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Motion of Charged Particles and Homogeneous Geodesics in Kähler C-Spaces with Two Isotropy Summands

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Abstract. Let $(M = G/K, g, J)$ be a Kähler C-space with two isotropy summands. We classify all such spaces. Thus, by using previous work of O. Ikawa, we obtain a large class of examples where the differential equation $\nabla_{\dot{x}}\dot{x} = kJ\dot{x}$ of the motion of a charged particle under the electromagnetic field kJ can be explicitly solved. In particular, geodesics curves in these spaces can also be described.

1. Introduction

Let (M, g) be a Riemannian manifold, F a closed 2-form, and X a vector field on M. We denote by $\iota_X : \Lambda^p(M) \to \Lambda^{p-1}(M)$ the interior product operator induced by X, and by $\mathcal{L} : TM \to T^*M$ the Legendre transformation defined by $u \mapsto \mathcal{L}(u)$, $\mathcal{L}(u)(v) = g(u, v)$ $(v \in TM)$. A curve $x(t)$ in M is called the *motion of a charged particle under the electromagnetic field F* if it satisfies the differential equation

$$\nabla_{\dot{x}}\dot{x} = -\mathcal{L}^{-1}(\iota_{\dot{x}}F)\,,$$

where ∇ is the Levi-Civita connection of M. This equation has its origin in general relativity ([Mi-Th-Wh]). When $F = 0$ then $x(t)$ is a geodesic in M. If M is a Kähler manifold with complex structure J there is a natural choice of an electromagnetic field F, namely a scalar multiple of the Kähler form ω, defined by $\omega(X, Y) = g(X, JY)$. Since $-\mathcal{L}^{-1}(\iota_X\omega) = JX$, a curve $x(t)$ is the motion of a charged particle under the electromagnetic field $\kappa\omega$ if and only if

$$\nabla_{\dot{x}}\dot{x} = kJ\dot{x}\,. \tag{1.1}$$

In a series of papers ([Ik1], [Ik2], [Ik3], [Ik4]) O. Ikawa studied the above equation for various homogeneous spaces. One of the questions he studied was the existence of periodic solutions of (1.1). He gave an affirmative answer for certain homogeneous semi-Riemannian

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manifolds, Sasakian manifolds, and Hermitian symmetric spaces. Note that it is a result of S. Kobayashi ([O'N]) that a geodesic in a homogeneous Riemannian manifold is simple.

Also, in [Ik3] Ikawa gave explicit solutions for (1.1) in the case that M is a Hermitian symmetric space (thus extending results of T. Adachi, S. Maeda, and S. Udagawa in [Ad-Ma-Ud]), and a Kähler C-space with certain conditions, one of which is that the second Betti number of M is $b_2(M) = 1$. In the present work we give a complete classification of the Kähler C-spaces $M = G/K$ considered in [Ik3], and for which equation (1.1) can be solved explicitly. This includes cases where G is either a classical or an exceptional simple Lie group.

Further, by combining our results with previous work of R. Dohira ([Doh]) it is possible to give a complete description of geodesic curves in such spaces. For an appropriate choice of the G-invariant metric on $M = G/K$ these geodesic curves are homogeneous, and their complete classification was given in the recent work [A-A] of D.V. Alekseevsky and the first author.

In particular we show:

THEOREM A. *Let $M = G/K$ be a Kähler C-space for which the isotropy representation decomposes into two non-equivalent irreducible submodules. Then M is locally isomorphic to one of the spaces presented on Table* 1.

TABLE 1. The Kähler C-spaces with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

G	G/K
$B_\ell = SO(2\ell+1)$	$SO(2\ell+1)/U(i) \times SO(2(\ell-i)+1)$ $(\ell > 0, i \neq 1)$
$C_\ell = Sp(\ell)$	$Sp(\ell)/U(i) \times Sp(\ell-i)$ $(\ell > 0, i \neq \ell)$
$D_\ell = SO(2\ell)$	$SO(2\ell)/U(i) \times SO(2(\ell-i))$ $(\ell > 0, i \neq 1, i \neq \ell)$
G_2	$G_2/U(2)$ ($U(2)$ is represented by the short root of G_2)
F_4	$F_4/SO(7) \times U(1)$
	$F_4/Sp(3) \times U(1)$
E_6	$E_6/SU(6) \times U(1)$
	$E_6/SU(2) \times SU(5) \times U(1)$
E_7	$E_7/SU(7) \times U(1)$
	$E_7/SU(2) \times SO(10) \times U(1)$
	$E_7/SO(12) \times U(1)$
E_8	$E_8/E_7 \times U(1)$
	$E_8/SO(14) \times U(1)$

THEOREM B. *For the Kähler C-spaces presented in Theorem A, the equation of motion of a charged particle under the electromagnetic field is given in Theorem* 1 *below. In particular, geodesic curves on these spaces are explicity described by Theorem* 2.

2. Motion of charged particles and geodesic curves in Riemannian homogeneous spaces

We will review some of the results in [Ik3] and [Doh]. Let $(M = G/K, g)$ be a Riemannian homogeneous space, where G is a connected Lie group and K a compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Since K is compact there exists an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. It is known that the tangent space T_oM of M at the origin $o = eK$ of M can be identified with $\mathfrak{m}$. The G-invariant metric g corresponds to an $\mathrm{Ad}(K)$-invariant inner product $\langle\, , \,\rangle$ on $\mathfrak{m}$. We assume that the following conditions are satisfied:

There exist $\mathrm{Ad}(K)$-invariant subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of $\mathfrak{m}$ such that

(1) $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$,

(2) $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k} \oplus \mathfrak{m}_2$, $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}$, $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$, $[\mathfrak{k}, \mathfrak{m}_i] \subset \mathfrak{m}_i$ $(i = 1, 2)$,

