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Higher Dimensional Compacta with Algebraically Closed Function Algebras

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Abstract. For a compact Hausdorff space X, $C(X)$ denotes the ring of all complex-valued continuous functions on X. We say that $C(X)$ is *algebraically closed* if every monic algebraic equation with $C(X)$-coefficients has a root in $C(X)$. Modifying the construction of [2], we show that, for each $m = 1, 2, \ldots, \infty$, there exists an m-dimensional compact Hausdorff space $X(m)$ such that $C(X(m))$ is algebraically closed.

1. Introduction and Main Theorem

For a compact Hausdorff space X, $C(X)$ denotes the ring of all complex-valued continuous functions on X. We say that $C(X)$ is *algebraically closed* if every monic algebraic equation with $C(X)$-coefficients has a root in $C(X)$. Also, for a positive integer n, we say that $C(X)$ is *n-th root closed* if, for each $f \in C(X)$, the equation $z^n - f = 0$ with respect to z has a root in $C(X)$. Clearly the algebraic closedness implies the n-th root closedness for each n.

A topological characterization of the first-countable compact Hausdorff space X such that $C(X)$ is algebraically closed has been obtained by Countryman, Jr. [3] and Miura-Niijima [9] (see also [7] for a generalization). In particular, such spaces must be at most one-dimensional. On the other hand, there exist higher dimensional compact Hausdorff spaces X such that $C(X)$ is n-th root closed for each n ([2, Theorem 6.2, Corollary 6.3]). In this note we modify the construction of [2] to prove the following theorem.

MAIN THEOREM. *For each $m = 1, 2, \ldots, \infty$, there exists an m-dimensional compact Hausdorff space $X(m)$ such that $C(X(m))$ is algebraically closed.*

As in [2], our construction is based on the Cole construction and the transfer homomorphisms [1, Corollary 14.6]. In what follows, familiarity with the paper [2] is assumed.

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2. Proof of Main Theorem

We start with a preliminary consideration, following [4] and [5]. Let $\mathcal{P}_n$ be the set of all monic polynomial of degree n with complex coefficients. Each element $p(z)$ of $\mathcal{P}_n$ has the form

$$p(z) = z^n + \sum_{i=n-1}^{0} a_i z^i$$

where $a_i \in \mathbf{C}$ for each i. Throughout, z refers to the variable of polynomials. The correspondence

$$(a_0, \ldots, a_{n-1}) \longleftrightarrow p(z)$$

yields a bijection

$$\Phi : \mathbf{C}^n \to \mathcal{P}_n .$$

We define a map $\pi_n : \mathbf{C}^n \to \mathbf{C}^n$ as follows. For each $i = 1, \ldots, n$, let σ_i be the i-th elementary symmetric function of n-variables: e.g. $\sigma_1(x_1, \ldots, x_n) = \sum_i x_i$, $\sigma_2(x_1, \ldots, x_n) = \frac{1}{2} \sum_{i \neq j} x_i x_j$, etc. For a point $\alpha = (\alpha_1, \ldots, \alpha_n)$, let

$$\pi_n(\alpha) = ((-1)^i \sigma_i(\alpha_1, \ldots, \alpha_n))_{i=1}^n$$

Notice that $\Phi(\pi_n(\alpha)) = \prod_{i=1}^n (z - \alpha_i)$.

Let $D(\alpha)$ be the discriminant of the polynomial $\Phi(\pi_n(\alpha))$. By Fundamental Theorem of Algebra and Rouché's Theorem, the map π_n is an n-fold branched covering map, branched over the variety $\{D = 0\}$.

The symmetric group Σ_n of degree n naturally acts on $\mathbf{C}^n$ as the permutation of coordinates:

$$\sigma \cdot (\alpha_1, \ldots, \alpha_n) = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}) , \quad \sigma \in \Sigma_n , \quad (\alpha_1, \ldots, \alpha_n) \in \mathbf{C}^n .$$

Clearly $\pi_n \circ \sigma = \pi_n$ for each $\sigma \in \Sigma_n$.

For a compact Hausdorff space X and a continuous map $a = (a_0, \ldots, a_{n-1}) : X \to \mathbf{C}^n$, let $P_a(z) = z^n + \sum_{i=n-1}^{0} a_i z^i \in C(X)[z]$. Examining the identification given by $\Phi : \mathbf{C}^n \to \mathcal{P}_n$, we see that the following two statements are equivalent.

