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C_n-moves and V_n-equivalence for Links

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Abstract. K. Habiro defined a C_n-move which is a local move on oriented links. He also proved that two knots are not distinguished by any Vassiliev invariants of order less than n if and only if they are related by a finite sequence of C_n-moves. In the case of $n \geq 3$, it is known that the result does not hold for links. In this note we will introduce a special C_n-move and give a "geometrical" necessary and sufficient condition using the terms of C_n-moves for that two links are not distinguished by any Vassiliev invariants of order less than 3 or 4.

1. Introduction

In this paper we consider oriented links in the 3-sphere S^3. A *C_n-move* ($n \geq 2$) is a local move indicated in Figure 1.1 on oriented links defined by K. Habiro [5].

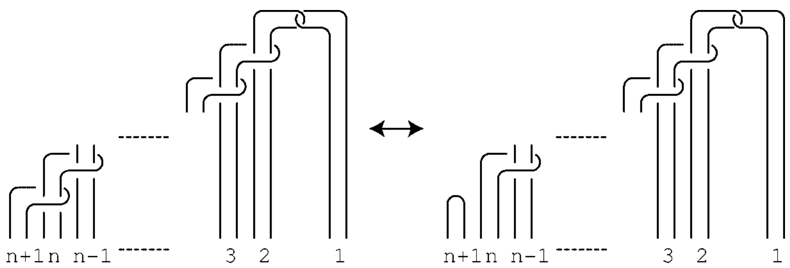

FIGURE 1.1

A C_1-move is defined as a crossing change. We remark that a C_2-move is equivalent to a Δ-move in Figure 1.2 (i) and a C_3-move is also called a clasp-pass move ([6]) in Figure 1.2 (ii).

We say that two links (or knots) are *V_n-equivalent* ([14]) if they are not distinguished by any Vassiliev invariants of order less than or equal to n. A C_n-move is closely related to Vassiliev invariants. In fact M. N. Goussarov and Habiro showed the following theorem independently:

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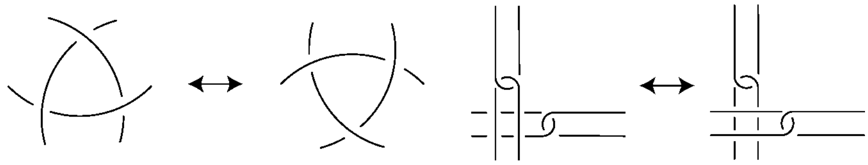


(i) (ii)

FIGURE 1.2

THEOREM 1.1 ([4, 7]). *Let K and K' be oriented knots and n a positive integer. Then K and K' can be transformed into each other by a finite sequence of C_{n+1}-moves if and only if they are V_n-equivalent.*

In this paper we consider the link case.

Since a C_2-move is a Δ-move and a Vassiliev invariant of order 1 for a μ-component link($\mu \geq 2$) is generated by the linking numbers of 2-component sublinks of the link ([11]), the result of H. Murakami and Y. Nakanishi is rewritten to the following:

THEOREM 1.2 ([12]). *Let L and L' be oriented links. Then L and L' can be transformed into each other by a finite sequence of C_2-moves if and only if they are V_1-equivalent.*

In the case of $n \geq 2$, the following theorem holds:

THEOREM 1.3 ([3, 15, 18]). *Let L and L' be oriented links in S^3 and n a positive integer. If L and L' can be transformed into each other by a finite sequence of C_n-moves, then $v(L) = v(L')$ for any Vassiliev invariant v of order less than n.*

It is known that the V_n-equivalence for links does not imply the C_{n+1}-equivalence which is the equivalence relation generated by C_{n+1}-moves and ambient isotopies. For example, the Whitehead link and the 2-component trivial link are V_2-equivalent. However they are not C_3-equivalent(see §5).

