TOKYO J. MATH.
VOL. 32, NO. 2, 2009

Three-Interval Exchange Transformation and its Odometer Representation

Yuko ICHIKAWA and Shunji ITO

Tokyo National College of Technology and Kanazawa University

(Communicated by H. Nakada)

Abstract. It is well known that 2-interval exchange transformation is isomorphic to the odometer transformation on some symbolic space and isomorphism is given by Ostrowski representation formula. The purpose of this paper is to extend these results to the case of 3-interval exchange transformation.

1. Introduction

For each irrational number $\alpha,\ 0 < \alpha < 1$, let us introduce 2-interval exchange transformation R_α of $I_\alpha = (-\alpha, 1]$ by

$$R_\alpha(x) = \begin{cases} x + \alpha & \text{if } -\alpha < x < 1 - \alpha\,, \\ x - 1 & \text{if } 1 - \alpha < x \le 1\,. \end{cases}$$

For the transformation R_α, it is a well-known fact that there exist a symbolic space Ω_α, a symbolic dynamics Od called odometer transformation and the isomorphic map $\Phi : \Omega_\alpha \to I_\alpha$ which connects Od and R_α as explained in Theorem 1 below.

The symbolic space Ω_α and the odometer transformation Od can be obtained by the following procedure. First, let us introduce the continued fraction transformation $T : [0, 1) \to [0, 1)$ by

$$T(\alpha) = \frac{1}{\alpha} - a(\alpha)\,,$$

where $a(\alpha) = \lfloor \frac{1}{\alpha} \rfloor$, and put $\alpha_0 = \alpha,\ \alpha_n = T(\alpha_{n-1})$, and $a_n = a(\alpha_{n-1})$. Using the sequence of digits $(a_1, a_2, \dots)$ continued fraction expansion of α, we secondly obtain the following expansion of $\beta \in (-\alpha, 1]$ called Ostrowski type expansion ([3], [2], [6]):

$$\beta = \sum_{N=1}^{\infty} (-1)^{N-1} \alpha\alpha_1 \cdots \alpha_{N-1} b_N\,,$$

Received July 17, 2008

where digits $(b_1, b_2, \dots)$ satisfy

$$b_n \leq a_n\,, \quad \text{if } b_n = a_n \text{ then } b_{n-1} = 0\,,$$

which is obtained by the skew product algorithm of continued fraction transformation. (See [2],[3] for detail). Using digits $(b_1, b_2, \dots)$, let us define a symbolic space Ω_α by

$$\Omega_\alpha = \{\mathbf{b} = (\cdots b_2 b_1) \mid b_n \leq a_n\,, \quad \text{if } b_n = a_n \ \text{ then } \ b_{n-1} = 0\}\ .$$

Finally by using a partial order $\prec$ on Ω_α introduced below, the odometer transformation Od is defined as follows: for $\mathbf{b} = (\cdots b_n \cdots b_1)$, $\mathbf{b}' = (\cdots b'_n \cdots b'_1) \in \Omega_\alpha$, we say that $\mathbf{b} \prec \mathbf{b}'$ if there exists n_0 such that

$$b_n = b'_n\,, \quad \text{if} \quad n > n_0 \quad \text{and} \quad b_{n_0} < b'_{n_0}\,.$$

Then, define

$$\mathrm{Od}(\mathbf{b}) = \min\{\mathbf{b}' \mid \mathbf{b} \prec \mathbf{b}', \mathbf{b}' \in \Omega_\alpha\}\,.$$

Then we have the following symbolic odometer representation of the 2-interval exchange transformation R_α.

THEOREM 1. *The following diagram commutes*:

$$\begin{tikzcd}
\Omega_\alpha \arrow[r, "\mathrm{Od}"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega_\alpha \arrow[d, "\Phi"] \\
I_\alpha \arrow[r, "R_\alpha"'] & I_\alpha
\end{tikzcd}$$

where $\Phi(\mathbf{b}) = \sum_{N=1}^{\infty} (-1)^{N-1} \alpha\alpha_1 \cdots \alpha_{N-1} b_N$.

The purpose of the paper is to study the symbolic odometer representation of a 3-interval exchange transformation $R_{\alpha,\beta}$ which is shown in Figure 1. The precise definition is given in Section 2.

To study the odometer representation of a 3-interval exchange transformation, we need to introduce some algorithm for induction, related to a 3-interval exchange transformation,

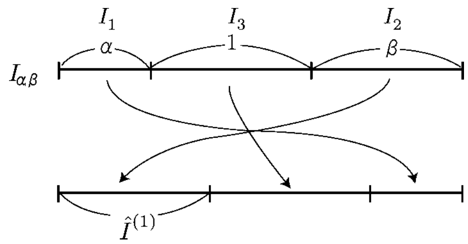


FIGURE 1. Figure of the 3-interval exchange transformation $R_{\alpha,\beta}$.

instead of the continued fraction algorithm related to the 2-interval exchange transformation R_α. By virtue of NIR-algorithm (see [4]), we obtain the symbolic space $\Omega_{\alpha,\beta}$ and the Ostrowski type expansion of $x \in I_{\alpha,\beta}$ are obtained in Section 5 and 6. And using the expansion and symbolic space we arrive at the main theorem in Section 6.

2. Three interval exchange transformations and their induced transformations

Let us introduce the following 3-interval exchange transformation (See Figure 1). For real numbers α, β $(0 < \alpha < \beta)$, let us consider the interval $I_{\alpha,\beta} = [x_{\alpha,\beta}, x_{\alpha,\beta} + \alpha + 1 + \beta]$ and partition $\{I_i \,|\, i = 1, 2, 3\}$ of $I_{\alpha,\beta}$ given by $I_1 = [x_{\alpha,\beta}, x_{\alpha,\beta} + \alpha]$, $I_3 = (x_{\alpha,\beta} + \alpha, x_{\alpha,\beta} + \alpha + 1), I_2 = [x_{\alpha,\beta} + \alpha + 1, x_{\alpha,\beta} + \alpha + 1 + \beta]$ and let us define the 3-interval exchange transformation $R_{\alpha,\beta}$ by

$$R_{\alpha,\beta}(x) := \begin{cases} x + 1 + \beta & x \in I_1\,, \\ x + \beta - \alpha & x \in I_3\,, \\ x - (1 + \alpha) & x \in I_2\,, \end{cases} \tag{1}$$

where the endpoint $x_{\alpha,\beta}$ of $I_{\alpha,\beta}$ is given explicitly later (the equation (6) in Section 3).

Now let us consider the induced transformation $R^{(1)}_{\alpha,\beta}$ of $R_{\alpha,\beta}$ induced on $\hat{I}^{(1)} := [x_{\alpha,\beta}, x_{\alpha,\beta} + \beta]$ which is the image of I_2 by $R_{\alpha,\beta}$. If $\frac{1}{\beta-\alpha} \notin \mathbf{N}$, then $R^{(1)}_{\alpha,\beta}$ is a 3-interval exchange transformation again. We name the middle interval $\hat{I}^{(1)}_3$ and shorter (longer) of the

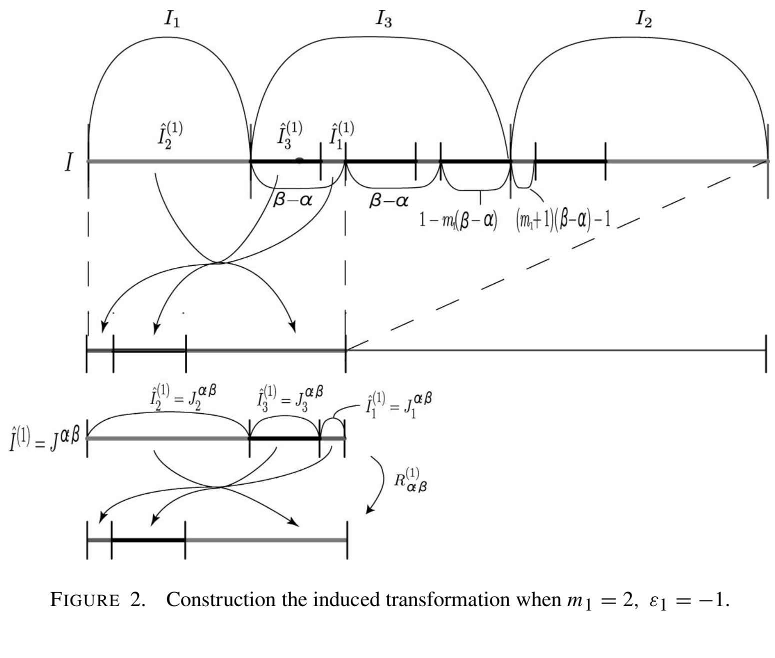


FIGURE 2. Construction the induced transformation when $m_1 = 2,\ \varepsilon_1 = -1$.