(3) There exists a non-zero constant $c \in \mathbf{R}$ such that

$$\langle [X, Y]_{\mathfrak{m}_2}, Z\rangle + c\langle X, [Z, Y]\rangle = 0$$

for all $X, Y \in \mathfrak{m}_1$, $Z \in \mathfrak{m}_2$ (here X_i denotes the $\mathfrak{m}_i$-component of $X \in \mathfrak{g}$), and

(4) For $W \in Z(\mathfrak{k})$, the center of $\mathfrak{k}$, we define an endomorphism $I : \mathfrak{m} \to \mathfrak{m}$ by

$$I(X) = [W, X_1] + \frac{1}{c}[W, X_2]\,.$$

Since $\mathrm{Ad}(k)I = I\,\mathrm{Ad}(k)$ for all $k \in K$, the endomorphism I can be extended to a G-invariant $(1, 1)$ tensor I. Then we have that $g(IX, Y) + g(X, IY) = 0$ for all $X, Y \in \mathcal{X}(M)$. We denote a homogeneous space satisfying conditions $(1) - (4)$ by $(G/K, g, \mathfrak{m}_1 \oplus \mathfrak{m}_2, J, c)$.

Let k be a constant. A curve $x(t)$ is called the *motion of a charged particle under the electromagnetic field kI* if it satisfies the differential equation

$$\nabla_{\dot{x}}\dot{x} = kI\dot{x}\,. \tag{2.2}$$

Note that if $k = 0$ then $x(t)$ is a geodesic.

THEOREM 1 [[Ik3]]. *Let $(G/K, g, \mathfrak{m}_1 \oplus \mathfrak{m}_2, J, c)$ be a Riemannian homogeneous space as defined above. Let $x(t)$ be the motion of a charged particle defined by* (2.2) *under the electromagnetic field kI with initial condition $x(0) = o$ and $\dot{x}(0) = X_1 + X_2$ $(X_1 \in \mathfrak{m}_1, X_2 \in \mathfrak{m}_2)$. Then $x(t)$ is given by*

$$x(t) = \exp t(X_1 + cX_2 + kW)\exp t(1 - c)\left(X_2 + \frac{k}{c}W\right)\cdot o\,.$$

If $x(t)$ intersects itself, then it is simply closed.

In the case where $k = 0$ the above theorem reduces to the following:

THEOREM 2 [[Doh]]. *Let $(G/K, g, \mathfrak{m}_1 \oplus \mathfrak{m}_2, c)$ be a Riemannian homogeneous space as defined above. Then any geodesic $\gamma(t)$ in G/K with $\gamma(0) = o$ and $\dot{\gamma}(0) = X_1 + X_2$ is given by*

$$\gamma(t) = \exp t(X_1 + cX_2)\exp t(1 - c)X_2 \cdot o\,.$$

Note that if $c = 1$, that is g is the standard metric on G/K, then $\gamma(t) = \exp t(X_1 + X_2)\cdot o$ is a homogeneous geodesic in G/K. These have been studied extensively by several authors (see the references in [A-A]).

3. Description of Kähler C-spaces

A C-space is a compact and simply connected complex homogeneous space (cf. [Wan]). By a Kähler C-space we mean a C-space M which admits a Kähler metric g and a complex structure J, such that the group $\mathrm{Aut}(M, J, g)$ of holomorphic isometries acts transitively on M. Hermitian symmetric spaces of compact type are typical examples of such a space. Kähler C-spaces are called also generalized flag manifolds. In this section we recall the construction of Kähler C-spaces. For more details we refer to [B-H], [It] and [Tak].

Let G be a compact and connected semisimple Lie group. We denote by $\mathfrak{g}$ the corresponding Lie algebra and by $\mathfrak{g}^{\mathbf{C}}$ its complexification. We chose a maximal torus T in G, and let $\mathfrak{h}$ be the Lie algebra of T. The complexification $\mathfrak{h}^{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$. We denote by $R \subset (\mathfrak{h}^{\mathbf{C}})^*$ the root system of $\mathfrak{g}^{\mathbf{C}}$ relative to $\mathfrak{h}^{\mathbf{C}}$ and we take the root space decomposition

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}^{\mathbf{C}}_{\alpha} .$$

Here, $\mathfrak{g}^{\mathbf{C}}_{\alpha} = \mathbf{C}E_{\alpha}$ are the 1-dimensional root spaces whose elements E_{α} satisfy the equation

$$\mathrm{ad}(H)E_{\alpha} = \alpha(H)E_{\alpha} , \quad (H \in \mathfrak{h}^{\mathbf{C}}) .$$

Let $\Pi = \{\alpha_1, \dots, \alpha_{\ell}\}$ $(\dim \mathfrak{h}^{\mathbf{C}} = \ell)$ be a fundamental system of R. We fix a lexicographic ordering on $(\mathfrak{h}^{\mathbf{C}})^*$ and we denote by R^+ the set of positive roots. It is well known that for any $\alpha \in R$ we can choose root vectors $E_{\alpha} \in \mathfrak{g}^{\mathbf{C}}_{\alpha}$ such that $B(E_{\alpha}, E_{-\alpha}) = -1$ and $[E_{\alpha}, E_{-\alpha}] = -H_{\alpha}$, where $H_{\alpha} \in \mathfrak{h}^{\mathbf{C}}$ is determined by the equation $B(H, H_{\alpha}) = \alpha(H)$, for all $H \in \mathfrak{h}^{\mathbf{C}}$. By using the last equation we obtain a natural isomorphism between $\mathfrak{h}^{\mathbf{C}}$ and the dual space $(\mathfrak{h}^{\mathbf{C}})^*$. The normalized root vectors E_{α} also satisfy the relation

$$[E_{\alpha}, E_{\beta}] = \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta}, & \text{if } \alpha, \beta, \alpha+\beta \in R \\ 0, & \text{if } \alpha, \beta \in R, \alpha+\beta \notin R \end{cases}$$

where $0 \neq N_{\alpha,\beta} = N_{-\alpha,-\beta} \in \mathbf{R}$ $(\alpha, \beta \in R)$. Then we obtain (cf. [Hel])

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R^+} (\mathbf{R}A_{\alpha} + \mathbf{R}B_{\alpha}) ,$$

where $A_{\alpha} = E_{\alpha} + E_{-\alpha}, B_{\alpha} = \sqrt{-1}(E_{\alpha} - E_{-\alpha}), \alpha \in R^+$. The complex conjugation τ on $\mathfrak{g}^{\mathbf{C}}$ with respect to the compact real form $\mathfrak{g}$ satisfies the relations $\tau(E_{\alpha}) = E_{-\alpha}$ and $\tau(E_{-\alpha}) = E_{\alpha}$.