We define a special C_n-move which is called an *SC_n-move* as follows: Let $\alpha_1, \ldots, \alpha_{n+1}$ be the arcs shown in a C_n-move and $c(\alpha_i)$ denote the component of the link which contains α_i for each i with $i = 1, 2, \ldots, n+1$. If there is an arc α_k such that $c(\alpha_k) \neq c(\alpha_i)$ for all i with $i \neq k$, we call the C_n-move an SC_n-move. Using this local move, we obtain the necessary and sufficient conditions for that two links are V_2-equivalent or V_3-equivalent:

THEOREM 1.4. *Let L and L' be oriented links. Then L and L' can be transformed into each other by a finite sequence of C_3-moves and SC_2-moves if and only if they are V_2-equivalent.*

In the case of $n = 3$, we obtain the following result for 2-component links:

THEOREM 1.5. *Let L and L' be 2-component oriented links. Then L and L' can be transformed into each other by a finite sequence of C_4-moves and SC_3-moves if and only if they are V_3-equivalent.*

REMARK. After this work, J.-B. Meilhan and A. Yasuhara refine Theorem 1.5 by showing that an SC_3-move is generated by a finite sequence of C_4-moves ([10]).

2. Vassiliev invariants and C_n-moves

2.1. Vassiliev invariants for links. We can define Vassiliev invariants for links as for knots. Let v be an invariant for links which takes values in an abelian group. Then we can extend v to an invariant for singular links by the Vassiliev skein relation:

$$v(L_D) = v(L_+) - v(L_-)\,,$$

where a singular link is an immersion of a disjoint union of oriented circles in S^3 which has only transversal double points as its singularities, and L_D, L_+ and L_- denote the diagrams of singular links which are identical except near one point as is shown in Figure 2.1. A singular link is considered as a flat vertex graph.

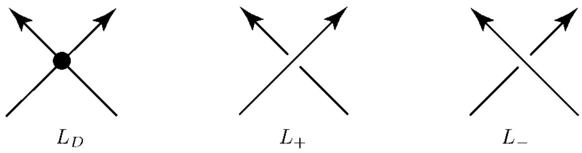


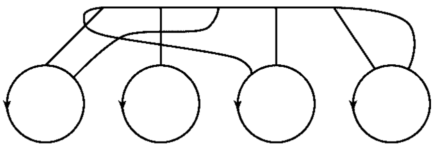
FIGURE 2.1

An invariant v is called a *Vassiliev invariant of order n* if $v(L) = 0$ for any singular link L with more than n double points and there exists a singular link L' with n double points such that $v(L') \neq 0$.

A *chord diagram of order n* is a disjoint union of counterclockwise oriented circles with n chords. A chord diagram of order n can be associated to a singular link with n double points by connecting the preimage of each double point with a chord. It is known that a Vassiliev invariant of order n is determined by a set of axioms and initial data called an actually table containing values for the chord diagrams of order n and immersions that respect them of order less than n. Moreover any invariant of order n is determined, modulo the invariants of order less than n, by its values on chord diagrams of order n ([2]).

Jacobi diagrams, generalizations of chord diagrams, were introduced by D. Bar-Natan [1]. K. Y. Ng and T. Stanford defined a certain kind of Jacobi diagrams called a one-branch tree diagram in [14] (see also [16]). A *one-branch tree diagram of order n* is a trivalent graph with $2n$ vertices as shown in Figure 2.2. It consists of μ circles and a graph G, where G is isomorphic to the standard n-tree in Figure 2.3. All univalent vertices in G are attached to a circle.

The set of all Vassiliev invariants of order less than n forms a vector space. A basis of the vector space of Vassiliev invariants of order less than or equal to 2 is determined by Murakami [11]. From this result, we can see that two μ-component links $L = K_1 \cup K_2 \cup \cdots \cup K_\mu$ and