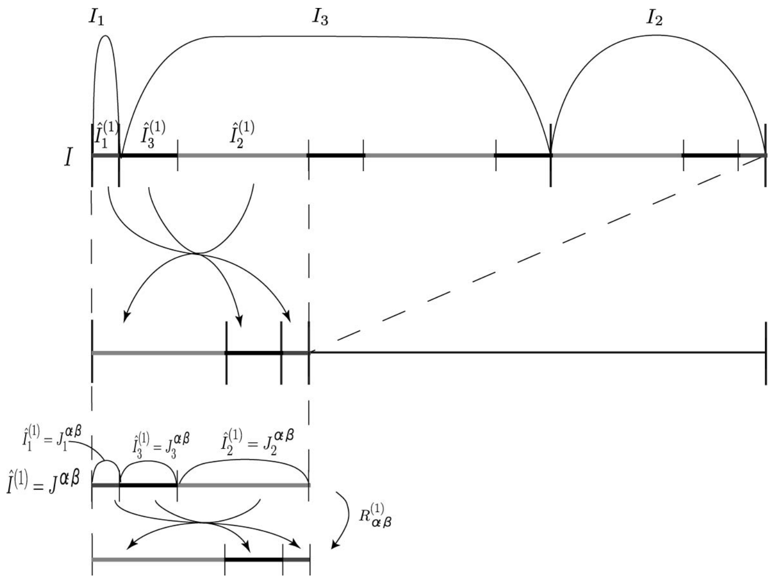


FIGURE 3. Construction the induced transformation when $m_1 = 2,\ \varepsilon_1 = 1$.

other two intervals $\hat{I}_1^{(1)}(\hat{I}_2^{(1)})$, respectively, (see Figure 2 and 3), and whose length are denoted by

$$|\hat{I}_1^{(1)}| = \hat{\alpha}_1,\ |\hat{I}_2^{(1)}| = \hat{\beta}_1,\ |\hat{I}_3^{(1)}| = \hat{\lambda}_1 .$$

These are concretely given as follows.

$$\begin{aligned}
\hat{\lambda}_1 \quad &= 1 - m_1(\beta - \alpha) \quad = |\hat{I}_3^{(1)}| , \\
\hat{\alpha}_1 &= \min\{\alpha, \beta - \alpha - \hat{\lambda}_1\} = |\hat{I}_1^{(1)}| , \\
\hat{\beta}_1 &= \max\{\alpha, \beta - \alpha - \hat{\lambda}_1\} = |\hat{I}_2^{(1)}| ,
\end{aligned}$$

where $m_1 = \left\lfloor \frac{1}{\beta-\alpha} \right\rfloor$, and define

$$\varepsilon_1 = \begin{cases} 1 & \text{if } \hat{\alpha}_1 = \alpha , \\ -1 & \text{if } \hat{\beta}_1 = \alpha . \end{cases}$$

In Section 4, as we need another notation of these intervals, here we prepare the notations $J^{\alpha,\beta} = \hat{I}^{(1)}, J_i^{\alpha,\beta} = \hat{I}_i^{(1)},\ i = 1, 2, 3$. For convenience, we set $\hat{\lambda}_0 = \varepsilon_0 = 1$. If $\hat{\lambda}_1 \neq 0$, that is equivalent to $\frac{1}{\beta-\alpha} \notin \mathbf{N}$, and $\hat{\alpha}_1 \neq \hat{\beta}_1$, then we can repeat this process to construct the induced transformation $R^{(2)}_{\alpha,\beta}$ of $R^{(1)}_{\alpha,\beta}$ on $\hat{I}^{(2)} := R^{(1)}_{\alpha,\beta}(\hat{I}_2^{(1)})$. In general, let us assume that the 3-interval exchange transformation $R^{(k-1)}_{\alpha,\beta}$ on $I^{(k-1)}$ has been given, then we obtain the 3-interval exchange transformation $R^{(k)}_{\alpha,\beta}$ which is induced from $R^{(k-1)}_{\alpha,\beta}$ on $\hat{I}^{(k)} := R^{(k-1)}_{\alpha,\beta}(\hat{I}^{(k-1)}_2)$. Intervals associated with $R^{(k)}_{\alpha,\beta}$ are denoted by $\{\hat{I}^{(k)}_i \mid i = 1, 2, 3\}$ and their length $|\hat{I}^{(k)}_1| = \hat{\alpha}_k,\ |\hat{I}^{(k)}_2| = \hat{\beta}_k,\ |\hat{I}^{(k)}_3| = \hat{\lambda}_k$ are given by

$$\begin{aligned}
\hat{\lambda}_k &= \hat{\lambda}_{k-1} - m_k(\hat{\beta}_{k-1} - \hat{\alpha}_{k-1})\,,\\
\hat{\alpha}_k &= \min\{\hat{\alpha}_{k-1}, \hat{\beta}_{k-1} - \hat{\alpha}_{k-1} - \hat{\lambda}_k\}\,,\\
\hat{\beta}_k &= \max\{\hat{\alpha}_{k-1}, \hat{\beta}_{k-1} - \hat{\alpha}_{k-1} - \hat{\lambda}_k\}\,,
\end{aligned} \tag{2}$$

where $m_k = \left\lfloor \frac{\hat{\lambda}_{k-1}}{\hat{\beta}_{k-1} - \hat{\alpha}_{k-1}} \right\rfloor$ and put

$$\varepsilon_k = \begin{cases} 1 & \text{if } \hat{\alpha}_k = \hat{\alpha}_{k-1}\,, \\ -1 & \text{if } \hat{\beta}_k = \hat{\alpha}_{k-1}\,. \end{cases}$$

Moreover, we see that $\hat{I}^{(k)}_1$ and $\hat{I}^{(k)}_2$ are closed intervals and $\hat{I}^{(k)}_3$ is an open interval for all k. Then, we have the following formulae.

PROPOSITION 1. *Define matrices* $A_{\left[\begin{smallmatrix} m \\ \varepsilon \end{smallmatrix}\right]}$ *as*

$$A_{\left[\begin{smallmatrix} m \\ 1 \end{smallmatrix}\right]} := \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & m & m+1 \end{pmatrix}, \quad A_{\left[\begin{smallmatrix} m \\ -1 \end{smallmatrix}\right]} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ m & 0 & m+1 \end{pmatrix},$$

then

$$\begin{bmatrix} \hat{\alpha}_{k-1} \\ \hat{\beta}_{k-1} \\ \hat{\lambda}_{k-1} \end{bmatrix} = A_{\left[\begin{smallmatrix} m_k \\ \varepsilon_k \end{smallmatrix}\right]} \begin{bmatrix} \hat{\alpha}_k \\ \hat{\beta}_k \\ \hat{\lambda}_k \end{bmatrix}. \tag{3}$$

PROOF. By the equation (2) we know that

$$\begin{aligned}
\hat{\alpha}_k &= \min\{\hat{\alpha}_{k-1}, (m_k+1)(\hat{\beta}_{k-1} - \hat{\alpha}_{k-1}) - \hat{\lambda}_{k-1}\}\,,\\
\hat{\beta}_k &= \max\{\hat{\alpha}_{k-1}, (m_k+1)(\hat{\beta}_{k-1} - \hat{\alpha}_{k-1}) - \hat{\lambda}_{k-1}\}\,,
\end{aligned}$$

then the the formulae (2) can be written as follows: if $\varepsilon_k = 1$ then

$$\begin{bmatrix} \hat{\alpha}_k \\ \hat{\beta}_k \\ \hat{\lambda}_k \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -(1+m_k) & 1+m_k & -1 \\ m_k & -m_k & 1 \end{pmatrix} \begin{bmatrix} \hat{\alpha}_{k-1} \\ \hat{\beta}_{k-1} \\ \hat{\lambda}_{k-1} \end{bmatrix} = A^{-1}_{\left[\begin{smallmatrix} m_k \\ 1 \end{smallmatrix}\right]} \begin{bmatrix} \hat{\alpha}_{k-1} \\ \hat{\beta}_{k-1} \\ \hat{\lambda}_{k-1} \end{bmatrix},$$

and if $\varepsilon_k = -1$ then

$$\begin{bmatrix} \hat{\alpha}_k \\ \hat{\beta}_k \\ \hat{\lambda}_k \end{bmatrix} = \begin{pmatrix} -(1+m_k) & 1+m_k & -1 \\ 1 & 0 & 0 \\ m_k & -m_k & 1 \end{pmatrix} \begin{bmatrix} \hat{\alpha}_{k-1} \\ \hat{\beta}_{k-1} \\ \hat{\lambda}_{k-1} \end{bmatrix} = A^{-1}_{\left[\begin{smallmatrix} m_k \\ -1 \end{smallmatrix}\right]} \begin{bmatrix} \hat{\alpha}_{k-1} \\ \hat{\beta}_{k-1} \\ \hat{\lambda}_{k-1} \end{bmatrix}.$$

Therefore, we obtain the conclusion. □

If $\hat{\lambda}_k \neq 0$ and $\hat{\alpha}_k \neq \hat{\beta}_k$ for any $k \in \mathbf{N}$, we obtain an infinite sequence

$$\begin{pmatrix} m_1 & m_2 & \cdots & m_k & \cdots \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_k & \cdots \end{pmatrix}.$$

We call it the digits of (α, β) from the induction of 3-interval exchange transformation $R_{\alpha,\beta}$.

PROPOSITION 2. *Let us assume that α, β $(0 < \alpha < \beta)$ produce the digits $\begin{pmatrix} m_1 & \cdots & m_k & \cdots \\ \varepsilon_1 & \cdots & \varepsilon_k & \cdots \end{pmatrix}$, that is, let us assume that $\hat{\lambda}_k \neq 0$, $\hat{\alpha}_k \neq \hat{\beta}_k$ for all $k \in \mathbf{N}$. Then we have*

(1) $\hat{\alpha}_n, \hat{\beta}_n, \hat{\lambda}_n \to 0$ *as* $n \to \infty$.
(2) *For infinitely many* $k \in \mathbf{N}$, $m_k \neq 0$.
(3) *For infinitely many* $k \in \mathbf{N}$, $\varepsilon_k = -1$.

PROOF. From the definition $\{\hat{\alpha}_n\}$,$\{\hat{\beta}_n\}$,$\{\hat{\lambda}_n\}$ are positive and not increasing. Suppose that there exists $C > 0$ such that $\hat{\alpha}_k + \hat{\lambda}_k > C$ for all $k \in \mathbf{N}$. Then by the equation (3) we know that

$$\hat{\beta}_k = \hat{\alpha}_{k+1} + \hat{\lambda}_{k+1} + \hat{\beta}_{k+1} > C + \hat{\beta}_{k+1} > nC + \hat{\beta}_{k+n}.$$

Thus we have $\hat{\beta}_k > nC$ for any $n \in \mathbf{N}$. This contradicts the fact that $\beta(\geq \hat{\beta}_k)$ is bounded. Therefore we have $\hat{\alpha}_k + \hat{\lambda}_k \searrow 0$, and then $\hat{\alpha}_k \searrow 0$ and $\hat{\lambda}_k \searrow 0$.

Suppose that $\varepsilon_k = 1$ for all $k \geq k_0$. Then $\hat{\alpha}_{k_0} = \hat{\alpha}_{k_0+1} = \cdots = \hat{\alpha}_k$. On the other hand

$$\begin{aligned} \hat{\beta}_{k_0+n} &= \hat{\beta}_{k_0+n-1} - \hat{\alpha}_{k_0+n} - \hat{\lambda}_{k_0+n} \\ &< \hat{\beta}_{k_0+n-1} - \hat{\alpha}_{k_0+n} = \hat{\beta}_{k_0+n-1} - \hat{\alpha}_{k_0} \\ &< \hat{\beta}_{k_0+n-2} - 2\hat{\alpha}_{k_0} < \cdots < \beta_{k_0} - n\hat{\alpha}_{k_0} < \beta - n\hat{\alpha}_{k_0}, \end{aligned}$$

for all $n \geq k_0$. This contradicts $\hat{\beta}_n > 0$. Therefore statement (3) holds.