We assume that G is simple. Let Π_K be a subset of Π and put

$$\Pi_M = \Pi \backslash \Pi_K = \{\alpha_{i_1}, \dots, \alpha_{i_m}\} , \quad (1 \leq i_1 \leq \cdots \leq i_m \leq \ell) .$$

We set

$$R_K = R \cap \langle \Pi_K \rangle , \quad R^+_K = R^+ \cap \langle \Pi_K \rangle , \quad R^+_M = R^+ \backslash R^+_K , \tag{3.3}$$

where by $\langle \Pi_K \rangle$ we denote the set of roots generated by Π_K. Then

$$\mathfrak{p} = \mathfrak{h}^{\mathbf{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha^{\mathbf{C}} \oplus \sum_{\alpha \in R_M^+} \mathfrak{g}_\alpha^{\mathbf{C}}$$

is a parabolic subgroup of $\mathfrak{g}^{\mathbf{C}}$ (cf. [Al1]). The set $\mathfrak{k} = \mathfrak{p} \cap \mathfrak{g}$ is a real subalgebra of $\mathfrak{g}$ determined as follows

$$\mathfrak{k} = \mathfrak{h} \oplus \sum_{\alpha \in R_K^+} (\mathbf{R} A_\alpha + \mathbf{R} B_\alpha) .$$

Let $G^{\mathbf{C}}$ be a simply connected complex simple Lie group whose Lie algebra is $\mathfrak{g}^{\mathbf{C}}$ and P be the parabolic subgroup of $G^{\mathbf{C}}$ generated by $\mathfrak{p}$. We know that the complex homogeneous manifold $G^{\mathbf{C}}/P$ is compact and simply connected, and that G acts transitively on it. Note that $K = G \cap P$ is a connected closed subgroup of G which corresponds to $\mathfrak{k}$. The canonical embedding $G \to G^{\mathbf{C}}$ gives a diffeomorphism of a compact homogeneous space $M = G/K$ to a simply connected complex homogeneous space $G^{\mathbf{C}}/P$, and M admits a G-invariant Kähler metric (cf. [Bo]). Therefore, $M = G/K$ is a Kähler C-space. By a result of A. Borel and F. Hirzebruch [B-H] the second Betti number of M is $\mathrm{b}_2(M) = m = |\Pi_M|$.

We define a linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ as follows:

$$\mathfrak{m} = \sum_{\alpha \in R_M^+} (\mathbf{R} A_\alpha + \mathbf{R} B_\alpha) .$$

Then with respect to the Killing form B we obtain the orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. This is a reductive decomposition of M, i.e. $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. We naturally identify $\mathfrak{m}$ with the tangent space $T_o M$ of M on the identity coset $o = eK$, and the isotropy representation $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ of K with the restriction of the adjoint representation of K on $\mathfrak{m}$, i.e. $\chi(K) = \mathrm{Ad}^K\big|_{\mathfrak{m}}$. We define a complex structure J on $\mathfrak{m}$ by

$$J A_\alpha = B_\alpha , \quad J B_\alpha = -A_\alpha \quad (\alpha \in R_M^+) .$$

This gives a G-invariant complex structure on $M = G/K$ and coincides with the canonical structure induced from the complex homogeneous space $G^{\mathbf{C}}/P$.

Next, we assume that $\Pi_M = \{\alpha_i : l \le i \le l\}$. For $n \in \mathbf{N}$, we set

$$R^+(\alpha_i, n) = \left\{ \alpha = \sum_{i=1}^{\ell} m_j \alpha_j \in R^+ : m_i = n \right\} ,$$

and define an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{m}_n$ of $\mathfrak{g}$ by

$$\mathfrak{m}_n = \sum_{\alpha \in R^+(\alpha_i, n)} (\mathbf{R} A_\alpha + \mathbf{R} B_\alpha) .$$

Then (cf. [Ik3])

$$\mathfrak{m} = \sum_{n\geq 1} \mathfrak{m}_n\,, \quad R_M^+ = \bigcup_{n\geq 1} R^+(\alpha_i, n)\,. \tag{3.4}$$

Note that $\mathfrak{m}_n$ are non-equivalent to each other. We set $\mathfrak{m}_0 = \mathfrak{k}$. Then for $n, m \geq 0$ the following are true:

$$[\mathfrak{k}, \mathfrak{m}_n] \subset \mathfrak{m}_n\,, \quad [\mathfrak{m}_n, \mathfrak{m}_m] \subset \mathfrak{m}_{n+n} + \mathfrak{m}_{|n-m|}\,, \quad [\mathfrak{m}_n, \mathfrak{m}_n] \subset \mathfrak{k} \oplus \mathfrak{m}_{2n}\,.$$

According to [Bo] the metric defined by

$$g\Big(\sum_{n\geq 1} X_n, \sum_{m\geq 1} Y_m\Big) = k\sum_{n\geq 1} n(-B(X_n, Y_n))\,, \quad (X_n \in \mathfrak{m}_n, Y_m \in \mathfrak{m}_m)$$

is a G-invariant Kähler metric on M.