one-branch tree diagram of order 6

FIGURE 2.2

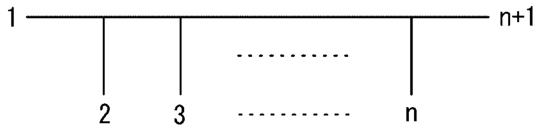


FIGURE 2.3

$L' = K'_1 \cup K'_2 \cup \cdots \cup K'_\mu$ are V_2-equivalent if and only if $lk(K_i \cup K_j) = lk(K'_i \cup K'_j)$ for any i and j with $1 \leq i < j \leq \mu$ and $a_2(K_i) = a_2(K'_i)$ for any i with $i = 1, \ldots, \mu$, where $lk(K_i, K_j)$ and $a_k(K_i)$ denote the linking number of $K_i \cup K_j$ and the k-th coefficient of the Conway polynomial for K_i, respectively. In the case of order less than or equal to 3, T. Kanenobu, Y. Miyazawa and A. Tani [8] obtained a basis of the vector space. Using the result, we can see that two links L and L' are V_3-equivalent if and only if they are V_2-equivalent and $V^{(3)}(K_i) = V^{(3)}(K'_i)$ for any i with $i = 1, \ldots, \mu$ and $a_3(K_i \cup K_j) = a_3(K'_i \cup K'_j)$ for any i and j with $1 \leq i < j \leq \mu$, where $V^{(3)}(K_i)$ denotes the third derivative at $t = 1$ of the Jones polynomial of K_i.

2.2. Band description of C_n-moves. As stated above, a C_1-move is defined as a crossing change. Figure 2.4 shows that a crossing change is realized by fusion with a Hopf link. Similarly Figure 2.5 shows that a Δ-move (a C_2-move) is realized by fusion with Borromean rings. Generally it is shown that a C_n-move is realized by fusion with an $(n + 1)$-component link which has a Brunnian property by K. Taniyama and Yasuhara in [20]. This $(n + 1)$-component link is called a *C_n-link*. We remark that a C_1-link is a Hopf link and a C_2-link is Borromean rings. A C_3-link is a 4-component link as illustrated in Figure 2.6.

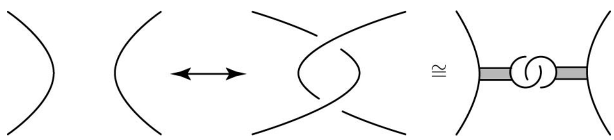


FIGURE 2.4

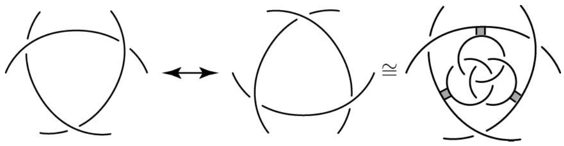


FIGURE 2.5

Murakami and Nakanishi [12] showed that a fusion-band with a Hopf link can leap over a subarc by a single Δ-move (Figure 2.7). Similarly it is shown that a fusion-band with a C_n-link can leap over a subarc by a finite sequence of C_{n+1}-moves by Taniyama and Yasuhara in [20].

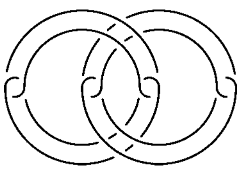


FIGURE 2.6

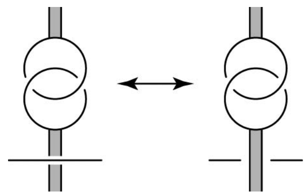


FIGURE 2.7

Let F be the result of fusion for a link L with some links $L_1, L_2, \ldots, L_n$. In this paper, the link which is represented by F is denoted by $\tilde{F}$. We call the intersection arc of a fusion-band and the link L a *root* of the fusion-band.

2.3. SC_n**-moves and Vassiliev invariants of order** n **for links.** We state a relation between SC_n-moves and Vassiliev invariants of order n for links. Let L and L' be two links which are related by a single C_n-move and v a Vassiliev invariant of order n. Then there exists a one-branch tree diagram T such that $v(L) - v(L') = \pm v(T)$ ([16, 17]). Furthermore, by the result of Y. Ohyama, it follows that $v(T_s) = 0$ for the one-branch tree diagram T_s corresponding to an SC_n-move ([16]). Hence we can obtain the following proposition:

PROPOSITION 2.1. *Let L and L' be two links which are related by a finite sequence of SC_n-moves. Then $v(L) = v(L')$ for any Vassiliev invariant v of order n.*

2.4. Self Delta-moves for links. If three arcs in a Δ-move are contained in the same component, the Δ-move is called a *self Δ-move*. If two links can be transformed into each other by a finite sequence of self Δ-moves, they are said to be *Δ-link homotopic* (or *self Δ-equivalent*). For 2-component links, Nakanishi and Ohyama gave the self Δ-equivalence classification completely.

