regular in } T^{i(1)\cdots i(t)} \\ 0 & \text{otherwise} \end{cases}$$

for the monomial $X = (z^1)^{n_1}(\bar{z}^1)^{n'_1} \cdots (z^m)^{n_m}(\bar{z}^m)^{n'_m} \in C^{alg}(\mathbf{R}^{2m}_\theta)$.

In the following we denote $\Psi^{i(1)\cdots i(t)}_{x_1\cdots x_t}$ simply by Ψ^i_x.

REMARK 23. If $x_1 = \cdots = x_t = 1$, then $\Psi^i_x = \Psi^i$.

We have the following Propositions.

PROPOSITION 24. *If X and Y are regular, then*

$$\Psi^i_x(XY) = \Psi^i_x(X)\Psi^i_x(Y)\,.$$

PROPOSITION 25. $\Psi^i_x(xy) = \Psi^i_x(yx)$ *for* $\forall x, y \in C^{alg}(\mathbf{R}^{2m}_\theta)$.

We can show the following in a similar way to the proof of Proposition 20.

THEOREM 26. Ψ^i_x *is a tracial state on* $C^{alg}(\mathbf{R}^{2m}_\theta)$.

REMARK 27. The unital $*$-algebra $C^{alg}(\mathbf{R}^{2m+1}_\theta)$ is defined by adding a selfadjoint generator z^{m+1} to $C^{alg}(\mathbf{R}^{2m}_\theta)$ with relations $z^i z^{m+1} = z^{m+1} z^i$ $(1 \le i \le m)$. We can construct tracial states on $C^{alg}(\mathbf{R}^{2m+1}_\theta)$ in the same way to the case of $C^{alg}(\mathbf{R}^{2m}_\theta)$.

5. Extreme points of the tracial state space

We would introduce extreme points of the tracial state space of $C^{alg}(\mathbf{R}^{2m}_\theta)$ in the case that deformation parameter θ_{ij} $(1 \le i \lneq j \le m)$ are irrational numbers. Let $\Psi^{(2)}$ be the tracial state, which is assumed to be $t = 2$ in $\Psi^{i(1)\cdots i(t)}_{x_1\cdots x_t}$. Namely it holds $\Psi^{(2)} = \Psi^{i(1)i(2)}_{x_1 x_2}$, where $1 \le i(1) \lneq i(2) \le m,\ x_1, x_2 \gneq 0$.

PROPOSITION 28. *If $\theta_{i(1)i(2)}$ is an irrational number, then $\Psi^{(2)}$ is an extreme point of the tracial state space of $C^{alg}(\mathbf{R}^{2m}_\theta)$.*

PROOF. We assume that there exist tracial states Ψ_1, Ψ_2 on $C^{alg}(\mathbf{R}^{2m}_\theta)$ such that

$$\Psi^{(2)} = (1-s)\Psi_1 + s\Psi_2 \quad (0 \lneq s \lneq 1)\,. \tag{10}$$

We prove this Proposition in three steps.

1ST STEP: Let K be a monomial formed from the set of generators

$$\{z^1, \bar{z}^1, \ldots, z^m, \bar{z}^m\} - \{z^{i(1)}, \bar{z}^{i(1)}, z^{i(2)}, \bar{z}^{i(2)}\}\,.$$

Suppose that $L = \overline{K}K$. By definition of $\Psi^{(2)}$ and (10), we have

$$(1-s)\Psi_1(L) + s\Psi_2(L) = 0\,.$$

Since a state is positive

$$\Psi_1(L) \ge 0 \qquad \Psi_2(L) \ge 0\,.$$

Hence

$$\Psi_1(L) = \Psi_2(L) = 0\,.$$

Let $\lambda_1, \lambda_2 \in \mathbf{C} - \{0\}$. We assume $\Psi_1(K) \neq 0$. Then we have

$$\Psi_1(\overline{(\lambda_1 \mathbf{1} + \lambda_2 K)}(\lambda_1 \mathbf{1} + \lambda_2 K)) = |\lambda_1|^2 + 2\mathrm{Re}(\overline{\lambda_1}\lambda_2 \Psi_1(K)) \geq 0\,. \tag{11}$$

However, if we take

$$\lambda_2 = -\lambda_1 \frac{1}{\Psi_1(K)}\,,$$

then we have

$$|\lambda_1|^2 + 2\mathrm{Re}(\overline{\lambda_1}\lambda_2 \Psi_1(K)) = -|\lambda_1|^2 \lneq 0\,. \tag{12}$$

(12) contradicts (11). Hence we get $\Psi_1(K) = 0$. As well as Ψ_1, we get $\Psi_2(K) = 0$.