We are interested in Kähler C-spaces with two isotropy summands. By (3.4) it is sufficient to set $R^+(\alpha_i, n) = 0$ for $n \geq 3$. Then $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and the Kähler metric is given by

$$g(X_1 + X_2, Y_1 + Y_2) = -kB(X_1, Y_1) - 2kB(X_2, Y_2)\,, \quad (X_i, Y_i \in \mathfrak{m}_i,\ i = 1, 2)\,.$$

Note that Hermitian symmetric spaces do not belong to this class of spaces since they are irreducible. (cf. [It]).

In the following we denote by $\mathfrak{k}^{\mathbf{C}}$, $\mathfrak{m}^{\mathbf{C}}$ and $\mathfrak{m}_n^{\mathbf{C}}$ the complexifications of $\mathfrak{k}$, $\mathfrak{m}$ and $\mathfrak{m}_n$ respectively. These are complex linear subspaces of $\mathfrak{g}^{\mathbf{C}}$ and we have that

$$\mathfrak{k}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C}} \oplus \sum_{\alpha\in R_K^+} (\mathbf{C}E_\alpha + \mathbf{C}E_{-\alpha})\,, \quad \mathfrak{m}^{\mathbf{C}} = \sum_{\alpha\in R_M^+} (\mathbf{C}E_\alpha + \mathbf{C}E_{-\alpha})\,.$$

We extend the G-invariant complex structure J, without any change of notation, to $\mathfrak{m}^{\mathbf{C}}$ by complex linearity. For the tangent space $\mathfrak{m}^{\mathbf{C}}$ we obtain the direct sum $\mathfrak{m}^{\mathbf{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$, where the subspaces $\mathfrak{m}^+ = \sum_{\alpha\in R_M^+} \mathbf{C}E_\alpha$ and $\mathfrak{m}^- = \sum_{\alpha\in R_M^+} \mathbf{C}E_{-\alpha}$ are the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. This induces the following decomposition

$$\mathfrak{m}_n^{\mathbf{C}} = \mathfrak{m}^{+n} \oplus \mathfrak{m}^{-n}\,,$$

where $\mathfrak{m}^{+n} = \sum_{\alpha\in R^+(\alpha_i,n)} \mathbf{C}E_\alpha$ and $\mathfrak{m}^{-n} = \sum_{\alpha\in R^+(\alpha_i,n)} \mathbf{C}E_{-\alpha} = \tau(\mathfrak{m}^{+n})$. Since $\mathrm{ad}(\mathfrak{k}^{\mathbf{C}})\mathfrak{m}^{\pm n} \subset \mathfrak{m}^{\pm n}$ the representation $(\mathrm{ad}(\mathfrak{k}^{\mathbf{C}}), \mathfrak{m}^{\pm})$ decomposes into a direct sum of irreducible submodules $\mathfrak{m}^{\pm n}$.

Kähler C-spaces can been classified by using the notion of *painted Dynkin diagrams*. We will describe this briefly following [Al2], [A-A] and [B-F-R]. Let $M = G/K$ be a Kähler C-space and Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a simple root system of R, and $\Pi_K = \{\alpha_1, \ldots, \alpha_k\}$ be a basis of the root system R_K. The pair (Π, Π_K) can be represented graphically by a *painted Dynkin diagram.*

DEFINITION 1. Let $\Gamma = \Gamma(\Pi)$ be the Dynkin diagram of the simple root system Π. By painting nodes of Γ corresponding to $\Pi \setminus \Pi_K$ in black, we obtain the *painted Dynkin diagram* of $M = G/K$. In this diagram the system Π_K is determined as the subdiagram of white roots.

The painted Dynkin diagram determines the reductive decomposition $\mathfrak{g}^{\mathbf{C}} = \mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{m}^{\mathbf{C}}$, and hence the space G/K completely. In particular for the complex Lie algebra $\mathfrak{k}^{\mathbf{C}}$ we obtain the decomposition

$$\mathfrak{k}^{\mathbf{C}} = \mathfrak{z} \oplus \mathfrak{k}'^{\mathbf{C}} = \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1) \oplus \mathfrak{k}'^{\mathbf{C}}$$

with $l-k$ copies of $\mathfrak{u}(1)$. Here $\mathfrak{k}'^{\mathbf{C}}$ is the semisimple part of $\mathfrak{k}^{\mathbf{C}}$ which is generated by $\Pi_K = \{\alpha_1, \ldots, \alpha_k\}$, and $\mathfrak{z}$ is the center of $\mathfrak{k}^{\mathbf{C}}$ which is generated by the remaining $l-k$ nodes of $\Gamma(\Pi)$ which have been painted black. Note that if all nodes of $\Gamma(\Pi)$ have been painted black this gives the space G/T, where T is a maximal torus in G.

We recall that two G-manifolds $M = G/K$ and $M' = G/K'$ are called *isomorphic* if there exists an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(K) = K'$.

PROPOSITION 1 [A12]. *Different painted connected Dynkin diagrams Γ and Γ_1 (except for the case of D_ℓ) define equivalent Kähler C spaces G/K and G/K', if the subdiagrams Γ' and Γ_1' of white roots corresponding to Π_K and $\Pi_{K'}$ are isomorphic.*

If G is a simple Lie group, Proposition 1 can be used to list all Kähler C-spaces G/K, (up to isomorphism). For the classical Lie groups there are four families and 101 non-isomorphic spaces corresponding to the exceptional Lie groups (cf. [A-A], [B-F-R]). In the exceptional case we can read all these spaces as follows. We denote by $G(\alpha_1, \ldots, \alpha_k)$ $(k \leq \ell)$ the Kähler C-space $M = G/K$ with $\Pi_K = \{\alpha_1, \ldots, \alpha_k\}$ For example, $F_4(\alpha_2, \alpha_3, \alpha_4)$ corresponds to the Kähler C-space $F_4/SO(7) \times U(1)$.

4. Decomposition of the isotropy representation into irreducible submodules

In this article we treat Kähler C-spaces which satisfy the relation

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \,. \tag{4.5}$$

In order to classify these spaces we use an important invariant of a Kähler C-space, namely the system of $\mathfrak{t}$-roots $R_{\mathfrak{t}}$.