Suppose that $m_k = 0$ for all $k \geq k_0$. Then from the definition we get $\hat{\lambda}_k = \hat{\lambda}_{k-1} - m_k(\hat{\beta}_k - \hat{\alpha}_k) = \hat{\lambda}_{k-1}$, this contradicts the fact that $\hat{\lambda}_k \searrow 0$. Therefore the statement (2) holds.

Suppose that there exists $k_0 \in \mathbf{N}$ and $C > 0$ such that $\hat{\beta}_k > C > 0$ for all $k \geq k_0$. Then we see that there exists $k_1 \in \mathbf{N}$ such that $\hat{\beta}_k > C > \hat{\alpha}_k + \hat{\lambda}_k$ for all $k \geq k_1$ and then $\hat{\lambda}_k < \hat{\beta}_k - \hat{\alpha}_k$. This means that $m_k = 0$ for all $k > k_1$. We get the desired contradiction to the statement (2). Therefore, we have $\hat{\beta}_k \searrow 0$ and the the statement (1) holds. □

3. Renormalization of induced transformation and NIR-algorithm

In Section 2, we study the induced transformation $R^{(1)}_{\alpha,\beta}$ of $R_{\alpha,\beta}$ on $\hat{I}^{(1)} = R_{\alpha,\beta}(I_2)$ and $R^{(1)}_{\alpha,\beta}$ is given explicitly by Figure 2 and 3. Let us recall the length of interval $\hat{I}^{(1)}_i$ is given by

$$\begin{aligned}
|\hat{I}^{(1)}_3| &= \hat{\lambda}_1 = 1 - m_1(\beta-\alpha)\,,\\
|\hat{I}^{(1)}_1| &= \hat{\alpha}_1 = \min\{\alpha, \beta-\alpha-\hat{\lambda}_1\} = \min\{\alpha, (m_1+1)(\beta-\alpha)-1\}\,,\\
|\hat{I}^{(1)}_2| &= \hat{\beta}_1 = \max\{\alpha, \beta-\alpha-\hat{\lambda}_1\} = \max\{\alpha, (m_1+1)(\beta-\alpha)-1\}\,.
\end{aligned}$$

Now let us consider the renormalization of 3-interval exchange transformation by linear isomorphism θ_1 to keep the length of $\theta_1(I^{(1)}_3)$ equals 1. Namely, let us consider the following map $T : X \to X$, $T(\alpha,\beta) = (\alpha_1, \beta_1)$, where $X := \{(\alpha,\beta) |\, \alpha < \beta,\ \alpha,\beta > 0\}$ and

$$\alpha_1 = \frac{\hat{\alpha}_1}{\hat{\lambda}_1}\,, \quad \beta_1 = \frac{\hat{\beta}_1}{\hat{\lambda}_1}\,. \tag{4}$$

Define cylinder sets $D\begin{pmatrix} p\\ q\end{pmatrix} = \{(\alpha,\beta) \in X \big| m_1(\alpha,\beta) = p, \varepsilon_1(\alpha,\beta) = q\}$, then $\left\{D\begin{pmatrix} p\\ q\end{pmatrix}\middle| p = 0,1,2\ldots, q = \pm 1\right\}$ is a partition of X and we see that

$$\begin{aligned}
D\begin{pmatrix} m\\ 1\end{pmatrix} &= \left\{(\alpha,\beta)\in X \,\middle|\, \alpha + \frac{1}{m+1} < \beta < \alpha + \frac{1}{m}, \beta > \frac{m+2}{m+1}\alpha + \frac{1}{m+1}\right\}\,,\\
D\begin{pmatrix} m\\ -1\end{pmatrix} &= \left\{(\alpha,\beta)\in X \,\middle|\, \alpha + \frac{1}{m+1} < \beta < \alpha + \frac{1}{m}, \beta < \frac{m+2}{m+1}\alpha + \frac{1}{m+1}\right\}\,,
\end{aligned}$$

(See Figure 4.) We can see that the restriction of T on $D\begin{pmatrix} m\\ \varepsilon\end{pmatrix}$ is an one-to-one onto map to X. Moreover we have the following theorem.

THEOREM 2. *Let $R^{(1)}_{\alpha,\beta}$ be the induced transformation of $R_{\alpha,\beta}$ on $\hat{I}^{(1)}$, and let R_{α_1,β_1} be the* 3*-interval exchange transformation on $I_{\alpha_1,\beta_1} = [x_{\alpha_1,\beta_1}, x_{\alpha_1,\beta_1} + \alpha_1 + 1 + \beta_1]$ given by the equation* (1)*. Then $R^{(1)}_{\alpha,\beta}$ is isomorphic to R_{α_1,β_1} with the isomorphism $\theta_1 : \hat{I}^{(1)} \to I_{\alpha_1,\beta_1}$ given by*

$$\theta_1(x) = \frac{\varepsilon_1}{1 - m_1(\beta-\alpha)}x = \frac{\varepsilon_1}{\hat{\lambda}_1}x\,.$$

The proof is obtained by the following Lemma 1 and Corollary 2. The transformation T is called N-I-R algorithm in the paper [4]. Before giving the proofs of the Lemma 1 and Corollary 2, we prepare a few notations. Let us consider the set X_0 given by

$$X_0 := \{(\alpha,\beta) \in X \mid T^n(\alpha,\beta) \text{ belongs to the interior of } X,\ n = 0,1,2,\ldots\}$$

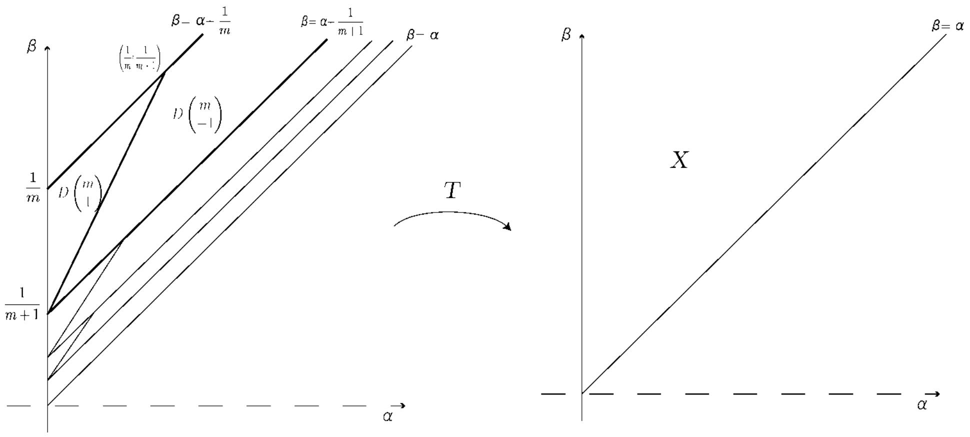


FIGURE 4. Cylinder sets $D\begin{pmatrix} m \\ \varepsilon \end{pmatrix}$ of X.

then for $(\alpha, \beta) \in X_0$ we can define $T^n(\alpha, \beta)$ for any $n \in \mathbf{N}$. Take $(\alpha, \beta) \in X_0$ and put

$$
\begin{aligned}
&(\alpha_0, \beta_0) := (\alpha, \beta)\,, \\
&(\alpha_{k+1}, \beta_{k+1}) := T(\alpha_k, \beta_k)\,, \\
&m_{k+1} := \left\lfloor \frac{1}{\beta_k - \alpha_k} \right\rfloor, \\
&\lambda_{k+1} := 1 - m_{k+1}(\beta_k - \alpha_k)\,,
\end{aligned} \tag{5}
$$

and rewrite

$$
\varepsilon_{k+1} = \begin{cases} +1 & \text{if } \alpha_k < (m_{k+1}+1)(\beta_k - \alpha_k) - 1\,, \\ -1 & \text{if } \alpha_k > (m_{k+1}+1)(\beta_k - \alpha_k) - 1\,, \end{cases}
$$

where $k = 0, 1, 2 \ldots$. By Proposition 3, this definition of m_{k+1} is equal to $\left\lfloor \frac{\hat{\lambda}_k}{\hat{\beta}_k - \hat{\alpha}_k} \right\rfloor$ which is defined in the previous section, so we use the same notation.

PROPOSITION 3. *The following relation holds*:

$$
\begin{aligned}
\hat{\alpha}_n &= \lambda_1 \cdots \lambda_n \alpha_n\,, \\
\hat{\beta}_n &= \lambda_1 \cdots \lambda_n \beta_n\,, \\
\hat{\lambda}_n &= \lambda_1 \cdots \lambda_n\,.
\end{aligned}
$$

PROOF. From the definition (4) and (5) we see that

$$
\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\beta}_1 \\ \hat{\lambda}_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} \alpha_1 \\ \beta_1 \\ 1 \end{bmatrix},
$$

and by the Proposition 1 we know

$$\begin{bmatrix}\alpha\\ \beta\\ 1\end{bmatrix} = A_{\begin{bmatrix}m_1\\ \varepsilon_1\end{bmatrix}} \cdots A_{\begin{bmatrix}m_n\\ \varepsilon_n\end{bmatrix}} \begin{bmatrix}\hat{\alpha}_n\\ \hat{\beta}_n\\ \hat{\lambda}_n\end{bmatrix},$$

and from the definition (4) and (5) we know

$$\begin{bmatrix}\alpha\\ \beta\\ 1\end{bmatrix} = \lambda_1 \cdots \lambda_n A_{\begin{bmatrix}m_1\\ \varepsilon_1\end{bmatrix}} \cdots A_{\begin{bmatrix}m_n\\ \varepsilon_n\end{bmatrix}} \begin{bmatrix}\alpha_n\\ \beta_n\\ 1\end{bmatrix}.$$