THEOREM 2.2 ([13]). *Let L and L' be 2-component links. Then L and L' are self Δ-equivalent if and only if $\delta_1(L) = \delta_1(L')$ and $\delta_2(L) = \delta_2(L')$, where $\delta_1(L)$ and $\delta_2(L)$*

are defined by $\delta_1(L) = a_1(L)(= lk(L))$ *and* $\delta_2(L) = a_3(L) - a_1(L)\{a_2(K_1) + a_2(K_2)\}$ *for* $L = K_1 \cup K_2$ *respectively.*

In [13], it is shown that for any self Δ-equivalence class, we can choose the result of fusion S for the $(2, 2l)$-torus link with at most one set of Borromean rings and a finite number of C_3-links such that $\tilde{S}$ is a representative element of the class (Figure 2.8). We call the fusion the *standard fusion* for the equivalence class in this paper.

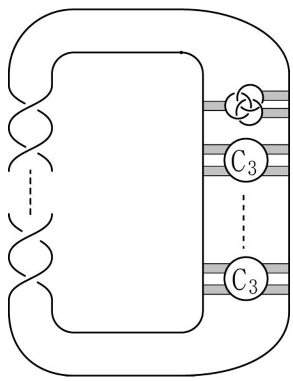


FIGURE 2.8

3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Let L and L' be oriented links which can be transformed into each other by a finite sequence of C_3-moves and SC_2-moves. If L and L' are related by a C_3-move, then they are V_2-equivalent by Theorem 1.3. If L and L' are related by an SC_2-move, then they are V_1-equivalent by Theorem 1.3 and $v(L) - v(L') = 0$ for any Vassiliev invariant v of order 2 by Proposition 2.1. In any cases, we can see that they are V_2-equivalent.

Next we show the 'if' part of Theorem 1.4. Here we prove the case of 2-component links. The proof for a general case can be done similarly. Assume that 2-component links $L = K_1 \cup K_2$ and $L' = K_1' \cup K_2'$ are V_2-equivalent, namely $lk(L) = lk(L')$ and $a_2(K_i) = a_2(K_i')$ $(i = 1, 2)$. By the result of [12], both of L and L' can be transformed into the $(2, 2l)$-torus link T_l by a finite sequence of C_2-moves (where $l = lk(L) = lk(L')$). Hence L and L' can be realized by the results of fusion for T_l with some Borromean rings. The results of fusion are denoted by L_1 and L_1' respectively. In the case of L and L' are μ-component links($\mu \geq 3$), they can be realized by the results of fusion for link as illustrated in Figure 3.1 with some Borromean rings, where $l_{i,j}$ denotes the linking number of i-th and j-th components of L for $1 \leq i < j \leq \mu$.

From now we deform L_1. At first we define types of Borromean rings in L_1. If the roots of three fusion-bands attached to a set of Borromean rings are contained in one component of