Let $M = G/K$ be a Kähler C-space and let R_K and R_M defined by (3.3). We call the elements of $R_M = R \backslash R_K$ *the complementary roots* of M.

We define the set $\mathfrak{t} = \mathfrak{z} \cap i\mathfrak{h}$, where $\mathfrak{z}$ is the center of $\mathfrak{k}^{\mathbf{C}}$ and $\mathfrak{h}$ is the real ad-diagonal subalgebra $\mathfrak{h} = \mathfrak{h}^{\mathbf{C}} \cap i\mathfrak{k}$. The space $\mathfrak{t}$ is the real form of the center $\mathfrak{z}$, and the complex reductive subalgebra $\mathfrak{k}^{\mathbf{C}}$ can be expressed as

$$\mathfrak{k}^{\mathbf{C}} = \mathfrak{z} \oplus \mathfrak{k}'^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} \oplus \mathfrak{k}'^{\mathbf{C}},$$

where $\mathfrak{k}'^{\mathbf{C}}$ is the semisimple part of $\mathfrak{k}^{\mathbf{C}}$. Let $\mathfrak{h}^*$ and $\mathfrak{t}^*$ be the dual spaces of $\mathfrak{h}$ and $\mathfrak{t}$ respectively. We consider the restriction map

$$\kappa : \mathfrak{h}^* \to \mathfrak{t}^* , \quad \alpha \mapsto \alpha\big|_{\mathfrak{t}}$$

and set $R_{\mathfrak{t}} = \kappa(R_M)$. Note that $\kappa(R_K) = 0$.

DEFINITION 2. The elements of $R_{\mathfrak{t}}$ are called $\mathfrak{t}$-roots.

A fundamental result about $\mathfrak{t}$-roots is that they correspond to irreducible submodules of the complexified isotropy representation of G/K. Next, we will consider the complexification $\mathfrak{m}^{\mathbf{C}}$ as a complex $\mathfrak{k}^{\mathbf{C}}$-module.

PROPOSITION 2 [A-P]. *There exists a* $1-1$ *correspondence between* $\mathfrak{t}$*-roots and irreducible* $\mathrm{ad}(\mathfrak{k}^{\mathbf{C}})$*-invariant submodules* $\mathfrak{m}_\xi$ *of* $\mathfrak{m}^{\mathbf{C}}$. *This correspondence is given by*

$$R_{\mathfrak{t}} \ni \xi \leftrightarrow \mathfrak{m}_\xi = \sum_{\kappa(\alpha)=\xi} \mathbf{C}E_\alpha .$$

Therefore, we obtain the following decomposition

$$\mathfrak{m}^{\mathbf{C}} = \sum_{\xi \in R_{\mathfrak{t}}} \mathfrak{m}_\xi . \tag{4.6}$$

By using Proposition 2 it is possible to obtain a complete description of the isotropy representation of K in the Kähler C-space M. In particular in [Al2], D. Alekseevsky found all irreducible submodules of the isotropy representation of all generalized flag manifolds G/K, where G is a classical simple Lie group.

We mention that the submodules $\mathfrak{m}_\xi$ ($\xi \in R_{\mathfrak{t}}$) are not equivalent. Indeed, if they were equivalent as $\mathfrak{k}^{\mathbf{C}}$-modules, then they would have been equivalent as $\mathfrak{h}^{\mathbf{C}}$-modules, but this is impossible because the roots of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{h}^{\mathbf{C}}$ are distinct and the roots spaces are 1-dimensional.

The complex conjugation τ of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{g}$ interchanges the spaces $\mathfrak{m}_\xi$ and $\mathfrak{m}_{-\xi}$. Due to this, a decomposition of the real $\mathrm{ad}(\mathfrak{k})$-module $\mathfrak{m} = (\mathfrak{m}^{\mathbf{C}})^\tau$ into irreducible submodules is given by

$$\mathfrak{m} = \sum_{\xi \in R_{\mathfrak{t}}^+} (\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau . \tag{4.7}$$

Here $\mathfrak{n}^\tau$ denotes the set of fixed points of τ of a vector subspace $\mathfrak{n} \subset \mathfrak{g}^{\mathbf{C}}$.

A useful way to find the $\mathfrak{t}$-roots of a Kähler C-space $M = G/K$ is as follows. Let Π and Π_K be bases of the root systems R and R_K respectively, given by

$$\Pi = \{\phi_1, \dots, \phi_k, \alpha_1, \dots, \alpha_m\} , \quad \Pi_K = \{\phi_1, \dots, \phi_k\} , \quad (\ell = k + m) . \tag{4.8}$$

We denote by $\Lambda_1, \dots, \Lambda_m$ the fundamental weights corresponding to the simple roots

$$\Pi_M = \Pi / \Pi_K = \{\alpha_1, \dots, \alpha_m\} , \tag{4.9}$$

i.e. the linear forms defined by the equations

$$\frac{2(\Lambda_j, \alpha_i)}{(\alpha_i, \alpha_i)} = \delta_{ij}\,, \quad (\Lambda_j, \phi_i) = 0\,.$$

The fundamental weights Λ_i $(i = 1, \dots, m)$ form a basis of the space $\mathfrak{t}^*$ (isomorphic to $\mathfrak{t}$ via the Killing form). Also, a basis of $\mathfrak{t}^*$ is given by $\{\bar{\alpha}_i = \kappa(\alpha i) : \alpha I \in \Pi M = \Pi/\Pi K\}$. Thus, the $\mathfrak{t}$-root $\kappa(\alpha) = \alpha|\,\mathfrak{t} = \bar{\alpha}$ associated to a complementary root $\alpha \in R_M$ is given by

$$\bar{\alpha} = k_1\bar{\alpha}_1 + \cdots + k_m\bar{\alpha}_m\,. \tag{4.10}$$

If α is a positive (resp. negative) complementary root, then all the coefficients k_i are non negative (resp. non positive).