Therefore, we have the conclusion of this proposition. □

From Theorem 2 we have

COROLLARY 1. *Let $R^{(n)}_{\alpha,\beta}$ be the induced transformation of $R_{\alpha,\beta}$ on $\hat{I}^{(n)}$, then $R^{(n)}_{\alpha,\beta}$ is isomorphic to R_{α_n,β_n} with the isomorphism $\theta_n : \hat{I}^{(n)} \to I_{\alpha_n,\beta_n}$ given by*

$$\theta_n(x) = \frac{\varepsilon_1}{\lambda_1}\frac{\varepsilon_2}{\lambda_2}\cdots\frac{\varepsilon_n}{\lambda_n}x = \frac{\varepsilon_1\varepsilon_2\cdots\varepsilon_n}{\hat{\lambda}_n}x\,.$$

LEMMA 1. *Put*

$$x_{\alpha,\beta} = \sum_{N=1}^{\infty} \delta_n \frac{1}{\varepsilon_1}\frac{\lambda_1}{\varepsilon_2}\frac{\lambda_2}{\varepsilon_3}\cdots\frac{\lambda_{N-1}}{\varepsilon_N}\beta_{N-1}\,, \tag{6}$$

where

$$\delta_k = \begin{cases}0 & \textit{if } \varepsilon_k = 1\,,\\ 1 & \textit{if } \varepsilon_k = -1\,.\end{cases}$$

Then $x_{\alpha,\beta}$ satisfies the relation

$$\theta_1(x_{\alpha,\beta}) = \begin{cases}x_{\alpha_1,\beta_1} & \textit{if } \varepsilon_1 = 1\,,\\ x_{\alpha_1,\beta_1} + (\alpha_1 + 1 + \beta_1) & \textit{if } \varepsilon_1 = -1\,.\end{cases}$$

PROOF. Assume that $\varepsilon_1 = 1$, then

$$\theta_1(x_{\alpha,\beta}) = \frac{\varepsilon_1}{\lambda_1}x_{\alpha,\beta} = \delta_1\frac{\beta_0}{\lambda_1} + \frac{\varepsilon_1}{\lambda_1}\sum_{N=2}^{\infty}\delta_N\frac{1}{\varepsilon_1}\frac{\lambda_1}{\varepsilon_2}\cdots\frac{\lambda_{N-1}}{\varepsilon_N}\beta_{N-1}\,,$$

$$= 0 + 1\cdot\sum_{N=2}^{\infty}\delta_N\frac{1}{\varepsilon_2}\frac{\lambda_2}{\varepsilon_3}\cdots\frac{\lambda_N}{\varepsilon_N}\beta_{N-1} = x_{\alpha_1,\beta_1}.$$

Assume that $\varepsilon_1 = -1$, then

$$\theta_1(x_{\alpha,\beta}) = \frac{\beta_0}{\lambda_1} + \sum_{N=2}^{\infty}\delta_N\frac{1}{\varepsilon_2}\frac{\lambda_2}{\varepsilon_3}\cdots\frac{\lambda_N}{\varepsilon_N}\beta_{N-1}$$

$$= \frac{\hat{\alpha}_1 + \hat{\lambda}_1 + \hat{\beta}_1}{\lambda_1} + x_{\alpha_1,\beta_1} = \alpha_1 + 1 + \beta_1 + x_{\alpha_1,\beta_1} .$$

□

From now on the left endpoint $x_{\alpha,\beta}$ of $I_{\alpha,\beta}$ is defined by the equation (6).

COROLLARY 2. *The map* $\theta_1(x) = \frac{\varepsilon_1}{\lambda_1}x$ *is bijective from* $\hat{I}^{(1)}(= R_{\alpha,\beta}(I_2))$ *to* I_{α_1,β_1}.

PROOF. By the Lemma 1, the left endpoint $x_{\alpha,\beta}$ of $\hat{I}^{(1)}$ is mapped to the point x_{α_1,β_1} or $x_{\alpha_1,\beta_1} + \alpha_1 + 1 + \beta_1$ respectively depending on the parity of ε_1, and it is the endpoint of I_{α_1,β_1}. Moreover from the fact that the ratio $|I_{\alpha_1,\beta_1}|/|\hat{I}^{(1)}|$ coincides with $1/\lambda_1$ we obtain the conclusion. □

COROLLARY 3. *The origine* 0 *is the only point included in* $\hat{I}^{(n)}$ *for all* n, *that is*,

$$\bigcap_{n=0}^{\infty} \hat{I}^{(n)} = \{0\} .$$

PROOF. By Proposition 3, we know $\hat{\beta}_n = \lambda_1\lambda_2\cdots\lambda_n\beta_n$, and we have

$$x_{\alpha,\beta} = \sum_{N=1}^{\infty} \delta_n\varepsilon_1\varepsilon_2\cdots\varepsilon_N\hat{\beta}_{N-1} .$$

We denote $a_n = \inf \hat{I}^{(n)}$, $b_n = \sup \hat{I}^{(n)}$, then we can see

$$(a_n, b_n) = \begin{cases} (a_{n-1}, a_{n-1} + \hat{\beta}_n) & \text{if } \hat{\varepsilon}_{n-1} = 1 , \\ (b_{n-1} - \hat{\beta}_n,\ b_{n-1}) & \text{if } \hat{\varepsilon}_{n-1} = -1 , \end{cases}$$

where

$$\hat{\varepsilon}_k = \begin{cases} 0 & \text{if } \varepsilon_1\varepsilon_2\cdots\varepsilon_k = 1 . \\ 1 & \text{if } \varepsilon_1\varepsilon_2\cdots\varepsilon_k = -1 . \end{cases}$$

We show that the following fomulae satisfy this reccurent equation.

$$a_n = \sum_{N=n+1}^{\infty} \delta_N\varepsilon_1\varepsilon_2\cdots\varepsilon_N\hat{\beta}_{N-1} - \hat{\varepsilon}_n\hat{\beta}_n , \tag{7}$$

$$b_n = \sum_{N=n+1}^{\infty} \delta_N\varepsilon_1\varepsilon_2\cdots\varepsilon_N\hat{\beta}_{N-1} + (1 - \hat{\varepsilon}_n)\hat{\beta}_n .$$

Consider the case when $\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1} = -1$ and $\varepsilon_n = -1$. In this case we have $\delta_n = 1$ and $\varepsilon_1\varepsilon_2\cdots\varepsilon_n = 1$, then from the equation (7) we get

$$b_{n-1} = \sum_{N=n}^{\infty} \delta_N\varepsilon_1\varepsilon_2\cdots\varepsilon_n\hat{\beta}_{n-1} + (1 - \hat{\varepsilon}_n)\hat{\beta}_n$$

$$= \delta_n \varepsilon_1 \cdots \varepsilon_n \hat{\beta}_{n-1} + \sum_{N=n+1}^{\infty} \delta_N \varepsilon_1 \varepsilon_2 \cdots \varepsilon_N \hat{\beta}_{N-1} + (1 - \hat{\varepsilon}_n)\hat{\beta}_n$$

$$= \hat{\beta}_{n-1} + \sum_{N=n+1}^{\infty} \delta_N \varepsilon_1 \varepsilon_2 \cdots \varepsilon_N \hat{\beta}_{N-1} .$$

On the other hand $a_n = \sum_{N=n+1}^{\infty} \delta_N \varepsilon_1 \varepsilon_2 \cdots \varepsilon_N \hat{\beta}_{N-1}$ then $a_n = b_{n-1} - \hat{\beta}_{n-1}$. We can show in the other case analoguesly.

Now we will see that a_n is negative for all n. In the equation (7), if $\hat{\varepsilon}_n = 0$ then $\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n = 1$, hence the first non-zero term in the alternating series is negative. If $\hat{\varepsilon}_n = 1$, then the first non-zero term in the alternating series is positive but the absolute value of this term is less than $\hat{\beta}_n$. Anyway we get that a_n is negative. Similarly, we can see b_n is positive. With Proposition 2, we get the conclusion. □

Moreover we will get the following fact by Lemma 3:

$$\bigcap_{n=0}^{\infty} \hat{I}_3^{(n)} = \{0\} . \tag{8}$$

EXAMPLE 1. Let $(\alpha, \beta) = (\frac{2\sqrt{6}}{3} - 1, 1)$, then we get

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \cdots \\ \beta_0 & \beta_1 & \cdots \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{6}}{3} - 1 & \frac{2\sqrt{6}-3}{5} & \frac{2\sqrt{6}-3}{5} & \frac{2\sqrt{6}}{3} - 1 & \cdots \\ 1 & \frac{2\sqrt{6}+7}{5} & 1 & 1 & \cdots \end{pmatrix}$$

and

$$\begin{pmatrix} m_1 & m_2 & \cdots \\ \varepsilon_1 & \varepsilon_2 & \cdots \end{pmatrix} = \overline{\begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}} .$$

Therefore $(\alpha, \beta) = (\frac{2\sqrt{6}}{3} - 1, 1)$ is a purely periodic point with period 3 of NIR-algorithm T.

4. Substitutions and Partitions

Now, let us introduce the following substitution $\sigma_{\left[\begin{smallmatrix} m \\ \varepsilon \end{smallmatrix}\right]}$, $\varepsilon \in \{1, -1\}$ by

$$\sigma_{\left[\begin{smallmatrix} m \\ 1 \end{smallmatrix}\right]} : \begin{cases} 1 \to 12, \\ 2 \to \overbrace{3 \cdots 3}^{m\,\text{times}}\, 2 , \\ 3 \to \overbrace{3 \cdots 3}^{m+1\,\text{times}}\, 2 , \end{cases} \quad , \quad \sigma_{\left[\begin{smallmatrix} m \\ -1 \end{smallmatrix}\right]} : \begin{cases} 1 \to \overbrace{3 \cdots 3}^{m\,\text{times}}\, 2 , \\ 2 \to 12, \\ 3 \to \overbrace{3 \cdots 3}^{m+1\,\text{times}}\, 2 . \end{cases}$$

In general, let us denote $\sigma_{\left[\begin{smallmatrix} m_{n+1} \\ \varepsilon_{n+1} \end{smallmatrix}\right]}(j) = W^{(n)}_{\binom{j}{1}} W^{(n)}_{\binom{j}{2}} \cdots W^{(n)}_{\binom{j}{l_j^{(n)}}}$, $n = 0, 1, 2, \dots$ where $l_j^{(n)}$ is the length of word $\sigma_{\left[\begin{smallmatrix} m_{n+1} \\ \varepsilon_{n+1} \end{smallmatrix}\right]}(j)$.