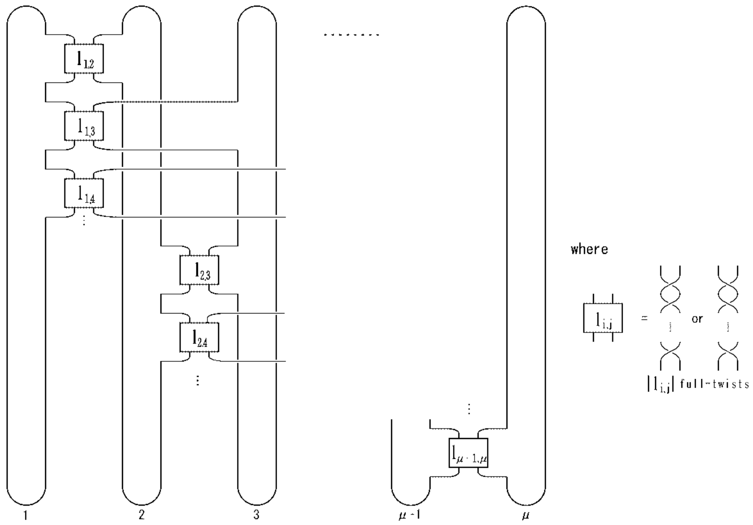


FIGURE 3.1

L_1, we say that the set of Borromean rings is of *type I*. If the roots of three bands attached to a set of Borromean rings are contained in two components of L_1, we say that the set of Borromean rings is of *type II*. A set of Borromean rings of type II can be removed by an SC_2-move. The result of fusion obtained from L_1 by removing all of the Borromean rings of type II is denoted by L_2. Obviously $\tilde{L_2}$ can be obtained from $\tilde{L_1}$ by applying SC_2-moves. Figure 3.2 shows an example of L_2.

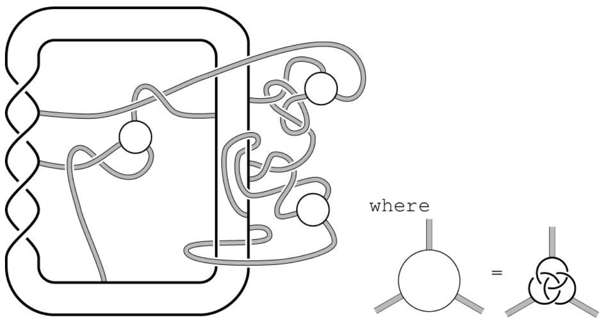


FIGURE 3.2

Let A_1 and A_2 be subarcs of T_l as shown in Figure 3.3. Take tubular neighborhoods N_1 and N_2 of A_1 and A_2 in S^3 respectively. We slide the root of each fusion-band attached to Borromean rings to A_1 or A_2 along T_l. Obviously the link obtained here is ambient isotopic to the link $\tilde{L_2}$. Next we move each Borromean rings and fusion-bands attached to the Borromean rings into N_1 or N_2 by applying C_3-moves. This is possible because a fusion-band attached to Borromean rings can leap over a subarc of the torus link T_l or another fusion-band by a finite sequence of C_3-moves. The result of fusion is denoted by L_3 (Figure 3.4).

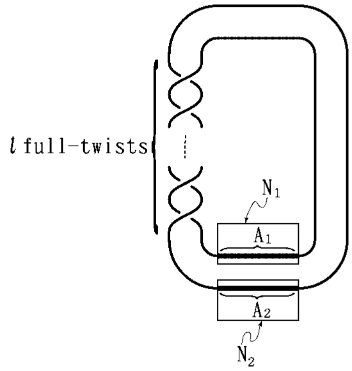


FIGURE 3.3

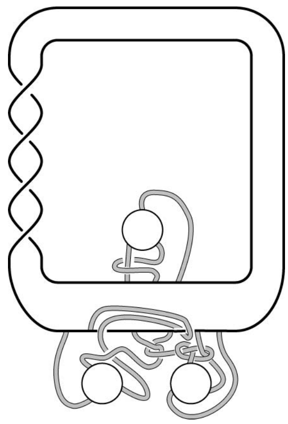


FIGURE 3.4

Similarly we deform L'_1 and obtain L'_3. We remark that $\tilde{L_3} = K_{3,1} \cup K_{3,2}$ and $\tilde{L'_3} = K'_{3,1} \cup K'_{3,2}$ are obtained from L and L' respectively by C_3-moves and SC_2-moves. As stated above, C_3-moves and SC_2-moves do not change the values of any Vassiliev invariants of order less than or equal to 2. Hence we can see that $K_{3,i}$ and $K'_{3,i}$ are V_2-equivalent for $i = 1, 2$. Since $K_{3,i} - N_i$ and $K'_{3,i} - N_i$ are trivial arc in the 3-ball $\overline{S^3 - N_i}$, $K_{3,i}$ and $K'_{3,i}$ are related by C_3-moves ($i = 1, 2$) by Theorem 1.1. Therefore we can conclude that L and L' can be related by a finite sequence of C_3-moves and SC_2-moves.

4. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. The idea of the proof is similar to that of Theorem 1.4.

Since we can see that the C_4-equivalence for links implies the V_3-equivalence, we assume that two 2-component links L and L' can be transformed into each other by an SC_3-move. Then we can see that they are V_2-equivalent by Theorem 1.3 and $v(L) = v(L')$ for any Vassiliev invariant v of order 3 from Proposition 2.1. Therefore we obtain that they are V_3-equivalent.

Next we show the 'if' part of Theorem 1.5. Assume that two 2-component links $L = K_1 \cup K_2$ and $L' = K'_1 \cup K'_2$ are V_3-equivalent, i.e. the following hold: $lk(L) = lk(L')$, $a_2(K_i) = a_2(K'_i)$ for any i with $i = 1, 2$, $a_3(L) = a_3(L')$ and $V^{(3)}(K_i) = V^{(3)}(K'_i)$ for any i with $i = 1, 2$. Then we have $\delta_1(L) = \delta_1(L')$ and $\delta_2(L) = \delta_2(L')$. By Theorem 2.2, we can conclude that L and L' are self Δ-equivalent, that is, they belong to the same self Δ-equivalence class. Let S be the standard fusion for the equivalence class which contains L and L', that is, S is the result of fusion for the $(2, 2l)$-torus link T_l (where $l = lk(L) = lk(L')$) with some links. Then both of L and L' can be realized by the results of fusion for S with some Borromean rings, where all of the Borromean rings are type I defined in §3. These results of fusion are denoted by L_1 and L'_1 respectively.

We take subarcs A_1 and A_2 of T_l and neighborhoods N_1 and N_2 of A_1 and A_2 respectively as same as in the proof of Theorem 1.4 (Figure 3.3). C_i denotes the component of T_l which contains A_i for $i(= 1, 2)$. Let $\mathcal{L}_0 = \{\beta_1, \dots, \beta_k\}$ be the set of Borromean rings and C_3-links connected with T_l in S and $\mathcal{L}_1$ the set of Borromean rings connected with S in L_1. For $i(= 1, 2)$, $\mathcal{L}_{1,i}$ denotes the subset of $\mathcal{L}_1$ which consists of elements each of which is connected with C_i. Let $\mathcal{B}_i$ be the set of fusion-bands attached to an element of $\mathcal{L}_i$ for $i(= 0, 1)$, and $\mathcal{B}_{1,i}$ the subset of $\mathcal{B}_1$ consists of an element which is attached to an element in $\mathcal{L}_{1,i}$ for $i(= 1, 2)$. For each β_i in $\mathcal{L}_0$, we take the 3-ball D_i which satisfies the following: Let $b_{i,1}, b_{i,2}, b_{i,3}$ and $b_{i,4}$ be the fusion-bands which attached to β_i, where if β_i is Borromean rings, then we set $b_{i,4} = \emptyset$. Then $S \cap D_i$ consists of β_i and $b'_{i,1}, \dots, b'_{i,4}$, where $b'_{i,j}$ is a small part of $b_{i,j}$ which is connected and contains the intersection $\beta_i \cap b_{i,j}$ for $j = 1, \dots, 4$ ($S \cap D_i$ looks like Figure 4.1).

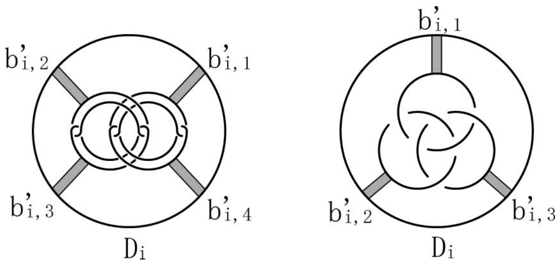


FIGURE 4.1

First we slide all of the roots of fusion-bands in $\mathcal{B}_1$ to A_1 or A_2 along $\tilde{S}$ as doing in the proof of Theorem 1.4. After that, we can remove the intersection between any fusion-band in $\mathcal{B}_1$ and any 3-ball D_i in the following way: Suppose that a fusion-band b in $\mathcal{B}_1$ and D_i intersect. Then $\tilde{S} \cap D_i$ is a trivial tangle. Hence we can sweep out b from D_i. As a result, if b and another band attached to β_i intersect, we can remove the intersection as in Figure 4.2. The result is denoted by L_2.