Recall that any root system R with a basis $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ contains a unique root $\mu = \sum_{i=1}^{\ell} m_i\alpha_i$ such that for any root $\alpha = \sum_{i=1}^{\ell} c_i\alpha_i$ in R we have $c_i \le m_i$ $(i = 1, \dots, \ell)$. The root μ is called the *highest root* in R and the coefficients $m_i \in \mathbf{Z}$ are called *heights* (also known as *Dynkin marks* of simple roots). We define the function

$$\mathrm{Mrk} : \Pi \to \mathbf{N}\,, \quad \alpha_i \mapsto \mathrm{Mrk}(\alpha_i) = m_i\,,$$

which maps each simple root $\alpha_i \in \Pi$ to its height m_i. Next we will use the description for the root systems of the classical and exceptional Lie algebras from the encyclopedia [G-O-V]. The expression of the highest root μ in terms of the simple roots is given in the following table.

TABLE 2. The highest root μ in terms of simple roots.

	Highest root μ
A_ℓ	$\alpha_1 + \cdots + \alpha_\ell$
B_ℓ	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell$
C_ℓ	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$
D_ℓ	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$
G_2	$3\alpha_1 + 2\alpha_2$
F_4	$2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$
E_6	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$
E_7	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$

Let $M = G/K$ be a Kähler C-space and let Π, Π_K and Π_M be defined by (4.8) and (4.9) respectively. By restricting a complementary root $\alpha \in R_M^+$ to $\mathfrak{t}$ by using (4.10) we obtain $0 \le k_i \le m_i$, where $\mathrm{Mrk}(\alpha_i) = m_i$ $(1 \le i \le m)$. We will use this remark and Proposition 2 to obtain the following:

PROPOSITION 3. *Let $M = G/K$ be a Kähler C-space with two isotropy summands. Then the set of positive $\mathfrak{t}$-roots has the form $R_{\mathfrak{t}}^+ = \{\xi, 2\xi\}$, where $0 \ne \xi \in \mathfrak{t}^*$.*

PROOF. By assumption we have $|R_{\mathfrak{t}}^+| = 2$. If $\Pi_M = \{\alpha_i\}$ $(i = 1, \dots, m)$ with $\mathrm{Mrk}(\alpha_i) = m_i$, then $\mathfrak{t}^* = \mathbf{R}\bar{\alpha}_i$ and any positive $\mathfrak{t}$-root has the form $\bar{\alpha} = k_i\bar{\alpha}_i$ with $1 \le k_i \le m_i$. Thus $R_{\mathfrak{t}}^+ = \{k_i\bar{\alpha}_i : 1 \le k_i \le m_i\}$ and $|R_{\mathfrak{t}}^+| = m_i$. It follows that $R_{\mathfrak{t}}^{+} = \{\bar{\alpha}_i, 2\bar{\alpha}_i\}$, where $\mathrm{Mrk}(\alpha_i) = 2$. If $\Pi_M = \{\alpha_i, \alpha_j\}$ with $\mathrm{Mrk}(\alpha_i) = m_i$ and $\mathrm{Mrk}(\alpha_j) = m_j$, then $\mathfrak{t}^* = \mathrm{span}_{\mathbf{R}}\{\bar{\alpha}_i, \bar{\alpha}_j\}$ and by using (4.10) it follows that $\bar{\alpha} = k_i\bar{\alpha_i} + k_j\bar{\alpha_j}$, for any $\alpha \in R_M^{+}$. However $0 \leq k_i \leq m_i$ and $0 \leq k_j \leq m_j$ where k_i, k_j cannot be simultaneously zero. By using Table 2 it should be $|R_{\mathfrak{t}}^{+}| \geq 3$ and this cannot happen by assumption. □

PROPOSITION 4. *Let G be a simple Lie group. Then there is a one-to-one correspondence between painted Dynkin diagrams with $\Pi_M = \{\alpha_i\}$ and* $\mathrm{Mrk}(\alpha_i) = 2$, *and Kähler C-spaces $M = G/K$ with two isotropy summands.*

PROOF. The case of A_ℓ is excluded because we cannot obtain a painted Dynkin diagram satisfying the above hypotheses. For the rest of the simple Lie groups we will examine each case separately. For such a procedure we need to describe the set of complementary roots. Let $\{e_i, \pi_j\}$ be an orthogonal basis of $\mathbf{R}^\ell$, where $i = 1, \ldots, \ell - m$ and $j = 1, \ldots, m$. Then

$$R = \{\pm\mu e_i, \pm e_i \pm e_j : i < j\} \cup \{\pm e_i \pm \pi_j : 1 \leq i \leq \ell - m, 1 \leq j \leq m\} \cup \{\pm\pi_i \pm \pi_j, \pm \mu\pi_j : i < j\},$$

$$R_K = \{\pm(e_i - e_j) : 1 \leq i < j \leq \ell - m\} \cup \{\pm\pi_i \pm \pi_j, \pm\mu\pi_j : 1 \leq i < j \leq m\},$$

where $\mu = 1$ in the case of the $B_\ell = \mathfrak{so}(2\ell + 1, \mathbf{C})$, $\mu = 2$ for $C_\ell = \mathfrak{sp}(\ell, \mathbf{C})$ and μ is absent for $D_\ell = \mathfrak{so}(2\ell, \mathbf{C})$.