Consider the induced transformation $R^{(1)}_{\alpha_n,\beta_n}$ of R_{α_n,β_n} onto $J^{\alpha_n,\beta_n} = R_{\alpha_n,\beta_n} I_2^{(n)} := J_1^{\alpha_n,\beta_n} \cup J_3^{\alpha_n,\beta_n} \cup J_2^{\alpha_n,\beta_n}$, where $J_i^{\alpha_n,\beta_n}$ $i = 1, 2, 3$ are defined in previous section.

For simplicity, let us denote $I^{(n)}$ instead of $I_{\alpha_n,\beta_n} = [x_{\alpha_n,\beta_n}, x_{\alpha_n,\beta_n} + \alpha_n + 1 + \beta_n]$, and denote $J^{(n)}$, $J_i^{(n)}$ instead of J^{α_n,β_n}, $J_i^{\alpha_n,\beta_n}$, $i = 1, 2, 3$, respectively (See Figure 5). Using these notations, let us define the finer partition of $\{I_i^{(n)} \mid i = 1, 2, 3\}$ as follows:

$$I^{(n)}_{\binom{j}{1}} := J_j^{(n)},$$

$$I^{(n)}_{\binom{j}{k}} := R_{\alpha_n,\beta_n} I^{(n)}_{\binom{j}{k-1}},$$

where $j \in \{1, 2, 3\}$, $k = 1, 2, \dots, l_j^{(n)}$, $n = 0, 1, 2, \dots$ and we see

$$l_j^{(n)} = \min\{N \mid R^N_{\alpha_n,\beta_n} J_j^{(n)} \subset R_{\alpha_n,\beta_n}\left(I_2^{(n)}\right), N \geq 1\}.$$

PROPOSITON 4. *The set* $\{I^{(n)}_{\binom{j}{k}} \mid j = 1, 2, 3,\ k = 1, 2, \dots, l_j^{(n)}\}$ *is a finer partition of* $\{I_i^{(n)} \mid i = 1, 2, 3\}$*, that is,*

$$I^{(n)} = \bigcup_{j=1,2,3} \bigcup_{k=1}^{l_k^{(n)}} I^{(n)}_{\binom{j}{k}} = \bigcup_{j=1,2,3} \bigcup_{k=1}^{l_k^{(n)}} R^k_{\alpha_n,\beta_n} I^{(n)}_{\binom{j}{1}}. \tag{9}$$

and intervals in the equation (9) *are pairwise disjoint. Moreover* $I^{(n)}_{\binom{j}{k}} \subset I^{(n)}_{W^{(n)}_{\binom{j}{k}}}$, $j = 1, 2, 3$, *and as we see in Figure* 5, $I_i^{(n)}$ *is decomposed into*

$$I_i^{(n)} = \bigcup_{W^{(n)}_{\binom{j}{k}} = i} I^{(n)}_{\binom{j}{k}}. \tag{10}$$

DEFINITION 1. For $x \in I_{\alpha,\beta}$, we call $s_1 s_2 \cdots$ the name of x under $R_{\alpha,\beta}$, if $R^{k-1}_{\alpha,\beta}(x) \in I_{s_k}$.

THEOREM 3. *Let* $s = (s_1, s_2 \cdots)$ *be the name of* 0, *then*

$$s = \lim_{n\to\infty} \sigma_{\left[\begin{smallmatrix} m_1 \\ \varepsilon_1 \end{smallmatrix}\right]} \circ \sigma_{\left[\begin{smallmatrix} m_2 \\ \varepsilon_2 \end{smallmatrix}\right]} \circ \cdots \circ \sigma_{\left[\begin{smallmatrix} m_n \\ \varepsilon_n \end{smallmatrix}\right]}(3).$$

PROOF. From the fact $\{0\} \in \bigcap_{n=0}^{\infty} \hat{I}_3^{(n)}$ which will be proved in Section 6, it follows that if we know the name of 0 under $R_{\alpha,\beta}^{(k)}$, denoted by $s_1^{(k)} s_2^{(k)} \cdots$, then the name of 0 under $R_{\alpha,\beta}^{(k-1)}$ is given by

$$s_1^{(k-1)} s_2^{(k-1)} \cdots = \sigma_{\begin{bmatrix} m_k \\ \varepsilon_k \end{bmatrix}} (s_1^{(k)} s_2^{(k)} \cdots) .$$

Therefore, we have

$$s_1 s_2 \ldots s_n \cdots = \sigma_{\begin{bmatrix} m_1 \\ \varepsilon_1 \end{bmatrix}} \circ \cdots \circ \sigma_{\begin{bmatrix} m_k \\ \varepsilon_k \end{bmatrix}} (s_1^{(k)} s_2^{(k)} \cdots) .$$

By the equation (8), $0 \in \hat{I}_3^{(n)}$ and $s_1^{(k)} = 3$ for all k, we have

$$s_1 s_2 \ldots s_k \cdots = \lim_{n \to \infty} \sigma_{\begin{bmatrix} m_1 \\ \varepsilon_1 \end{bmatrix}} \circ \cdots \circ \sigma_{\begin{bmatrix} m_n \\ \varepsilon_n \end{bmatrix}} (3) .$$

$\square$

5. Ostrowski type numerical expansion

To begin with, we prepare a family of maps $\{g_k\}_k$. Let $A^* = \bigcup_{n \geq 0} \{1, 2, 3\}^n$ be a set of finite sequences of $\{1, 2, 3\}$ and $g_n : A^* \to \mathbf{Z}|I_1^{(n)}| + \mathbf{Z}|I_2^{(n)}| + \mathbf{Z}|I_3^{(n)}|$ be a homomorphism satisfying:

$$\begin{aligned} g_n(\emptyset) &:= 0 , \\ g_n(i) &:= R_{\alpha_n,\beta_n}(x) - x \qquad x \in I_i^{(n)} \quad \text{if } i \in \{1, 2, 3\} , \qquad (11) \\ &= \begin{cases} g_n(1) = 1 + \beta_n , \\ g_n(2) = -(1 + \alpha_n) , \\ g_n(3) = \beta_n - \alpha_n , \end{cases} \\ g_n(uv) &:= g_n(u) + g_n(v) \quad \text{for any } u, v \in A^* . \qquad (12) \end{aligned}$$

Recall that $(\alpha_0, \beta_0) = (\alpha, \beta)$ and $I^{(0)} = I_{\alpha,\beta}$. Then, we know that $g_n(i)$ gives a translation of the interval $I_i^{(n)} \subset I^{(n)}$ by R_{α_n,β_n}.

By the equation (9), intervals $I_{\binom{j}{1}}$ and their images by $R_{\alpha,\beta}$ cover I. Hence, for any $x \in I_{\binom{j}{k}}$, we know that $R_{\alpha,\beta}^{-(k-1)}(x) \in I_{\binom{j}{1}}$. On the other hand, by Proposition 4 and the

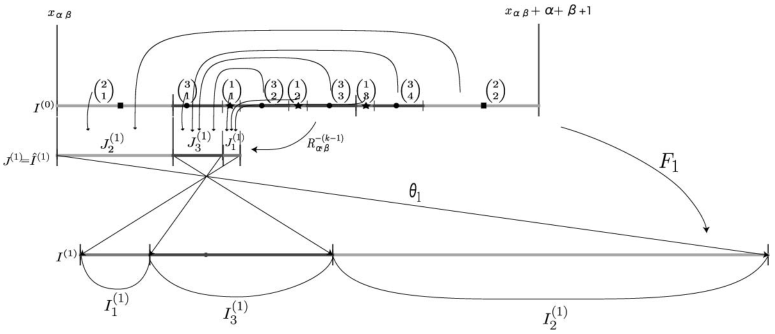


FIGURE 5. the map F_1 and the partition $\{I^{(0)}_{\binom{j}{k}}\}$ of $I^{(0)}$ in the case of $m_1 = 2, \varepsilon_1 = -1$.

equation (11), if $x \in I_{\binom{j}{k}}$, then $R^{-1}_{\alpha,\beta}(x) \in I_{\binom{j}{k-1}} \subset I_{W^{(0)}_{\binom{j}{k-1}}}$ and

$$R^{-1}_{\alpha,\beta}(x) = x - g_0(W^{(0)}_{\binom{j}{k-1}}) .$$

Thus we get by the equation (12) that

$$\begin{aligned} R^{-(k-1)}_{\alpha,\beta}(x) &= x - g_0(W^{(0)}_{\binom{j}{k-1}}) - g_0(W^{(0)}_{\binom{j}{k-2}}) - \cdots - g_0(W^{(0)}_{\binom{j}{1}}) \\ &= x - g_0(P^{(0)}_{\binom{j}{k}}) , \end{aligned}$$

where $P^{(n)}_{\binom{j}{k}} := W^{(n)}_{\binom{j}{1}} W^{(n)}_{\binom{j}{2}} \cdots W^{(n)}_{\binom{j}{k-1}}$ is the prefix word before $W^{(n)}_{\binom{j}{k}}$ in $\sigma_{\left[\begin{smallmatrix} m_{n+1} \\ \varepsilon_{n+1} \end{smallmatrix}\right]}(j)$. Then by Theorem 2, we can define the map $F_1 : I_{\alpha,\beta} \to I^{(1)}$ as follows (See Figure 5):

$$\begin{aligned} F_1(x) &= \theta_1(R^{-(k-1)}_{\alpha,\beta}(x)) \\ &= \frac{\varepsilon_1}{\lambda_1}\left(x - g_0(P^{(0)}_{\binom{j}{k}}) \right) \quad \text{if } x \in I_{\binom{j}{k}} . \end{aligned}$$

Similarly we define maps $F_n : I^{(n-1)} \to I^{(n)}$

$$F_n(x) = \frac{\varepsilon_n}{\lambda_n}\left(x - g_{n-1}(P^{(n-1)}_{\binom{j}{k}}) \right) \quad \text{if } x \in I^{(n-1)}_{\binom{j}{k}} . \tag{13}$$

Then we get

$$F_n I^{(n-1)}_{\binom{j}{k}} = I^{(n)}_j\,, \tag{14}$$

$$F_n^{-1} I^{(n)}_{\binom{j}{k}} \cap I^{(n-1)}_{\binom{j'}{k'}} \neq \emptyset \;\Rightarrow\; j' = W^{(n)}_{\binom{j}{k}}\,. \tag{15}$$

Now, let us define the map φ from $I_{\alpha,\beta}$ to some symbolic space $\Omega^0_{\alpha,\beta}$ as follows: for any $x \in I_{\alpha,\beta}$, we can find the sequence of the intervals $I^{(n)}_{\binom{j_n}{k_n}}$, $n = 0, 1, 2, \ldots$ satisfying $x \in I_{\binom{j_0}{k_0}}$, and $x_n := F_n(x_{n-1}) \in I^{(n)}_{\binom{j_n}{k_n}}$. From the indices of these intervals, we can define the map φ to the symbolic space by setting

$$\varphi(x) = \begin{pmatrix} \cdots & j_1 & j_0 \\ \cdots & k_1 & k_0 \end{pmatrix}.$$

Then we have the following Ostrowski type numerical representation.