(A) There exist continuous functions $\rho_1, \ldots, \rho_n \in C(X)$ such that $P_a(z) = \prod_{i=1}^n (z - \rho_i)$.

(B) There exists a continuous map $\rho : X \to \mathbf{C}^n$ such that $\pi_n \circ \rho = a$.

The above equivalence translates the algebraic closedness of $C(X)$ to the existence of a lift of an arbitrary map $X \to \mathbf{C}^n$ with respect to $\pi_n (n \geq 1)$.

Next we recall the Cole construction on the basis of [2] (cf. [10, Chapter 3, p.194-197]). For a compact Hausdorff space X, the set of all continuous maps $X \to \mathbf{C}^n$ is denoted by $\mathrm{Map}(X, \mathbf{C}^n)$. For a subset S of $\mathrm{Map}(X, \mathbf{C}^n)$ and an integer $n \geq 2$, let $R(X; n, S)$ be the space defined by

$$R(X; n, S) = \{(x, (z_a)_{a\in S}) \in X \times (\mathbf{C}^n)^S \mid a(x) = z_a \text{ for each } a \in S\}$$

and let $\pi^S_{X;n} : R(X; n, S) \to X$ be the projection given by $\pi^S_{X;n}(x, (z_a)_{a\in S}) = x$. The space $R(X; n, S)$ and the map $\pi^S_{X;n}$ form the pull-back diagram:

$$\begin{array}{ccc} R(X; n, S) & \xrightarrow{\Delta_{a\in S}\overline{a}} & (\mathbf{C}^n)^S \\ \pi^S_{X;n}\big\downarrow & & \big\downarrow (\pi_n)^S \\ X & \xrightarrow{\Delta_{a\in S}a} & (\mathbf{C}^n)^S \end{array}$$

where $\Delta_{a\in S}a : X \to (\mathbf{C}^n)^S$ denotes the map defined by $(\Delta_{a\in S}a)(x) = (a(x))_{a\in S}$. In particular, we see

(C) for each element $a : X \to \mathbf{C}^n$ of S, there exists a map $\overline{a} : R(X; n, S) \to \mathbf{C}^n$ such that $a \circ \pi^S_{X;n} = \pi_n \circ \overline{a}$.

For simplicity, the space $R(X; n, \mathrm{Map}(X, \mathbf{C}^n))$ and the projection $\pi^{\mathrm{Map}(X,\mathbf{C}^n)}_{X;n}$ are denoted by $R(X; n)$ and $\pi_{X;n} : R(X; n) \to X$ respectively.

When S is a finite subset, then the S-fold product action of symmetric group $(\Sigma_n)^S$ on $(\mathbf{C}^n)^S$ naturally induces an action on $R(X; n, S)$ given by:

$$(\sigma_a)_{a\in S} \cdot (x, (z_a)_{a\in S}) = (x, (\sigma_a \cdot z_a)_{a\in S}),$$
$$(\sigma_a)_{a\in S} \in (\Sigma_n)^S,\ (x, (z_a)_{a\in S}) \in R(X; n, S).$$

The same proof as the one of [2, Proposition 3.5] works to prove the following.

PROPOSITION 2.1. *For each integer $n > 1$, the projection $\pi_{X;n} : R(X; n) \to X$ induces a monomorphism $(\pi_{X;n})^* : \check{H}^*(X; \mathbf{Q}) \to \check{H}^*(R(n; X); \mathbf{Q})$ of the Čech cohomologies of rational coefficients.*

PROOF OF MAIN THEOREM. First we prove the theorem for $1 \le m < \infty$.

Starting with $X_0 = S^m$, the m-dimensional sphere, we construct, by a transfinite induction, an inverse spectrum $\mathcal{S}$ of length ω_1, the first uncountable ordinal.

The ordinal ω_1 is represented as a countable disjoint union $\cup_{n=2}^{\infty}\Lambda_n$ of uncountable sets Λ_n. For $\alpha < \omega_1$ with $\alpha \in \Lambda_n$, let $X_{\alpha+1} = R(X_\alpha; n)$ and let $p^{\alpha+1}_\alpha = \pi_{X_\alpha,n} : X_{\alpha+1} \to X_\alpha$. When $\beta < \omega_1$ is a limit ordinal, let $X_\beta = \lim_{\leftarrow}\{X_\alpha, p^\gamma_\alpha; \alpha < \gamma < \beta\}$. For each $\alpha < \beta$, let $p^\beta_\alpha : X_\beta \to X_\alpha$ be the limit projection.