CASE OF B_ℓ. Assume that the painted Dynkin diagram satisfies the above hypotheses. The simple root α_i can be any of $\alpha_2, \ldots, \alpha_\ell$, thus $\Pi_M = \{\alpha_i : 2 \leq i \leq \ell\}$. It determines the Kähler C-space $M = G/K = SO(2\ell + 1)/U(i) \times SO(2(\ell - i) + 1)$, $(2 \leq i \leq \ell)$. We choose the set of positive roots

$$R^{+} = \{e_i, e_i \pm e_j : 1 \leq i < j \leq \ell - m\} \cup \{e_i \pm \pi_j : 1 \leq i \leq \ell - m, 1 \leq j \leq m\} \cup \{\pi_i \pm \pi_j, \pi_j : 1 \leq i < j \leq m\}.$$

and $R_K^{+} = \{e_i - e_j, \pi_i \pm \pi_j, \pi_j : i < j\}$. Then the positive complementary roots are given by

$$R_M^{+} = R^{+} \backslash R_K^{+} = \{e_i, e_i + e_j : i < j\} \cup \{e_i \pm \pi_j : 1 \leq i \leq \ell - m, 1 \leq j \leq m\}.$$

We have $\mathfrak{t} = \mathbf{R}\bar{\alpha}_i$ $(1 < i \leq \ell)$ and $\kappa(\alpha) = k_i\bar{\alpha}_i$ for any $\alpha \in R_M^{+}$ with $1 \leq k_i \leq 2$. In particular, recalling that $\kappa(R_K) = 0$ we obtain $e_i\big|_{\mathfrak{t}} = (e_i \pm \pi_j)\big|_{\mathfrak{t}} = \bar{\alpha}_i$ and $(e_i + e_j)\big|_{\mathfrak{t}} = 2\bar{\alpha}_i$. Thus $R_{\mathfrak{t}}^{+} = \{\bar{\alpha}_i, 2\bar{\alpha}_i : 2 \leq i \leq \ell\}$. By using relation (4.6) we obtain the direct sum decomposition

$$\mathfrak{m}^{\mathbf{C}} = \mathfrak{m}_{-2\bar{\alpha}_i} \oplus \mathfrak{m}_{-\bar{\alpha}_i} \oplus \mathfrak{m}_{\bar{\alpha}_i} \oplus \mathfrak{m}_{2\bar{\alpha}_i}.$$

According to (4.7) this induces the real decomposition (4.5), where the irreducible submodule $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are such that

$$\mathfrak{m}_1^{\mathbf{C}} = \mathfrak{m}_{-\bar{\alpha}_i} \oplus \mathfrak{m}_{\bar{\alpha}_i}, \quad \mathfrak{m}_2^{\mathbf{C}} = \mathfrak{m}_{-2\bar{\alpha}_i} \oplus \mathfrak{m}_{2\bar{\alpha}_i}.$$

CASE OF C_ℓ. The simple root α_i can be any of $\alpha_1, \dots, \alpha_{\ell-1}$. This determines the flag manifold $M = G/K = Sp(\ell)/U(i) \times Sp(\ell - i)$. We choose

$$R^+ = \{2e_i, e_i \pm e_j : i < j\} \cup \{e_i \pm \pi_j : 1 \le i \le \ell - m, 1 \le j \le m\} \cup \{\pi_i \pm \pi_j, 2\pi_j : i < j\},$$

and let $R_K^+ = \{e_i - e_j, \pi_i \pm \pi_j, 2\pi_j : i < j\}$. We conclude that

$$R_M^+ = \{2e_i, e_i + e_j : 1 \le i < j \le \ell - m\} \cup \{e_i \pm \pi_j : 1 \le i \le \ell - m, 1 \le j \le m\}.$$

We have $\mathfrak{t} = \mathbf{R}\bar{\alpha}_i$ $(1 \le i \le \ell - 1)$ and $\kappa(\alpha) = k_i\bar{\alpha}_i$ where $1 \le k_i \le 2$. We obtain $e_i \pm \pi_j\big|_{\mathfrak{t}} = \bar{\alpha}_i$ and $(e_i + e_j)\big|_{\mathfrak{t}} = 2e_i\big|_{\mathfrak{t}} = 2\bar{\alpha}_i$. Thus $R_{\mathfrak{t}}^+ = \{\bar{\alpha}_i, 2\bar{\alpha}_i : 1 \le i \le \ell - 1\}$. By Proposition 2 and by a similar analysis in the case of B_ℓ we obtain (4.5).

CASE OF D_ℓ. The possible root $\alpha_i \in \Pi$ with $\mathrm{Mrk}(\alpha_i) = 2$ is one of $\alpha_2, \dots, \alpha_{\ell-2}$. This choice determines the flag manifold $M = G/K = SO(2\ell)/U(i) \times SO(2(\ell - i))$. We set

$$R^+ = \{e_i \pm e_j : 1 \le i < j \le \ell - m\} \cup \{e_i \pm \pi_j : 1 \le i \le \ell - m, 1 \le j \le m\}$$
$$\cup \{\pi_i \pm \pi_j : 1 \le i < j \le m\},$$

and let $R_K^+ = \{e_i - e_j, \pi_i \pm \pi_j : i < j\}$. Then

$$R_M^+ = \{e_i + e_j : 1 \le i < j \le \ell - m\} \cup \{e_i \pm \pi_j : 1 \le i \le \ell - m, 1 \le j \le m\}.$$

We have $\mathfrak{t} = \mathbf{R}\bar{\alpha}_i$ $(2 \le i \le \ell - 2)$. For any $\alpha \in R_M^+$ it holds $\kappa(\alpha) = k_i\bar{\alpha}_i$ with $1 \le k_i \le 2$. In particular we obtain $(e_i \pm \pi_j)\big|_{\mathfrak{t}} = \bar{\alpha}_i$ and $(e_i + e_j)\big|_{\mathfrak{t}} = 2\bar{\alpha}_i$. Thus $R_{\mathfrak{t}}^+ = \{\bar{\alpha}_i, 2\bar{\alpha}_i : 2 \le i \le \ell - 2\}$. By a similar analysis with B_ℓ we conclude (4.5).

Next we proceed to the exceptional groups. We present only the case of the groups G_2 and F_4 and for the other exceptional groups we refer to [A-A].