PROPOSITON 5. *For $x \in I_{\alpha,\beta}$ and $\varphi(x) = \begin{pmatrix} \cdots & j_1 & j_0 \\ \cdots & k_1 & k_0 \end{pmatrix}$ we have*

(1) *if $x, x' \in I_{\alpha,\beta}$ and $x \neq x'$ then $\varphi(x) \neq \varphi(x')$*

(2) *x can be represented by*

$$x = \sum_{N=1}^{\infty} \frac{\lambda_0}{\varepsilon_0}\frac{\lambda_1}{\varepsilon_1}\cdots\frac{\lambda_{N-1}}{\varepsilon_{N-1}} g_{N-1}(P^{(N-1)}_{\binom{j_{N-1}}{k_{N-1}}})\,.$$

PROOF. Statement (1) is obtained in Lemma 2 in the next section. From the fact that $x_1 = F_1(x),\ x_n = F_n(x_{n-1})$ and (13),

$$\begin{aligned} x &= \frac{\lambda_1}{\varepsilon_1}x_1 + g_0(P^{(0)}_{\binom{j_0}{k_0}}) && \text{if } x \in I_{\binom{j_0}{k_0}} \\ &= \frac{\lambda_1}{\varepsilon_1}\left(\frac{\lambda_2}{\varepsilon_2}x_2 + g_1(P^{(1)}_{\binom{j_1}{k_1}})\right) + g_0(P^{(0)}_{\binom{j_0}{k_0}}) && \text{if } x_1 \in I^{(1)}_{\binom{j_1}{k_1}} \\ &= \sum_{N=1}^{n} \frac{\lambda_0}{\varepsilon_0}\frac{\lambda_1}{\varepsilon_1}\cdots\frac{\lambda_{N-1}}{\varepsilon_{N-1}} g_{N-1}(P^{(N-1)}_{\binom{j_{N-1}}{k_{N-1}}}) + \frac{\lambda_0}{\varepsilon_0}\frac{\lambda_1}{\varepsilon_1}\cdots\frac{\lambda_n}{\varepsilon_n}x_n\,. \end{aligned}$$

Since $x_n \in I_{\alpha_n,\beta_n}$, we get $|x_n| < \alpha_n + 1 + \beta_n < 1 + 2\beta_n$, then

$$\left|\frac{\lambda_0}{\varepsilon_0}\frac{\lambda_1}{\varepsilon_1}\cdots\frac{\lambda_n}{\varepsilon_n}x_n\right| = \lambda_1\lambda_2\cdots\lambda_n|x_n| < \hat{\lambda}_n + 2\hat{\beta}_n\,.$$

By Proposition 2, we obtain the conclusion. □

6. Symbolic space and Odometer transformation

Let us introduce a directed graph $G = \{\mathcal{V}, \mathcal{E}, iv, tv\}$ shown in Figure 6 as follows: $\mathcal{V} = \bigcup_{n=0}^{\infty} \{(1)_n, (2)_n, (3)_n\}$, $\mathcal{E} = \bigcup_{n=0}^{\infty} \left\{ \begin{pmatrix} j_n \\ k_n \end{pmatrix} \,\middle|\, I^{(n)}_{\begin{pmatrix} j_n \\ k_n \end{pmatrix}} \subset I_{\alpha_n, \beta_n} \right\}$, $iv, tv : \mathcal{E} \to \mathcal{V}$ where

$$iv \begin{pmatrix} j_n \\ k_n \end{pmatrix} = (j_n)_{n+1}, \quad tv \begin{pmatrix} j_n \\ k_n \end{pmatrix} = (W^{(n)}_{\begin{pmatrix} j_n \\ k_n \end{pmatrix}})_n . \tag{16}$$

See Figure 6.

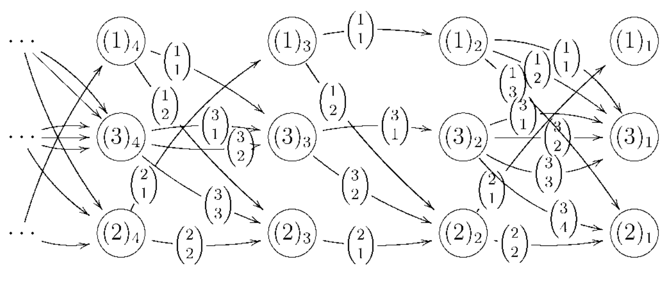
DEFINITION 2. An infinite sequence $\omega = \begin{pmatrix} \cdots & j_1 & j_0 \\ \cdots & k_1 & k_0 \end{pmatrix}$ is called a G-admissible sequence if

$$W^{(N)}_{\begin{pmatrix} j_N \\ k_N \end{pmatrix}} = j_{N-1} , \tag{17}$$

for all $N = 1, 2, 3, \ldots$. For a finite sequence $u = \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix}$, u is called a G-admissible word if (17) holds for $N = 1, 2, \ldots, n-1$.

Define the symbolic space $\Omega_{\alpha,\beta}$ by

$$\Omega_{\alpha,\beta} = \left\{ \omega = \begin{pmatrix} \cdots & j_1 & j_0 \\ \cdots & k_1 & k_0 \end{pmatrix} \,\middle|\, \omega \text{ is } G\text{-admissible} \right\} .$$



$$(m_3, \varepsilon_3) = (1, -1) \qquad (m_2, \varepsilon_2) = (0, 1) \qquad (m_1, \varepsilon_1) = (2, -1)$$

$$\sigma_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{cases} 1 \to 32 \\ 2 \to 12 \\ 3 \to 332 \end{cases} \qquad \sigma_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{cases} 1 \to 12 \\ 2 \to 2 \\ 3 \to 32 \end{cases} \qquad \sigma_{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \begin{cases} 1 \to 332 \\ 2 \to 12 \\ 3 \to 3332 \end{cases}$$

FIGURE 6. Graph $G = \{\mathcal{V}, \mathcal{E}, iv, tv\}$ and substitutions of Example 1.

Then we see that for any $x \in I_{\alpha,\beta}$, $\varphi(x) = \begin{pmatrix} \cdots & j_1 & j_0 \\ \cdots & k_1 & k_0 \end{pmatrix}$ is an element of $\Omega_{\alpha,\beta}$, that is, $\varphi(I_{\alpha,\beta}) \subset \Omega_{\alpha,\beta}$. In fact, let us assume that $x_{n-1} \in I^{(n-1)}_{\binom{j_{n-1}}{k_{n-1}}}$ and $F_n(x_{n-1}) \in I^{(n)}_{\binom{j_n}{k_n}}$, then by the equations (14) and (10)

$$I^{(n)}_{j_{n-1}} = \bigcup_{W^{(n)}_{\binom{j}{k}} = j_{n-1}} I^{(n)}_{\binom{j}{k}} .$$

This shows that the relation $W^{(n)}_{\binom{j_n}{k_n}} = j_{n-1}$ holds.

Let us define the set $\Omega^0_{\alpha,\beta}$ by

$$\Omega^0_{\alpha,\beta} = \varphi(I_{\alpha,\beta}) = \left\{ \varphi(x) \middle| \, x \in I_{\alpha,\beta} \right\} .$$

Oppositely, let us define the map $\Phi : \Omega_{\alpha,\beta} \to I_{\alpha,\beta}$ by setting

$$\Phi \begin{pmatrix} \cdots & j_1 & j_0 \\ \cdots & k_1 & k_0 \end{pmatrix} = \sum_{N=1}^{\infty} \frac{\lambda_0}{\varepsilon_0} \frac{\lambda_1}{\varepsilon_1} \cdots \frac{\lambda_{N-1}}{\varepsilon_{N-1}} g_{N-1}(P^{(N-1)}_{\binom{j_{N-1}}{k_{N-1}}}) . \tag{18}$$

For G-admissible word $u = \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix}$, let us define the cylinder set $\Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix}$ of $I_{\alpha,\beta}$ as follows:

$$\begin{aligned} \Delta(u) &= \Delta \begin{pmatrix} j'_{n-1} & \cdots & j'_0 \\ k'_{n-1} & \cdots & k'_0 \end{pmatrix} \\ &:= \left\{ x \middle| \, \varphi(x) = \begin{pmatrix} \cdots & j_{n+1} & j_n & j'_{n-1} & \cdots & j'_0 \\ \cdots & k_{n+1} & k_n & k'_{n-1} & \cdots & k'_0 \end{pmatrix} \right\} . \end{aligned}$$

For $\Delta(u)$, we call u the word of $\Delta(u)$. From the equations (14), (15) and (16), the cylinder set is an inverval and is denoted by

$$\Delta \begin{pmatrix} j'_{n-1} & \cdots & j'_0 \\ k'_{n-1} & \cdots & k'_0 \end{pmatrix} = \bigcap_{N=1}^{n} F_1^{-1} \circ F_2^{-1} \circ \cdots \circ F_{N-1}^{-1} I^{(N-1)}_{\binom{j'_{N-1}}{k'_{N-1}}} . \tag{19}$$

LEMMA 2. *The cylinder sets defined by the equation* (19) *satisfy the following conditions:*

(1) *Put for* $n = 0, 1, 2, \ldots$

$$\xi_n = \left\{ \Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix} \middle| \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix} \textit{ is G-admissible} \right\} ,$$

then ξ_n is a partition of $I_{\alpha,\beta}$, that is,

$$I_{\alpha,\beta} = \bigcup \Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix} \quad (\textit{disjoint})\,.$$

If $m > n$, ξ_m is a finer partition than ξ_n (*See Figure* 7).