2ND STEP: Let M be regular in $T^{i(1)i(2)}$. Then we have

$$|\Psi^{(2)}(M)|^2 = \Psi^{(2)}(\overline{M}M)$$

by definition of $\Psi^{(2)}$. Therefore there exists ω ($0 \leq \omega \lneq 2\pi$) that satisfies

$$\Psi^{(2)}(\overline{(|\Psi^{(2)}(M)|\mathbf{1} + e^{i\omega}M)}(|\Psi^{(2)}(M)|\mathbf{1} + e^{i\omega}M)) = 0\,. \tag{13}$$

We denote $|\Psi^{(2)}(M)|\mathbf{1} + e^{i\omega}M$ by S. By (10) and (13) we have

$$(1-s)\Psi_1(\overline{S}S) + s\Psi_2(\overline{S}S) = 0\,.$$

However, since a state is positive

$$\Psi_1(\overline{S}S) \geq 0 \qquad \Psi_2(\overline{S}S) \geq 0\,.$$

Hence

$$\Psi_1(\overline{S}S) = \Psi_2(\overline{S}S) = 0\,.$$

Let $r_1, r_2 \in \mathbf{C} - \{0\}$. Assuming that $\Psi^{(2)}(S) \neq 0$, then we have

$$\Psi^{(2)}((\overline{r_1 \mathbf{1} + r_2 S})(r_1 \mathbf{1} + r_2 S)) = |r_1|^2 + 2\mathrm{Re}(\overline{r_1} r_2 \Psi^{(2)}(S)) \geq 0\,. \tag{14}$$

However, if we take

$$r_2 = -\frac{r_1}{\Psi^{(2)}(S)}\,,$$

then

$$|r_1|^2 + 2\mathrm{Re}(\overline{r_1} r_2 \Psi^{(2)}(S)) = -|r_1|^2 \lneq 0 \tag{15}$$

(15) contradicts (14). Hence, we get

$$\Psi^{(2)}(S) = 0\,.$$

In the similar way, we get

$$\Psi_1(S) = \Psi_2(S) = 0\,.$$

Consequently, we obtain

$$\Psi^{(2)}(M) = \Psi_1(M) = \Psi_2(M)\,.$$

3RD STEP: Suppose that $P, Q \in T^{i(1)i(2)}$ and P is not regular in $T^{i(1)i(2)}$. Then we have

$$\begin{aligned}
&\Psi_1((\overline{P\Psi_1(\overline{Q}Q) - P\overline{Q}Q})(P\Psi_1(\overline{Q}Q) - P\overline{Q}Q)) \\
&\quad = \Psi_1(\overline{P}P)\Psi_1(\overline{Q}Q)^2 + \Psi_1(\overline{Q}Q\overline{P}P\overline{Q}Q) - 2\Psi_1(\overline{P}Q\overline{Q}P)\Psi_1(\overline{Q}Q) \\
&\quad = 0\,.
\end{aligned} \tag{16}$$

By (16), we get the following in the same way that the 2nd step.

$$\Psi_1(P\overline{Q}Q) = \Psi_1(P)\Psi_1(\overline{Q}Q)\,. \tag{17}$$

Considering (17), We see that $\Psi_1(P) = 0$ in order for Ψ_1 to be a tracial state in the case that $\theta_{i(1)i(2)}$ is an irrational number. Let m, n be in natural number. In fact, if $\Psi_1((z^{i(1)})^m(z^{i(2)})^n) \neq 0$, then we get

$$\Psi_1((z^{i(1)})^m(z^{i(2)})^n) \neq \Psi_1((z^{i(2)})^n(z^{i(1)})^m) \tag{18}$$

since $\theta_{i(1)i(2)}$ is an irrational number. On the other hand, if $\Psi_1((z^{i(1)})^m) \neq 0$, then (17) shows that

$$\Psi_1((z^{i(1)})^m)\Psi_1((\bar{z}^{i(2)})^n(z^{i(2)})^n) = \Psi_1((z^{i(1)})^m(\bar{z}^{i(2)})^n(z^{i(2)})^n) \neq 0\,.$$