CASE OF G_2. We fix an ordering so that the set of positive roots is

$$R^+ = \{-e_2, -e_3, e_2 - e_3, -(e_2 + e_3), -(e_2 + 2e_3), -(2e_2 + e_3)\}.$$

A basis of R is $\Pi = \{\alpha_1 = -e_2, \alpha_2 = e_2 - e_3\}$ and according Table 2 the maximal root is $\mu = -(e_2 + 2e_3) = 3\alpha_1 + 2\alpha_2$. The expression of any positive root in terms of simple roots is given by

$$-(2e_2 + e_3) = 3\alpha_1 + \alpha_2,\ -(e_2 + 2e_3) = 3\alpha_1 + 2\alpha_2,\ e_2 - e_3 = \alpha_2,$$
$$-(e_2 + e_3) = 2\alpha_1 + \alpha_2,\ -e_2 = \alpha_1,\ -e_3 = \alpha_1 + \alpha_2.$$

The root system of G_2 contains long and short roots whose relation is given by the Dynkin diagram of G_2. In particular the roots of G_2 are expressed by using vectors e_i, $(1 \le i \le 3)$ such that

$$\sum_{i=1}^{3} e_i = 0, \quad (e_i, e_j) = \begin{cases} \dfrac{2}{3}, & i = j, \\ -\dfrac{1}{3}, & i \ne j. \end{cases} \tag{4.11}$$

By using (4.11) we find that $(\alpha_1, \alpha_1) = \frac{2}{3}$ while $(\alpha_2, \alpha_2) = 2$, so $(\alpha_2, \alpha_2) = 3(\alpha_1, \alpha_1)$. The Kähler C-space with two isotropy summands is determined by $G_2(\alpha_1)$ and corresponds to the space $G_2/U(2)$ where $U(2)$ represented by the short root α_1. We obtain $\Pi_M = \{\alpha_2\}$ with $\mathrm{Mrk}(\alpha_2) = 2$ and $\mathfrak{t} = \mathbf{R}\bar{\alpha}_2$. Any positive $\mathfrak{t}$-root is expressed by $\kappa(\alpha) = k_2\bar{\alpha}_2$ $(\alpha \in R_M^+)$ with $1 \le k_2 \le 2$. Thus, $R_{\mathfrak{t}}^+ = \{\bar{\alpha}_2, 2\bar{\alpha}_2\}$ and by Proposition 2, the isotropy representation of M decomposes into a direct sum of two real irreducible submodules $\mathfrak{m}_1, \mathfrak{m}_2$.

CASE OF F_4. We fix the set of positive roots to be

$$R^+ = \left\{ e_1, e_2, e_3, e_4, e_i \pm e_j \ (i < j), \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

A system of simple roots is the set $\Pi = \{\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4), \alpha_2 = e_4, \alpha_3 = e_3 - e_4, \alpha_4 = e_2 - e_3\}$. By Table 2 the maximal root is given by $\mu = e_1 + e_2 = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$. Any positive root $\alpha \in R$ can be expressed as $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4$, with $|k_1| \le 2$, $|k_2| \le 4$, $|k_3| \le 3$, $|k_4| \le 2$ (cf. [A-A]).

(1) $F_4(\alpha_2, \alpha_3, \alpha_4)$: This determines the Kähler C-space $F_4/SO(7) \times U(1)$. We have $\Pi_M = \{\alpha_1\}$ and $\mathfrak{t} = \mathbf{R}\bar{\alpha}_1$. Any positive $\mathfrak{t}$-root is given by $\kappa(\alpha) = k_1\bar{\alpha}_1$ $(\alpha \in R_M^+)$ where $1 \le k_1 \le 2$. Therefore $R_{\mathfrak{t}}^+ = \{\bar{\alpha}_1, 2\bar{\alpha}_1\}$ and by Proposition 2 we conclude that $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

(2) $F_4(\alpha_1, \alpha_2, \alpha_3)$: This corresponds to the Kähler C-space $F_4/Sp(3) \times U(1)$. We have $\Pi_M = \{\alpha_4\}$ and $\mathfrak{t} = \mathbf{R}\bar{\alpha}_4$. Any positive $\mathfrak{t}$-root can be expressed as $\kappa(\alpha) = k_4\bar{\alpha}_4$ with $1 \le k_4 \le 2$. I Thus $R_{\mathfrak{t}}^+ = \{\bar{\alpha}_4, 2\bar{\alpha}_4\}$, or equivalently $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. By similar arguments applying to the rest of the exceptional Lie groups, we conclude that the assumption of $\Pi_M = \{\alpha_i\}$ with $\mathrm{Mrk}(\alpha_i) = 2$ implies that the corresponding Kähler C-space has two positive roots or equivalently two isotropy summands. These spaces appear in Table 2 and are the following:

$$E_8(\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8),\ E_8(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8),\ E_7(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6),$$
$$E_7(\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7),\ E_7(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7),\ E_6(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5),$$
$$E_6(\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6).$$

For the converse, we need to use some of the results of [Al-Me-To]. Let $M = G/K$ be a Kähler C-space which satisfies (4.5). By Proposition 3 we have $R_{\mathfrak{t}} = \{\pm\xi, \pm 2\xi\}$ and this defines a depth-two gradation

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

of the complex Lie algebra $\mathfrak{g}^{\mathbf{C}}$, given by $\mathfrak{g}_0 = \mathfrak{k}^{\mathbf{C}}$ and $\mathfrak{g}_i = \mathfrak{m}_{i\xi}$, $(i = \pm 1, \pm 2)$. According to Proposition 2, the $\mathfrak{k}^{\mathbf{C}}$-modules $\mathfrak{g}_i = \mathfrak{m}_{i\xi}$ are irreducible.

In [Al-Me-To] it was shown that such gradations with irreducible $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ correspond to a subset $\Pi_M = \{\alpha_i\}$ of a simple root system Π of R with $\mathrm{Mrk}(\alpha_i) = 2$. This completes the proof. $\square$

As a consequence we obtain Theorem A stated in the introduction.

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