(2) $\Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix}$ *is an open interval if $j_{n-1} = 3$, and is closed if $j_{n-1} = 1$ or* 2 *and its length is given by*

$$\left| \Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix} \right| = \begin{cases} \hat{\alpha}_n & \text{if } j_{n-1} = 1\,, \\ \hat{\beta}_n & \text{if } j_{n-1} = 2\,, \\ \hat{\lambda}_n & \text{if } j_{n-1} = 3\,. \end{cases}$$

(3) *Let E_n be the set of endpoints of intervals in ξ_n and $E = \bigcup_{n=0}^{\infty} E_n$. Then for a point $x \in E$, there exist $\omega \in \Omega^0_{\alpha,\beta}$ and $\omega' \in \Omega_{\alpha,\beta} \setminus \Omega^0_{\alpha,\beta}$ such that $\Phi(\omega) = \Phi(\omega') = x$.*

PROOF. By the equation (19), we have

$$\Delta \begin{pmatrix} j_n & \cdots & j_0 \\ k_n & \cdots & k_0 \end{pmatrix} = \Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix} \cap F_1^{-1} \circ F_2^{-1} \circ \cdots \circ F_n^{-1} I^{(n)}_{\binom{j_n}{k_n}}\,. \tag{20}$$

Therefore $\Delta \begin{pmatrix} j_n & \cdots & j_0 \\ k_n & \cdots & k_0 \end{pmatrix}$ is a subset of $\Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix}$. From Proposition 4 $\{-I_{\binom{j}{k}} - (n)\}$ is a partition of $I^{(n)}$ then

$$\begin{aligned} \Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix} &= \bigcup_{\binom{j_n}{k_n}} \left(\Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix} \cap F_1^{-1} \circ F_2^{-1} \circ \cdots \circ F_n^{-1} I^{(n)}_{\binom{j_n}{k_n}} \right) \\ &= \bigcup \Delta \begin{pmatrix} j_n & \cdots & j_0 \\ k_n & \cdots & k_0 \end{pmatrix}\,, \end{aligned}$$

Thus the statement (1) holds.

The set $F_1^{-1} \circ F_2^{-1} \circ \cdots \circ F_{n-1}^{-1} I^{(n-1)}_{\binom{j_{n-1}}{k_{n-1}}}$ is constructed by the union of several intervals which have the same length. If $j_{n-1} = 3$, then $I^{(n-1)}_{\binom{j_{n-1}}{k_{n-1}}}$ is open, therefore each interval in $F_1^{-1} \circ F_2^{-1} \circ \cdots \circ F_{n-1}^{-1} I^{(n-1)}_{\binom{j_{n-1}}{k_{n-1}}}$ is open. Similarly if $j_{n-1} = 1$ or 2 then they are closed. One of these intervals is whole included in $\varDelta\begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix}$ and the other is disjoint. The it is decided by j_{n-1} that whether $\varDelta\begin{pmatrix} j_n & \cdots & j_0 \\ k_n & \cdots & k_0 \end{pmatrix}$ is open or closed. By the definition of $I^{(n)}_{\binom{j}{k}}$, we have

$$\left|I^{(n-1)}_{\binom{j_{n-1}}{k_{n-1}}}\right| = \left|J^{(n-1)}_{j_{n-1}}\right| = \lambda_n\left|I^{(n)}_{j_n}\right|,$$

and $|F_n^{-1}I| = \lambda_n|I|$ then

$$\begin{aligned}\left|F_1^{-1}\circ F_2^{-1}\circ\cdots\circ F_{n-1}^{-1} I^{(n-1)}_{\binom{j_{n-1}}{k_{n-1}}}\right| &= \lambda_1\left|F_2^{-1}\circ F_3^{-1}\cdots\circ F_{n-1}^{-1} I^{(n-1)}_{\binom{j_{n-1}}{k_{n-1}}}\right| \\ &= \lambda_1\lambda_2\cdots\lambda_{n-1}\left|I^{(n-1)}_{\binom{j_{n-1}}{k_{n-1}}}\right| = \lambda_1\lambda_2\cdots\lambda_{n-1}\lambda_n\left|I^{(n)}_{j_n}\right|.\end{aligned}$$

By Proposition 2, statement (2) holds.

Suppose that $\varDelta\begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix}$ and $\varDelta\begin{pmatrix} j'_{n-1} & \cdots & j'_0 \\ k'_{n-1} & \cdots & k'_0 \end{pmatrix}$ are two adjoining intervals in ξ_n. Then one of j_{n-1} and j'_{n-1} is 3, and the other is 1 or 2. We denote these set $\varDelta$, $\varDelta'$ for simplicity. Without lost of generality, we can assume that $j_{n-1} = 3$ and $\sup\varDelta = \inf\varDelta'$. Put $x_\varDelta = \sup\varDelta$. Then $\varDelta$ is an open interval, and $\varDelta'$ is closed. For $m > n$, we denote the rightmost subinterval of ξ_m which is included in $\varDelta$ by $\varDelta_R(m)$ and the leftmost subinterval of ξ_m which is included in $\varDelta$ by $\varDelta'_L(m)$. Since $\varDelta_R(m)$ is open and $\varDelta'_L(m)$ is closed, we can write that

$$\begin{aligned}\varDelta_R(m) &= \varDelta\begin{pmatrix} 3 & \cdots & 3 & j_{n-1} & \cdots & j_0 \\ k_{m-1} & \cdots & k_n & k_{n-1} & \cdots & k_0 \end{pmatrix}, \\ \varDelta'_L(m) &= \varDelta\begin{pmatrix} j'_{m-1} & \cdots & j'_n & j'_{n-1} & \cdots & j'_0 \\ k'_{m-1} & \cdots & k'_n & k'_{n-1} & \cdots & k'_0 \end{pmatrix} \qquad j_n, \ldots, j_{m-1} \in \{1, 2\}.\end{aligned}$$

By statement (2), the lengths of $\varDelta_R(m)$ and $\varDelta'_L(m)$ goes to 0, then we have

$$x_\varDelta = \bigcap_{m>n}\overline{\varDelta_R(m)} = \bigcap_{m>n}\varDelta'_L(m)$$

where $\overline{A}$ denote a closure of set A. Thus the words associated with $\varDelta_R(m)$ and $\varDelta'_L(m)$ are mapped $x_\varDelta$ as m goes to infinity. □

By this lemma and Proposition 2, the length of each cylinder set shrinks to 0 as the length of its word goes to infinity. Therefore for $x, x' \in I_{\alpha,\beta}$, if $x \neq x'$ then x and x' belong to different cylinder sets corresponding to sufficiently long words. Then we complete the proof of Proposition 5.

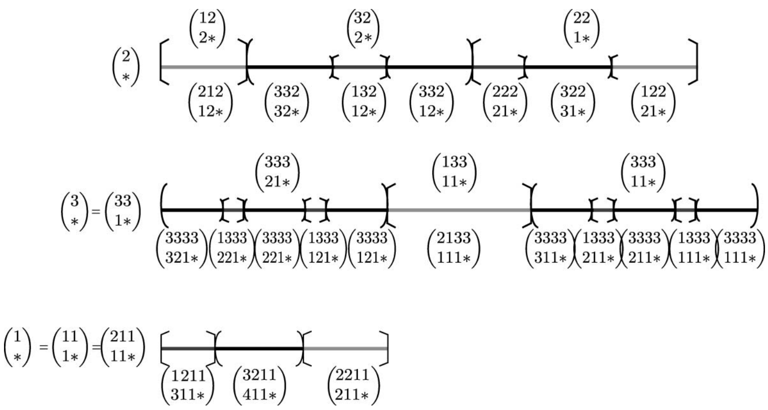


FIGURE 7. Cylinder sets in Example 1. $\begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix} := \Delta \begin{pmatrix} j_{n-1} & \cdots & j_0 \\ k_{n-1} & \cdots & k_0 \end{pmatrix}$

LEMMA 3. *For any* $(\alpha, \beta) \in X_0$, $\omega_0 = \begin{pmatrix} \cdots & 3 & 3 \\ \cdots & 1 & 1 \end{pmatrix}$ *is G-admissible*, $\omega_0 \in \Omega_0$ *and* $\Phi(\omega_0) = 0$.

PROOF. By $P^{(n)}_{\binom{3}{1}} = \emptyset$ for all $n \in \mathbf{N}$, it is obvious that $\Phi(\omega_0) = 0$. Denote $\Delta\underbrace{\begin{pmatrix} 3 & \cdots & 3 \\ 1 & \cdots & 1 \end{pmatrix}}_{n\text{times}}$ by Δ_n. It is sufficient to show that 0 belongs the interior of Δ_n. Take n satisfying $m_n \neq 0$, then we have $\Delta_n \subsetneqq \Delta_{n-1}$, and Δ_n is the rightmost (leftmost) subinterval in Δ_{n-1} respectively depending on the parity of $\hat{\varepsilon}_n$. By Proposition 2, we know that $m_k \neq 0$ for infinitely many k, then we can take $n_1 < n_2 < \cdots$ such that

$$\Delta_{n_{k-1}} = R_{n_k} \cup \Delta_{n_k} \cup L_{n_k} \quad \text{(disjoint)},$$

where $x < y < z$ for $\forall x \in R_{n_k}, \forall y \in \Delta_{n_k}, \forall z \in L_{n_k}$, and $|R_{n_k}| > 0$, $|L_{n_k}| > 0$. Then 0 is not a boundary point of any Δ_k. Therefore we get the equation (8), that is, $\bigcap_{n=0}^{\infty} \hat{I}_3^{(n)} = \{0\}$. □

Now, we introduce a partial order $\prec$ in $\Omega_{\alpha,\beta}$ as follows. Let $\omega = \begin{pmatrix} \cdots & j_1 & j_0 \\ \cdots & k_1 & k_0 \end{pmatrix}$, $\omega' = \begin{pmatrix} \cdots & j'_1 & j'_0 \\ \cdots & k'_1 & k'_0 \end{pmatrix}$. Then we say that $\omega \prec \omega'$ if there exists $n_0 \in \mathbf{N}$ such that

$$\begin{cases} \begin{pmatrix} j_n \\ k_n \end{pmatrix} = \begin{pmatrix} j'_n \\ k'_n \end{pmatrix} & n > n_0 \,, \\ \begin{pmatrix} j_{n_0} \\ k_{n_0} \end{pmatrix} \neq \begin{pmatrix} j'_{n_0} \\ k'_{n_0} \end{pmatrix} \text{ and } & k_{n_0} < k'_{n_0} \,. \end{cases}$$

Note that $j_{n_0} = j'_{n_0}$, because $\begin{pmatrix} j_{n_0+1} \\ k_{n_0+1} \end{pmatrix} = \begin{pmatrix} j'_{n_0+1} \\ k'_{n_0+1} \end{pmatrix}$.