Then we obtain

$$\Psi_1((z^{i(1)})^m(\bar{z}^{i(2)})^n(z^{i(2)})^n) \neq \Psi_1((z^{i(2)})^n(z^{i(1)})^m(\bar{z}^{i(2)})^n)\,.$$

as well as (18). Eventually, it turns out in these cases that Ψ_1 is not tracial. This contradicts to that Ψ_1 is a tracial state. Hence we see that $\Psi_1(P) = 0$ generally. As well as Ψ_1, we see that $\Psi_2(P) = 0$.

Eventually we obtain

$$\Psi^{(2)} = \Psi_1 = \Psi_2$$

from three steps. This completes the proof. □

We have the following Corollary in relation to Proposition 28.

COROLLARY 29. *Let $\mathcal{A}^{ab}$ be the quotient of $C^{alg}(\mathbf{R}^4_\theta)$ by the two-sided ideal generated by $\bar{z}^1 z^1 - a\mathbf{1}$ and $\bar{z}^2 z^2 - b\mathbf{1}$, $(a, b \gneq 0)$. If θ_{12} is an irrational number, then $\mathcal{A}^{ab}$ has the unique tracial state.*

We consider the case that deformation parameters θ_{ij} $(i, j = 1, \dots, m)$ of $C^{alg}(\mathbf{R}^{2m}_\theta)$ are irrational numbers and satisfy the following condition: For any integers k_{ij} $(i, j = 1, \dots, m)$ such that $\sum_{i \lneq j} |k_{ij}| \neq 0$, θ_{ij} satisfy $\sum_{i \lneq j} k_{ij}\theta_{ij} \notin \mathbf{Z}_{\geq 0}$. Then we have the following Lemma.

LEMMA 30. *Let x, y be monomials of $C^{alg}(\mathbf{R}^{2m}_\theta)$ which both are not scalar multiple of* **1**. *If a monomial xy is not regular and xy is formed by two or more different generators, then $xy \neq yx$.*

The following Proposition holds based on Lemma 30.

PROPOSITION 31. *If $t \geq 2$ for a tracial state $\Psi^i_x = \Psi^{i(1)\cdots i(t)}_{x_1\cdots x_t}$, then Ψ^i_x is an extreme point of the tracial state space of $C^{alg}(\mathbf{R}^{2m}_\theta)$.*

The proof is obtained by the similar method to Proposition 28.

6. Pure state

We give non-trivial pure states on $C^{alg}(\mathbf{R}^{2m}_\theta)$. Suppose that $n_1, n'_1, \dots, n_m, n'_m \in \mathbf{Z}_{\geq 0}$, $t \in \mathbf{C} - \{0\}$, $k = 1, \dots, m$.

DEFINITION 32. Let Φ^k_t be the linear functional defined by setting

$$\Phi^k_t(X) := \begin{cases} t^{n_k}\bar{t}^{n'_k} & \text{if } X \in T^k\,, \\ 0 & \text{otherwise} \end{cases}$$

for the monomial $X = (z^1)^{n_1}(\bar{z}^1)^{n'_1} \cdots (z^m)^{n_m}(\bar{z}^m)^{n'_m} \in C^{alg}(\mathbf{R}^{2m}_\theta)$.

It is obvious that Φ^k_t is a positive linear functional. Then we have the following.

THEOREM 33. *Φ^k_t is a pure state on $C^{alg}(\mathbf{R}^{2m}_\theta)$.*

The proof is obtained by the similar method to Proposition 28.

REMARK 34. We can construct tracial states on the θ-deformed sphere and so on in a similar way to the θ-deformed plane.

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Present Address:
DEPARTMENT OF MATHMATICS, FACULTY OF SCIENCE AND TECHNOLOGY,
KEIO UNIVERSITY,
YOKOHAMA, 223–8522 JAPAN.
e-mail: miyakenn@a6.keio.jp