Here we prove the following lemmas to continue the deformation. By the result of Taniyama and Yasuhara [20], we have the following lemmas.

LEMMA 4.1. *A fusion-band attached to Borromean rings can pass through a fusion-band attached to another set of Borromean rings or a C_3-link by a finite sequence of C_4-moves.*

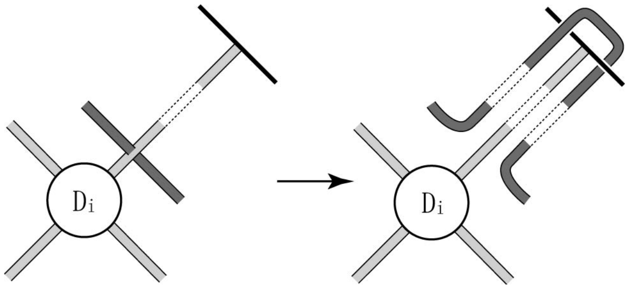


FIGURE 4.2

LEMMA 4.2. *A fusion-band in* $\mathcal{B}_{1,1}$ (*resp.* $\mathcal{B}_{1,2}$) *can leap over a subarc of the component* C_2 (*resp.* C_1) *of* T_l *by an* SC_3*-move.*

PROOF OF LEMMA 4.1. In [5, 20], it is proved that the local moves in Lemma 4.1 are realized by C_5-moves or C_6-moves respectively. Since a C_k-move is realized by twice C_{k-1}-moves for any k, the moves in Lemma 4.1 are realized by a finite sequence of C_4-moves. □

PROOF OF LEMMA 4.2. A fusion-band of Borromean rings can leap over a subarc of the link by a C_3-move. If the band has a root in A_1, then the other two bands of the Borromean rings also have roots in A_1. Hence the C_3-move is an SC_3-move. □

Using Lemmas 4.1 and 4.2, we can move any element of $\mathcal{L}_1$ and all fusion-bands in $\mathcal{B}_1$ into the neighborhood N_1 or N_2 by applying C_4-moves and SC_3-moves in the following way: We begin with moving each element of $\mathcal{L}_{1,i}$ into N_i by an isotopy. Next we collect fusion-bands in $\mathcal{B}_{1,i}$ into N_i. We consider the following cases (Figure 4.3):

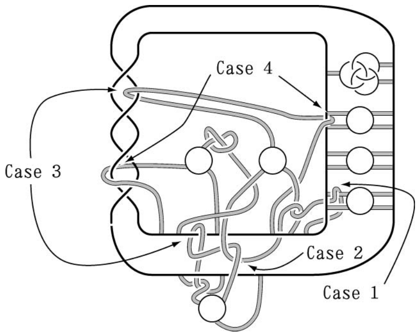


FIGURE 4.3

Case 1. Suppose that a fusion-band b in $\mathcal{B}_1$ is linked to another fusion-band b' in $\mathcal{B}_0$. Then b and b' are attached to the different Borromean rings or C_3-link. Hence b can pass through b' by a finite sequence of C_4-moves from Lemma 4.1.

Case 2. Suppose that a fusion-band b in $\mathcal{B}_{1,1}$ is linked to another fusion-band b' in $\mathcal{B}_{1,2}$. Then b can pass through b' by a finite sequence of C_4-moves by the same reason as in Case 1.

Case 3. Suppose that a fusion-band b in $\mathcal{B}_{1,1}$ (resp. $\mathcal{B}_{1,2}$) is linked to a subarc of the component C_2 (resp. C_1). From Lemma 4.2 the band b can leap over the subarc by an SC_3-move.