Let $\omega_{\alpha,\beta} = \begin{pmatrix} \cdots & j_1 & j_0 \\ \cdots & k_1 & k_0 \end{pmatrix}$ be a maximal element in $\Omega_{\alpha,\beta}$ under $\prec$, that is,

$$j_n = 2, \;\; k_n = l_{j_n}^{(n)} \,, \quad n = 0, 1, 2, \dots \,,$$

and put

$$M = \bigcup_{n=0}^{\infty} \mathrm{Od}^{-n}(\omega_{\alpha,\beta}) \,.$$

Then we can define the odometer transformation $\mathrm{Od} : \Omega^0_{\alpha,\beta} \setminus M \to \Omega^0_{\alpha,\beta} \setminus M$.

$$\mathrm{Od}(\omega) = \min\{\omega' \in \Omega_{\alpha,\beta} \mid \omega' \succ \omega\} \,.$$

More concretely, $\mathrm{Od}(\omega)$ is given by the following formula:

$$\mathrm{Od} : \begin{pmatrix} \cdots & j_n & \cdots & j_0 \\ \cdots & k_n & \cdots & k_0 \end{pmatrix}$$

$$\to \begin{cases} \begin{pmatrix} \cdots & j_n & \cdots & j_0 \\ \cdots & k_n & \cdots & k_0 + 1 \end{pmatrix} & \text{if } \; k_1 < l_{j_0}^{(0)} \\ \begin{pmatrix} \cdots & j_{n+1} & j_n & j'_{n-1} & \cdots & j'_0 \\ \cdots & k_{n+1} & k_n + 1 & 1 & \cdots & 1 \end{pmatrix} & \text{if } \begin{cases} k_1 = l_{j_p}^{(p)}, \\ \quad (0 \le p \le n-1), \\ k_n < l_{j_n}^{(n)}, \end{cases} \end{cases}$$

where $j'_{n-1} = W^{(n)}_{\binom{j_n}{k_n+1}}$, $j'_p = W^{(p)}_{\binom{1}{p}}$ $(0 \le p \le n-1)$.

THEOREM 4. *The following commutative diagram holds*:

$$\begin{tikzcd} \Omega^0_{\alpha,\beta} \setminus M \arrow[r, "\mathrm{Od}"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^0_{\alpha,\beta} \setminus M \arrow[d, "\Phi"] \\ I_{\alpha,\beta} \arrow[r, "R_{\alpha,\beta}"'] & I_{\alpha,\beta} \end{tikzcd}$$

where $\Phi : \Omega^0_{\alpha,\beta} \setminus M \to I_{\alpha,\beta}$ *is given by the equation* (18) *and it is bijection.*

PROOF. First we prove the case when the carry over does not occur. For $\omega = \begin{pmatrix} \cdots & j_n & \cdots & j_0 \\ \cdots & k_n & \cdots & k_0 \end{pmatrix}$, we have

$$\mathrm{Od}(\omega) = \begin{pmatrix} \cdots & j_n & \cdots & j_0 \\ \cdots & k_n & \cdots & k_0 + 1 \end{pmatrix} .$$

By Proposition 4, if $\Phi(\omega) \in I_{\binom{j_0}{k_0}} \subset I_{W^{(0)}_{\binom{j_0}{k_0}}}$, then we know that

$$\begin{aligned}
\Phi(\mathrm{Od}(\omega)) - \Phi(\omega) &= \sum_{N=1}^{\infty} \frac{\lambda_0}{\varepsilon_0} \frac{\lambda_1}{\varepsilon_1} \cdots \frac{\lambda_{N-1}}{\varepsilon_{N-1}} g_{N-1}(P^{(N-1)}_{\binom{j_{N-1}}{k_{N-1}}}) + g_0(P^{(0)}_{\binom{j_0}{k_0+1}}) \\
&\qquad - \sum_{N=1}^{\infty} \frac{\lambda_0}{\varepsilon_0} \frac{\lambda_1}{\varepsilon_1} \cdots \frac{\lambda_{N-1}}{\varepsilon_{N-1}} g_{N-1}(P^{(N-1)}_{\binom{j_{N-1}}{k_{N-1}}}) \\
&= g_0(W^{(0)}_{\binom{k_0}{j_0}}) .
\end{aligned}$$

Since $x + g_0(i) = R_{\alpha,\beta}(x)$ for $x \in I_{\binom{i}{*}}$, we get

$$\Phi(\mathrm{Od}(\omega)) = R_{\alpha,\beta}(\Phi(\omega)) .$$

Next, let us consider the case when the carry over occurs at the first place, that is, $\omega = \begin{pmatrix} \cdots & j_1 & j_0 \\ \cdots & k_1 & l^{(0)}_{j_0} \end{pmatrix}$, $k_1 \neq l^{(1)}_{k_1}$,

$$\mathrm{Od}(\omega) = \begin{pmatrix} \cdots & j_n & \cdots & j_1 & j'_0 \\ \cdots & k_n & \cdots & k_1 + 1 & 1 \end{pmatrix} .$$

Then we have

$$\begin{aligned}
&\Phi(\mathrm{Od}(\omega)) - \Phi(\omega) \\
&= \sum_{N=2}^{\infty} \frac{\lambda_1}{\varepsilon_1} \frac{\lambda_2}{\varepsilon_2} \cdots \frac{\lambda_{N-1}}{\varepsilon_{N-1}} g_{N-1}(P^{(N-1)}_{\binom{j_{N-1}}{k_{N-1}}}) + \frac{\lambda_1}{\varepsilon_1} g_0(P^{(0)}_{\binom{j_0}{k_0+1}}) \\
&\quad - \left(\sum_{N=1}^{\infty} \frac{\lambda_0}{\varepsilon_0} \frac{\lambda_1}{\varepsilon_1} \cdots \frac{\lambda_{N-1}}{\varepsilon_{N-1}} g_{N-1}(P^{(N-1)}_{\binom{j_{N-1}}{k_{N-1}}}) + g_0(P^{(0)}_{\binom{j_0}{l^{(0)}_{j_0}}}) \right)
\end{aligned}$$

$$= \frac{\lambda_1}{\varepsilon_1} g_1(W^{(1)}_{\binom{j_1}{k_1}}) - g_0(P^{(0)}_{\binom{j_0}{l^{(0)}_{j_0}}}) .$$

Because $g_0(W^{(0)}_{\binom{k_0}{j_0}})$ is the translation of $I_{\binom{k_0}{j_0}}$ under $R_{\alpha,\beta}$, by Theorem 2 and the definition of $\sigma_{\left[\begin{smallmatrix} m \\ \varepsilon \end{smallmatrix}\right]}$, we get

$$g_1(i) = \theta_1(g_0(\sigma_{\left[\begin{smallmatrix} m_1 \\ \varepsilon_1 \end{smallmatrix}\right]}(i)))$$

and

$$\frac{\lambda_1}{\varepsilon_1} g_1(i) = g_0(\sigma_{\left[\begin{smallmatrix} m_1 \\ \varepsilon_1 \end{smallmatrix}\right]}(i)) = g_0(P^{(0)}_{\binom{i}{l^{(0)}_i}}) + g_0(W^{(0)}_{\binom{i}{l^{(0)}_i}}) .$$

Therefore

$$\frac{\lambda_1}{\varepsilon_1} g_1(W^{(1)}_{\binom{j_1}{k_1}}) - g_0(P^{(0)}_{\binom{j_0}{l^{(0)}_i}}) = g_0(W^{(0)}_{\binom{j_0}{l^{(0)}_{j_0}}}) .$$

This leads to the conclusion of the theorem in the case the carry over occurs at the first place. We get the proof for in general cases analogously. □

References

[1] P. ARNOUX, *Substitutions in Dynamics, Arithmetics and Combinatorics* (by N. PYTHEAS Fogg) Chapter 6, Springer Lecture Notes in Math. Vol. 1794.

[2] Y. HARA, SH. ITO, On real quadratic fields and periodic expansions, Tokyo J. Math. **12** (1989), No. 2, 357–370.

[3] SH. ITO, Some skew product transformations associated with continued fractions and their invariant measures, Tokyo J. Math. **9** (1986), 115–133.

[4] S. ITO, K. NAKAZAWA, H. RAO, An Algorithm Related to Three-Interval-Exchange Transformation, preprint.

[5] G. RAUZY, Exchanges D'intervalles et transformations induites, Acta. Arith. **34** (1979), 315–328.

[6] A. OSTROWSKI, Bemerkungen zur Theorie der Diophantischen Approximation I, II, Abh. Math. Sem. Hamburg **1** (1922), 77–98, 250–251.

Present Addresses:
YUKO ICHIKAWA
TOKYO NATIONAL COLLEGE OF TECHNOLOGY,
KUNUGIDA-MACHI, HACHIOJI-SHI, TOKYO, 193–0997 JAPAN.
e-mail: yuko@tokyo-ct.ac.jp

SHUNJI ITO
KANAZAWA UNIVERSITY,
KADOMA-MACHI, KANAZAWA-SHI, ISHIKAWA, 920–1192 JAPAN.
e-mail: ito@t.kanazawa-u.ac.jp