Let $\mathcal{S}$ be the resulting inverse spectrum and let $X(m) = \lim_{\longleftarrow} \mathcal{S}$. For each $\alpha < \omega_1$, let $p_\alpha : X(m) \to X_\alpha$ be the limit projection. As in the proof of [2, Theorem 6.2], we can make use of Proposition 2.1 to prove that $\dim X(m) = m$. In order to show that $C(X(m))$ is algebraically closed, we take an arbitrary integer $n \ge 1$ and choose an arbitrary monic polynomial $P(z) = z^n + \sum_{i=n-1}^{0} f_i z^i$ of degree n in $C(X(m))[z]$, where $f_i \in C(X(m))$ for each $i = 0, \dots, n-1$. For notational simplicity, let $P(x,z) = z^n + \sum_{i=n-1}^{0} f_i(x)z^i \in \mathbf{C}[z]$ for each $x \in X(m)$. Now let $a = (a_0, \dots, a_{n-1}) : X(m) \to \mathbf{C}^n$ be the map satisfying $\Phi(a(x)) = P(x,z)$ for each $x \in X(m)$. It is easy to see that a is actually continuous. Since $X(m)$ is the limit of an inverse spectrum of uncountable length (and since $\mathbf{C}^n$ is second countable), there exists an ordinal $\alpha < \omega_1$ with $\alpha \in \Lambda_n$ and a map $a_\alpha : X_\alpha \to \mathbf{C}^n$ such that $a = a_\alpha \circ p_\alpha$.

By the definition of $X_{\alpha+1} = R(X_\alpha; n)$ and (C), there exists a map $\rho_\alpha : X_\alpha \to \mathbf{C}^n$ such that $\pi_n \circ \rho_\alpha = a_\alpha \circ p_\alpha^{\alpha+1}$. Then the map $\rho := \rho_\alpha \circ p_\alpha = (\rho_1, \dots, \rho_n) : X(m) \to \mathbf{C}^n$ satisfies

$$\pi_n \circ \rho = \pi_n \circ \rho_\alpha \circ p_\alpha = a_\alpha \circ p_\alpha = a\,.$$

In view of the equivalence of (A) and (B), this means that the polynomial $P(x,z)$ satisfies

$$P(x,z) = \Phi(a(x)) = \prod_{i=1}^{n} (z - \rho_i(x))$$

for each $x \in X(m)$. In other words, $P(z)$ admits a factorization $P(z) = \prod_{i=1}^{n}(z - \rho_i)$. Therefore $C(X(m))$ is algebraically closed.

In order to obtain an infinite dimensional space $X(\infty)$, we take the topological sum $\bigoplus_{m=1}^{\infty} X(m)$ and let $X(\infty) = \beta(\bigoplus_{m=1}^{\infty} X(m))$, the Stone-Čech compactification of $\bigoplus_{m=1}^{\infty} X(m)$. It is easy to see that $X(\infty)$ is the desired space.

This completes the proof.

REMARK 2.2. If a compact Hausdorff space X has the n-th root closed $C(X)$, then the first Čech cohomology $\check{\mathrm{H}}^1(X; \mathbf{Z})$ is n-divisible. Hence if $C(X)$ is algebraically closed, then $\check{\mathrm{H}}^1(X; \mathbf{Z})$ must be divisible. For the relationship between the divisibility of $\check{\mathrm{H}}^1(X; \mathbf{Z})$ and the approximate n-th root closedness of $C(X)$, see [2, section 4]. Gorin and Lin constructed a 2-dimensional compact metric space X such that $\check{\mathrm{H}}^1(X; \mathbf{Z})$ is divisible, while there exists an algebraic equation of degree 4 with $C(X)$-coefficients which has no root in $C(X)$ ([4, Theorem 3.4]). This suggests the following conjecture.

CONJECTURE. *There exists a compact Hausdorff space Y such that $C(Y)$ is n-th root closed for each $n \geq 2$, but not algebraically closed.*

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