Case 4. Suppose that a fusion-band b in $\mathcal{B}_{1,i}$ is linked to a subarc of the component C_i. Then we slide the part of b which is linked to the subarc along C_i into N_i. They can be done by a finite sequence of C_4-moves and SC_3-moves because the band b can pass through another band in $\mathcal{B}_0$ and subarcs of the component C_{i+1} (where C_3 means C_1) by the moves (cf. Cases 1 and 2). See Figure 4.4.

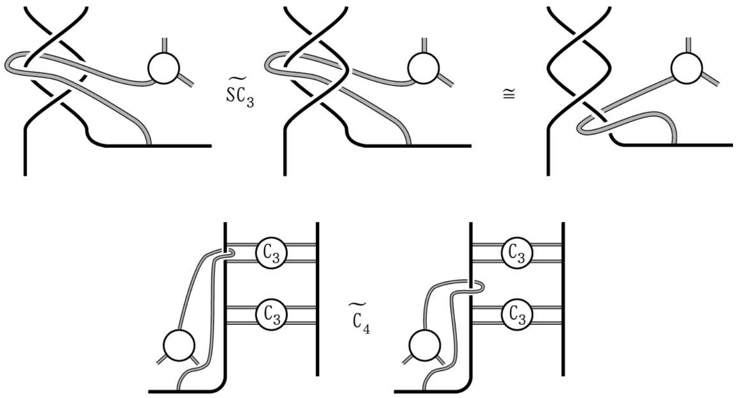


FIGURE 4.4

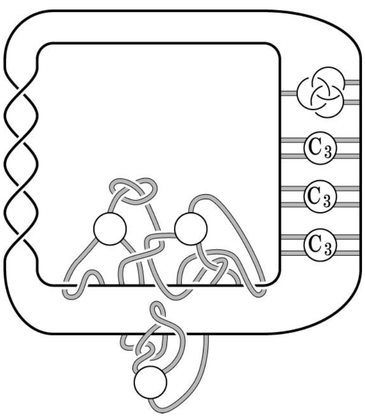


FIGURE 4.5

As a result, we obtain a fusion as in Figure 4.5, for example. This is denoted by L_3.

We deform L_1' as doing for L_1 and obtain L_3'. We remark that $\tilde{L_3}$ and $\tilde{L_3'}$ can be obtained from L and L' respectively by a finite sequence of C_4-moves and SC_3-moves. Therefore each component of $\tilde{L_3}$ and $\tilde{L_3'}$ are V_3-equivalent respectively. Hence they can be transformed into each other by C_4-moves by Theorem 1.1 and this completes the proof.

5. Example

The Whitehead link $W = W_1 \cup W_2$ and the 2-component trivial link $U = U_1 \cup U_2$ are V_2-equivalent because of $lk(W) = lk(U) = 0$ and $a_2(W_i) = a_2(U_i) = 0$ for $i(= 1, 2)$. On the other hand, we can see that they cannot be transformed into each other by a finite sequence of C_3-moves using the following theorem proved by Taniyama and Yasuhara:

THEOREM 5.1 ([19]). *Let $L = K_1 \cup K_2$ and $L' = K_1' \cup K_2'$ be 2-component links. Then L and L' can be transformed into each other by a finite sequence of C_3-moves if and only if $lk(L) = lk(L')$, $a_2(K_i) = a_2(K_i')$ for $i = 1, 2$ and $a_3(L) \equiv a_3(L')$* (mod 2).

In fact $a_3(W)$ is not congruent to $a_3(U)$ modulo 2. However Figure 5.1 shows that W can be transformed into U by an SC_2-move.

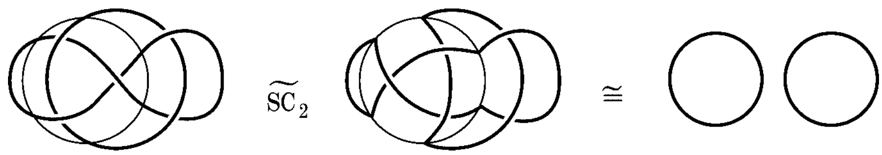


FIGURE 5.1

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